Functional Equations

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1 Introduction

Functional equations are a part of algebra where you're given a relation about a function and are asked to find all the solutions. You don't get to assume anything nice about the function (such as continuity or monotonicity), and yet the relation will reduce to a very small class of functions. In this lecture, we'll be looking at functional equations whose domains resemble the real numbers, so you have to find ways to prove stuff about $f(\pi)$, for instance. The trick to solving functional equations is to manipulate what you're given into something like $f(x) = x$. This may take a lot of work, and substituting in specific values for variables is often fruitful. Good luck!

1.1 Notation

Here's a list of things which you should know to be able to read functional equation problems

- $\mathbb{N}$ is the set of positive integers (excluding 0 for the purposes of contest math).
- $\mathbb{Z}$ is the set of integers.
- $\mathbb{Q}$ is the set of rational numbers.
- $\mathbb{R}$ is the set of real numbers.
- $\mathbb{C}$ is the set of complex numbers.
- Unless otherwise stated, $\mathbb{R}^+$ is the set of positive reals (excluding 0) and $\mathbb{R}_{\geq 0}$ is the set of nonnegative reals (including 0).
- $f : S_1 \to S_2$ (read f from S_1 to S_2) means that f is a function whose domain is S_1 and whose codomain is S_2 ($f(x) = y$ makes sense for $x \in S_1$ and $y \in S_2$).

1.2 Warm up

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) + y = f(x + y)$.
2. What if $f : \mathbb{R}^+ \to \mathbb{R}^+$?

1.3 Cauchy's Functional Equation

Cauchy's functional equation looks quite simple. It is $f(x+y) = f(x) + f(y)$. The obvious solutions are $f(x) = cx$. Indeed, it is relatively easy to show that $f(x) = cx$ for all $x \in \mathbb{Q}$. However, there are other solutions for $f : \mathbb{R} \to \mathbb{R}$, which can be constructed using the axiom of choice. Contest problems will never contain these functions as solutions, so you must find another way to eliminate them as possibilities. However, because of the obscene nature of these other functions, any one of the following conditions implies that $f(x) = cx$: monotonicity, continuity, locally bounded. This leads us to...

1.4 Zeb's Three Rules

1. No limits.
2. Cauchy's equation is RIDICULOUS
3. *NO LIMITS.*

2 Problems

1. (Traditional) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2 + y) = xf(x) + y$
2. (MOP 2007) Find all functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $f(x^2 + y) = xf(x) + f(y)$.
3. (1989 SL GRE 3) Let $g : \mathbb{C} \to \mathbb{C}, \omega \in \mathbb{C},\ a \in \mathbb{C},\ \omega^3 = 1\ (\omega \neq 1)$. Show that there is one and only one function $f : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) + f(\omega z + a) = g(z), \quad z \in \mathbb{C}.$$

   Find the function f.

4. (1992 SL CHN 1) Let $\mathbb{R}^+$ be the set of all nonnegative real numbers. Given two positive real numbers a and b, suppose that a mapping $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the functional equation

$$f(f(x)) + af(x) = b(a+b)x.$$

   Prove that there exists a unique solution of this equation.

5. (1992 SL MON 1 / 1994 SL A4) Find all the functions $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying the identity

$$f(x)f(y) = y^\alpha f\left(\frac{x}{2}\right) + x^\beta f\left(\frac{y}{2}\right), \quad x, y \in \mathbb{R}^+,$$

   where α, β are given real numbers.

6. (2003 SL A2 (AUS)) Find all nondecreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that

   (i) $f(0) = 0, f(1) = 1$;

   (ii) $f(a)+f(b) = f(a)f(b)+f(a+b-ab)$ for all real numbers a, b such that $a < 1 < b$.

7. (1992 IMO 2 / SL IND 2) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + f(y)) = y + f(x)^2 \quad \text{for all } x, y \text{ in } \mathbb{R}.$$

3 The Hard Problem

1. (MOP 2008) Determine whether there exists a polynomial $P(x)$ with real coefficients, not identically zero, for which we can find a function $f : \mathbb{R} \to \mathbb{R}$ that satisfies the relation

$$f(x) - \frac{x^2}{3} \cdot f\left(\frac{3x-3}{3+x}\right) = P\left(\frac{3x+3}{3-x}\right) \quad \text{for all irrational numbers } x.$$