Functional Equations

Mitchell Lee

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1 From last week

1. (T. Mildorf) Let $P(x)$ be a polynomial with positive coefficients such that $P(1) \geq 1$. Prove that
$$P\left(\frac{1}{x}\right) \geq \frac{1}{P(x)}$$
for all $x > 0$.

2. (USAMO 2009) For $n \geq 2$ let $a_1, a_2, \ldots a_n$ be positive real numbers such that
$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \leq \left(n + \frac{1}{2}\right)^2.$$

3. (ISL 1996) Let $a_1, a_2 ... a_n$ be non-negative reals, not all zero. Show that
(a) The polynomial $p(x) = x^n - a_1x^{n-1} + ... - a_{n-1}x - a_n$ has precisely 1 positive real root R.
(b) Let $A = \sum_{i=1}^n a_i$ and $B = \sum_{i=1}^n ia_i$. Show that $A^A \leq R^B$.

2 General strategies

2.1 Guess a solution

This should be the first thing that you try. The solutions to most Olympiad functional equations will not be a ridiculous composition of random things you've never heard about. (There, of course, are some exceptions, but they are rare.) Try plugging in degree 0 and 1 polynomials, powers of x, and exponential functions for the functions you are trying to solve for. If these don't work, then you should try more exotic functions - but don't neglect the possibility that there are no solutions to the functional equation. Don't neglect the possibility that there are multiple solutions, either.

2.2 Find the value of the function at identities

Once you've guessed your solution g for the function f, prove that $f(0) = g(0)$. Sometimes, you should also prove that $f(1) = g(1)$. This is useful because 0 and 1 are identities, and multiplying something by 0 gives you 0.

2.3 Plug in things that will help you

There will almost always be two or more free variables in your functional equation - try plugging in small values like 0 and 1, or seeing what would happen if some of the free variables had the same value, or were additive or multiplicative inverses.

2.4 Use symmetry

Try plugging in things that will cause one or both sides of the equation to be symmetric in its variables. Then flip the variables.

2.5 Prove some properties of your function

A function f is called **injective** or **one-to-one** if $f(x) = f(y)$ implies $x = y$ where x, y are any reals. A function f is called **surjective** or **onto** if for any y in the codomain of f, there exists an x with $f(x) = y$. A function is called **bijective** if it is injective and surjective. A function f is called **increasing** if $f(x) < f(y)$ for all $x < y$. A function f is called **decreasing** if $f(x) > f(y)$ for all $x < y$. Proving that your function satisfies one or more of these properties is often a key step to solving a functional equation.

Also, keep in mind that increasingness or decreasingness implies injectivity. Additionally, if $f(g(x))$ is surjective, then so is $f(x)$. If it is injective, then so is $g(x)$.

3 Pitfalls

3.1 Prove that your answer actually works

For your solution to be correct in a functional equation, you must state all of the functions which satisfy the equation, prove that no other functions satisfy the equation, and prove that the functions which you claim to be the answer actually satisfy the equation. Usually, you can just write ”it can be easily verified that these functions satisfy the equation,” but sometimes you need to be more careful.

3.2 Don’t use limits unless you have proven continuity

Seriously, just don’t.

3.3 Don’t use inverse functions unless you have proven injectivity

Again, just don’t do it.

4 Cauchy’s equation

One of the simplest examples of a functional equation is Cauchy’s functional equation, $f(x)+f(y) = f(x+y)$, which you will encounter often. This equation would appear to have the solutions $f(x) = cx$ where c is a constant. However, despite its simple appearance, it has an infinitude of pathological solutions $f : \mathbb{R} \mapsto \mathbb{R}$.

Fortunately, not all hope is lost. Note that $f(nx) = nf(x)$ for all positive integers n, which can be proven by induction. Thus, $qf\left(\frac{p}{q}x\right) = f(px) = pf(x)$, and $f\left(\frac{p}{q}x\right) = \frac{p}{q}f(x)$ for all integers p, q. Letting $x = 1$, we find that $f(r) = rf(1)$ for all rational numbers r. If we are also given that f is an increasing function, then we can use the fact that between any two reals there exists a rational to show that this holds for irrational r as well.
In fact, if you can prove ANY of the following :

1. that $f(x)$ is a monotonic function
2. that $f(x) \geq 0$ for all $x \geq 0$
3. that $f(x)$ is bounded in some interval
4. that $xf(x) = f(x^2)$ for all real numbers x

then the only solutions will be $f(x) = cx$.

5 Problems

1. Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ for which $2f(x) + 3f(1 - x) = 5x^2 - 3x - 11$ for all real x.
2. Find all increasing functions $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $f(xy) = f(x)f(y)$ for all real x, y.
3. Find all functions $f : \mathbb{Q} \mapsto \mathbb{Q}$ for which $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for all real x, y.
4. Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that $y^2 f(x) - x^2 f(y) = y^2 - x^2$ for all real x, y.
5. Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that $(x + y)f(x)f(y) = xyf(x + y)$ for all real x, y and $|f(x)| \geq x$ for all x.
6. Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that $f(f(x) + y) = 2x + f(f(y) - x)$ for all real x, y.
7. Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that $f(x^2 + f(y)) = y + (f(x))^2$ for all real x, y.
8. Find a function $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ which satisfies
$$f(xf(y)) = \frac{f(x)}{y}$$
for all positive rationals x, y.