Functional Equations Solutions

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7. First, let $x = 0$ and define $c = f(0)$. With this, we get

$$f(f(y)) = y + c^2$$

Note also that $f(x^2 + c) = f(x)^2$, so setting $x = 0$ gives us $f(c) = c^2$. Now, take the original equation, and set $y = 0, x = c$. This results in

$$\begin{aligned}
f(c^2 + c) &= f(c)^2 \\
c^2 + f(c^2 + c) &= f(c) + f(c)^2 \\
f(c^2 + f(c^2 + c)) &= f(f(c) + f(c)^2) \\
c^2 + c + f(c)^2 &= c + f(f(c))^2 \\
c^2 + c^4 &= (c + c^2)^2 \\
c^2 + c^4 &= c^4 + 2c^3 + c^2 \\
2c^3 &= 0
\end{aligned}$$

so $c = 0$. This means that $f(f(y)) = y$, so f is an involution. If we replace $f(y)$ with y and set $x = a$ constant, we can write

$$f(a^2 + y) = f(y) + f(a)^2 \geq f(y)$$

so f is monotonically increasing. Now, if $f(x) \geq x$, this implies that $f(f(x)) \geq f(x) \Leftrightarrow x \geq f(x)$. If $f(x) \leq x$, this also implies that $x \leq f(x)$. This means that the only possibile solution is $\boxed{f(x) = x}$, and it is trivial to show that it works.

The Hard Problem

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