GEOMETRIC INEQUALITIES

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WOLTERS-NOORDHOFF PUBLISHING GRONINGEN 1969
THE NETHERLANDS

LTERS-NOORDHOFF PUBLISHING, GRONINGEN, 1968

Printed in The Netherlands

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Preface

This is a collection of inequalities in elementary plane geometry, the majority of which deals with triangles and the remarkable lines associated with them.

Geometric inequalities are as old as geometry itself. The first book of Euclid's *Elements* contains several theorems on inequalities for the sides and the angles of a triangle, the most important of which is perhaps Proposition XX: the sum of two sides is greater than the third. It seems permissible to state that almost all geometric inequalities are based in some way or another on this theorem.

A multitude of inequalities has been found during the last two centuries. One of the oldest is that with respect to the radii of the circumcircle and the incircle: $R \geqslant 2r$, given by Euler in 1765. It still maintains its position of high quality in this field, because it shows two properties for mathematical appreciation: it is simple, but by no means trivial. Many others have been given in the course of time, but while the elementary geometry of remarkable points, lines and circles (sometimes disparagingly referred to as the microscopy of the triangle) had its climax in the later part of the nineteenth century, the interest in inequalities has been increasing in, say, the last thirty years. The results are rather disjointed: they are scattered at random over many books, journals, problem sections and examination papers. In this book we tried to collect and present them to the reader, according to a certain classification, although we are fully aware that the latter had to be imperfect in view of the data we had to deal with.

The collection offers a variety of theorems on geometric inequalities. We inserted a few which at first glance may seem unsympathetic by their complicity or may even be considered as bizarre. But the greater part is in our opinion attractive and interesting and among them are some real gems. The effort involved in collecting, checking and classifying the contents of this book was more than compensated for by the pleasure experienced from so many ingenious theorems and elegant proofs and we have admired the skill and the imagination of that multitude of mathematicians from former and recent years, whose contributions fill the following pages.

With a few exceptions we were able to add bibliographic references to the theorems. As some identical or similar inequalities have appeared, obviously independent of each other, at different times and places, we are not at all certain that we have always given due honour to priority claims. To many theorems we added a complete proof, sometimes we restricted ourselves to a hint and in many other cases the proof was omitted altogether. We had no strict rules for that; we were guided by the importance of the inequality, the nature of the proof and the accessability of the reference. For all theorems, but especially for those without an added proof, we liked to think that the inequality concerned could be a challenge to the reader.

This is not a new book. A considerable part of it is a translation of a publication in serbo-croatian *Geometrijske nejednakosti*, by all the authors with the exception of myself. It appeared as number 31 of the *Matematička Biblioteka* at Beograd in 1966. The four members of the Yugoslave group have done the tedious field work; they have consulted innumerous volumes of likely and unlikely mathematical journals. After selecting the material they classified it and checked the proofs. They continued in this manner when preparing the English version and during this second period they unearthed more valuable treasures. It was an honour and a pleasure to be invited to join their group and to be given an advisory task in the making of this edition as well as contributing thereto. Although each of the five authors mentioned in alphabetical order on the title page contributed his own share, the edition is essentially the result of close operation and team work, A comparison with the original version shows that the collection of inequalities has been extended and classified anew. On the other hand we decided on several grounds to delete all theorems on solid geometry: they are interesting, but rather isolated and we wished to give our work a certain unity and brevity. The character of the book, however, has not been changed. It emphasizes algebraic (including goniometric) methods and with some exaggeration one may state that all our theorems on triangles give inequalities for three real numbers a, b, c, which are corollaries of the system $-a+b+c>0$, $a-b+c>0$, $a+b-c>0$. Consequently this book on *geometric* inequalities does not contain any drawings.

O. B.

The authors intend to keep a systematic check on the further development of geometric inequalities and to make an annual report on their findings in the journal: Univerzitet u Beogradu, Publikacije Elektrotehničkog fakulteta, Serija Matematika i fizika or in some other suitable journal. This report will inform the reader about the state of research in the field. The authors welcome suggestions and critical remarks on the book, so as to make the report more complete. Such material should be sent to: Professor D. S. Mitrinović, Department of Mathematics, Electrotechnical Faculty, University of Beograd, Yugoslavia.

Notations

In this book the following notations are used:

A, B, C	vertices of a triangle
a, b, c	sides BC, CA, AB
α, β, γ	its angles
h_a, h_b, h_c	altitudes
m_a, m_b, m_c	medians
w_a, w_b, w_c	angle-bisectors
O	circumcentre
R	radius of circumcircle
I	incentre
r	radius of incircle
H	orthocentre
G	centroid
s	semi-perimeter
F	area of triangle ABC
P	point in the interior of a triangle
I_a, I_b, I_c	excentres
r_a, r_b, r_c	radii of excircles
R_1, R_2, R_3	distances from P to the vertices of ABC
r_1, r_2, r_3	distances from P to the sides of ABC
w_1, w_2, w_3	angle-bisectors of the angles BPC, CPA, APB
r'_1, r'_2, r'_3	Cevian segments PD, PE, PF

$$R_i(P) = R_i$$

$$r_i(P) = r_i$$

$$Q = (b-c)^2 + (c-a)^2 + (a-b)^2$$

$$MN^2 = \overline{MN}^2 = |MN|^2$$

$M_k(x, y, z)$	$= (xyz)^{1/3}$	for $k = 0$
	$= (\frac{1}{3}(x^k+y^k+z^k))^{1/k}$	for $k \neq 0$,
	$= \min(x, y, z)$	for $k = -$
	$= \max(x, y, z)$	for $k = +$
A, B, C, D	vertices of a quadrilateral	
a, b, c, d	sides AB, BC, CD, DA	
$\alpha, \beta, \gamma, \delta$	its angles	
p, q	diagonals AC, BD	
$l = 2s$	perimeter of $ABCD$	
F	area of quadrilateral $ABCD$	
P	intersection of the diagonals	

1. Inequalities involving only the sides of a triangle

1.1 $3(bc+ca+ab) \leqslant (a+b+c)^2 < 4(bc+ca+ab).$

Equality holds if and only if the triangle is equilateral.

PROOF. We commence with $2bc \leqslant b^2+c^2$, $2ca \leqslant c^2+a^2$, $2ab \leqslant a^2+b^2$. By adding together these inequalities, and then adding $4(bc+ca+ab)$, we get

$$3(bc+ca+ab) \leqslant (a+b+c)^2,$$

where equality holds only if $a = b = c$.

Since a, b and c are the sides of a triangle, we have

$$|b-c| < a,\ |c-a| < b,\ |a-b| < c,$$

i.e.

$$(b-c)^2 < a^2,\ (c-a)^2 < b^2,\ (a-b)^2 < c^2.$$

By addition, we obtain

$$a^2+b^2+c^2 < 2(bc+ca+ab),$$

i.e.

$$(a+b+c)^2 < 4(bc+ca+ab).$$

F. E. Wood, Problem E 345, Amer. Math. Monthly 45 (1938), 551.

1.2 $a^2+b^2+c^2 \geqslant \dfrac{36}{35}\left(s^2+\dfrac{abc}{s}\right).$

Equality holds if and only if the triangle is equilateral.

PROOF. Using the inequalities

$$a^2+b^2+c^2 \geqslant \tfrac{1}{3}(a+b+c)^2$$

and

$$abc \leqslant [\tfrac{1}{3}(a+b+c)]^3,$$

$M_k(x, y, z)$	$= (xyz)^{1/3}$	for $k = 0$
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and

$$abc \leqslant [\tfrac{1}{3}(a+b+c)]^3,$$

where the equalities hold if and only if $a = b = c$, we have

$$a^2+b^2+c^2 \geqslant \frac{4}{3}s^2 = \frac{36}{35}\left[s^2 + (\tfrac{2}{3}s)^3 \cdot \frac{1}{s}\right]$$

$$= \frac{36}{35}\left[s^2 + \left(\frac{a+b+c}{3}\right)^3 \cdot \frac{1}{s}\right]$$

$$\geqslant \frac{36}{35}\left(s^2 + \frac{abc}{s}\right),$$

with equality only if $a = b = c$.

J. F. Darling-W. Moser, Problem E 1456, Amer. Math. Monthly 68 (1961), 294 and 930.

1.3 $8(s-a)(s-b)(s-c) \leqslant abc.$ (1)

Equality holds if and only if the triangle is equilateral.

PROOF. We have

$$\sqrt{a^2-(b-c)^2} \leqslant a, \quad \sqrt{b^2-(c-a)^2} \leqslant b, \quad \sqrt{c^2-(a-b)^2} \leqslant c,$$

i.e.

$$\sqrt{(a+b-c)^2(b+c-a)^2(c+a-b)^2} \leqslant abc.$$

Since $b+c-a$, $c+a-b$, $a+b-c$ are positive numbers, we get

$$(a+b-c)(b+c-a)(a+b-c) \leqslant abc.$$

Equality in (1) holds if and only if $a = b = c$.

A. Padoa, Period. Mat. (4) 5 (1925), 80–85.

1.4 $8abc \leqslant (a+b)(b+c)(c+a),$

with equality if and only if the triangle is equilateral.

E. Cesàro, Nouvelle Correspondance Math. 6 (1880), 140.

REMARK. This inequality also holds for all non-negative real numbers a, b, c.

1.5 $3(a+b)(b+c)(c+a) \leqslant 8(a^3+b^3+c^3),$

with equality if and only if $a = b = c$.

A. Padoa, Period. Mat. (4) 5 (1925), 80–85.

REMARK. This inequality also holds for all non-negative real numbers a, b, c.

1.6 $2(a+b+c)(a^2+b^2+c^2) \geqslant 3(a^3+b^3+c^3+3abc)$,

with equality if and only if $a = b = c$.

M. Colins, Educational Times 13 (1870), 30–31.

1.7 $abc < a^2(s-a) + b^2(s-b) + c^2(s-c) \leqslant \frac{3}{2}abc.$ (1)

PROOF. Since

$$2 \sum a^2(s-a) = \sum a^2(b+c-a) = - \sum a^3 + \sum b^2c$$

and

$$(b+c-a)(c+a-b)(a+b-c) = - \sum a^3 + \sum b^2c-2abc$$

we have

$$2 \sum a^2(s-a) = (b+c-a)(c+a-b)(a+b-c) + 2abc. \qquad (2)$$

From (2) and **1.3** follows (1).

Equality holds if and only if the triangle is equilateral.

1.8 $bc(b+c) + ca(c+a) + ab(a+b) \geqslant 48(s-a)(s-b)(s-c).$ (1)

Equality holds if and only if the triangle is equilateral.

PROOF. Since

$$b+c \geqslant 2\sqrt{bc},\ c-a \geqslant 2\sqrt{ca},\ a+b \geqslant 2\sqrt{ab},$$

we have

$$bc(b+c) + ca(c+a) + ab(a+b) \geqslant 2[(bc)^{3/2} + (ca)^{3/2} + (ab)^{3/2}].$$

By arithmetic-geometric mean inequality we get

$$(bc)^{3/2} + (ca)^{3/2} + (ab)^{3/2} \geqslant 3abc,$$

so that

$$bc(b+c) + ca(c+a) + ab(a+b) \geqslant 6abc. \qquad (2)$$

From (2) and **1.3** follows (1).

1.9 $a^3(s-a) + b^3(s-b) + c^3(s-c) \leqslant abcs.$ (1)

PROOF. This inequality is equivalent to

$$a^2(a-b)(a-c) + b^2(b-c)(b-a) + c^2(c-a)(c-b) \geqslant 0. \qquad (2)$$

Inequality (2) holds because it is a particular case of Schur's inequality (see: G. H. Hardy, J. E. Littlewood and G. Pólya: Inequalities, Cambridge 1934, p. 64).

Equality in (1) holds if and only if $a = b = c$.

J. Andersson, Problem E 1779, Amer. Math. Monthly 59 (1952), 41.

1.20 For all real t,

$$a^t(s-a) + b^t(s-b) + c^t(s-c) \leqslant \tfrac{1}{2}abc(a^{t-2}+b^{t-2}+c^{t-2}).$$

Equality holds if and only if $a = b = c$.

This result is due to R. Ž. Djordjević.

1.21 $a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geqslant 0,$

where equality holds if and only if the triangle is equilateral.

E. Catalan, Educational Times, N.S. 10 (1906), 57.

1.22 $64s^3(s-a)(s-b)(s-c) \leqslant 27a^2b^2c^2.$

Equality occurs if and only if the triangle is equilateral.

A. Padoa, Period. Mat. (4) 6 (1926), 38–40.

1.23 If $q = (Q/2)^{1/2}$, then

$$2(s-q)(2s+q)^2 \leqslant 27abc \leqslant 2(s+q)(2s-q)^2. \qquad (1)$$

The first and second equality in (1) hold respectively for isosceles triangles whose base is the smallest and largest of the three sides; of course, both equality signs apply when the triangle is equilateral, since in that case $q = 0$.

R. Frucht, Canad. J. Math. 9 (1957), 227–231.

1.24 $$\frac{2s}{abc} \leqslant \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}. \qquad (1)$$

Equality holds if and only if the triangle is equilateral.

Proof. Since

$$(bc)^2 + (ca)^2 + (ab)^2 \geqslant (ca)(ab) + (ab)(bc) + (bc)(ca),$$

i.e.,

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geqslant \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab},$$

and

$$\frac{2s}{abc}=\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab},$$

we conclude that (1) holds.

Gaz. Mat. B 10 (1959), 162.

1.15 $$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\geqslant\frac{9}{s}.$$

Equality holds if and only if the triangle is equilateral.

1.16 $$\frac{3}{2}\leqslant\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<2. \tag{1}$$

Equality occurs if and only if $a=b=c$.

Proof. By means of arithmetic-harmonic mean inequality,

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\geqslant\frac{9}{2(a+b+c)}, \tag{2}$$

with equality if and only if $a=b=c$.

Since

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)-3,$$

we conclude by (2) that the first inequality in (1) holds.

Since

$$b+c>\tfrac{1}{2}(a+b+c),\ c+a>\tfrac{1}{2}(a+b+c),\ a+b>\tfrac{1}{2}(a+b+c),$$

we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<\frac{2(a+b+c)}{a+b+c}=2.$$

This proves the second inequality in (1).

A. M. Nesbitt, Problem 15114, Educational Times (2) 3 (1903), 37–38.

M. Petrović, Računanje sa brojnim razmacima, Beograd 1932, p. 79.

REMARK. The first inequality in (1) also holds for all non-negative real numbers. See, for example, D. S. Mitrinović, Elementary Inequalities, Groningen 1964, 144–147.

1.17 $$\frac{15}{4} \leqslant \frac{s+a}{b+c} + \frac{s+b}{c+a} + \frac{s+c}{a+b} < \frac{9}{2}.$$

Equality occurs if and only if $a = b = c$.
Gaz. Mat. B 7 (1956), 438.

1.18 $$\left(1-\frac{a}{b}\right)\left(1-\frac{a}{c}\right)+1 > \frac{c-a}{b}+\frac{b-a}{c}.$$

Gaz. Mat. B 15 (1964), 422.

1.19 $$\frac{1}{3} \leqslant \frac{a^2+b^2+c^2}{(a+b+c)^2} < \frac{1}{2}.$$

Equality occurs if and only if $a = b = c$.
M. Petrović, Ens. Math. 18 (1916), 153–163.

1.20 $$\sqrt{s} < \sqrt{s-a}+\sqrt{s-b}+\sqrt{s-c} \leqslant \sqrt{3s}.$$

Equality holds if and only if the triangle is equilateral.
A. Santalò, Math. Notae 3 (1943), fasc. 2, 65–73.
E. G. Gotman, Matematika v škole 1965, No. 1, 76.

1.21 For all real t,

$$2F\cdot\min[(abc)^{-2/3}, (\max(a,b,c)\min(a,b,c))^{-1}]M_t(a,b,c)$$
$$\leqslant M_t(h_a, h_b, h_c)$$
$$\leqslant 2F\cdot\max[(abc)^{-2/3}, (\max(a,b,c)\min(a,b,c))^{-1}]M_t(a,b,c).$$

O. Reuter, Elem. Math. 18 (1963), 34–35.

1.22 For all real t,

$$(a^2+b^2+c^2)(b^{t-2}c^{t-2}+c^{t-2}a^{t-2}+a^{t-2}b^{t-2})$$
$$< 2(a^t+b^t+c^t)(a^{t-2}+b^{t-2}+c^{t-2}).$$

PROOF. Consider the point P which has the barycentric coordinates a^t, b^t, c^t. Then

$$PA^2 = \frac{(a^t+b^t+c^t)(b^tc^2+c^tb^2)-a^{2t}c^t-a^tb^2c^t-a^tb^tc^2}{(a^t+b^t+c^t)^2},$$

whence

$$\frac{PA^2}{b^2c^2} = \frac{(a^t+b^t+c^t)(b^{t-2}+c^{t-2})-a^2(b^{t-2}c^{t-2}+c^{t-2}a^{t-2}+a^{t-2}b^{t-2})}{(a^t+b^t+c^t)^2},$$

with the analogous formulae for $\frac{PB^2}{c^2a^2}$ and $\frac{PC^2}{a^2b^2}$.

Therefore

$$(a^t+b^t+c^t)^2\left(\frac{PA^2}{b^2c^2}+\frac{PB^2}{c^2a^2}+\frac{PC^2}{a^2b^2}\right)$$
$$= 2(a^t+b^t+c^t)(a^{t-2}-b^{t-2}+c^{t-2})$$
$$-(a^2+b^2+c^2)(b^{t-2}c^{t-2}+c^{t-2}a^{t-2}+a^{t-2}b^{t-2}).$$

Since the first member of this equality is positive, the second member will also be positive.

L. Bénézech, Problem 412, J. Math. Elém. Paris (4) 1 (1892), 234.

1.23 $(a+b+c)^3 \leqslant 5[bc(b+c)+ca(c+a)+ab(a+b)]-3abc,$ (1)

with equality if and only if the triangle is equilateral.

PROOF. We have

$$(a+b+c)^3 = a^3+b^3+c^3+6abc$$
$$+3[bc(b+c)+ca(c+a)+ab(a+b)]-3abc. \quad (2)$$

Since (see: **1.6**)

$$a^3+b^3+c^3+9abc \leqslant 2[bc(b+c)+ca(c+a)+ab(a+b)],$$

from (2) we obtain (1).

S. Reich, Problem E 1930, Amer. Math. Monthly 73 (1966), 1017–1018.

1.24 $\left(1+\frac{a}{b}\right)\left(1+\frac{a}{c}\right) < 3+\frac{c+b}{c}+\frac{a+c}{b}.$

C. Ionescu-Tiu, Gaz. Mat. B 15 (1964), 272.

2. Inequalities for the angles of a triangle

2.1 $0 < \sin\alpha + \sin\beta + \sin\gamma \leqslant \frac{3}{2}\sqrt{3}$. (1)

Equality holds if and only if the triangle is equilateral.

PROOF. Starting from

$$a^2+b^2+c^2 = 8R^2(1+\cos\alpha\cos\beta\cos\gamma)$$

and

$$\frac{(a+b+c)^2}{3} \leqslant a^2+b^2+c^2,$$

we have

$$\frac{(a+b+c)^2}{3} \leqslant a^2+b^2+c^2 = 8R^2(1+\cos\alpha\cos\beta\cos\gamma) \leqslant 9R^2, \quad (2)$$

where we used the inequality **2.23.**

From (2) and the sine law, we get (1).

A. Padoa, Period. Mat. (4) 5 (1925), 80–85.

T. R. Curry, Problem E 1644, Amer. Math. Monthly 70 (1963), 1099 and 71 (1964), 915–916.

2.2 $0 < \sin\alpha + \sin\beta + \sin\gamma \leqslant \frac{3}{2}\sqrt{3}$ in each triangle. (1)

$2 < \sin\alpha + \sin\beta + \sin\gamma \leqslant \frac{3}{2}\sqrt{3}$ in acute triangles. (2)

$0 < \sin\alpha + \sin\beta + \sin\gamma < 1+\sqrt{2}$ in obtuse triangles. (3)

O. Bottema, Euclides 30 (1954/55), 114–116.

REMARK. Inequalities (1) and (2) are also proved in:

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.3 $0 < \sin^2\alpha + \sin^2\beta + \sin^2\gamma \leqslant \frac{9}{4}$ in each triangle.

$2 < \sin^2\alpha + \sin^2\beta + \sin^2\gamma \leqslant \frac{9}{4}$ in acute triangles.

$0 < \sin^2\alpha + \sin^2\beta + \sin^2\gamma < 2$ in obtuse triangles.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.4 $\sin\alpha + \sin\beta + \sin\gamma \geqslant \sin 2\alpha + \sin 2\beta + \sin 2\gamma$.

Equality holds if and only if the triangle is equilateral.

PROOF. Applying the sine law we obtain

$$\sin\alpha+\sin\beta+\sin\gamma = \frac{a+b+c}{2R} = \frac{F}{rR}.$$

Also

$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma = 2(\sin\alpha\cos\alpha+\sin\beta\cos\beta+\sin\gamma\cos\gamma)$$
$$= \frac{1}{R}(a\cos\alpha+b\cos\beta+c\cos\gamma).$$

Since

$$a\cdot\cos\alpha+b\cdot\cos\beta+c\cdot\cos\gamma = \frac{2F}{R},$$

we have

$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma = \frac{2F}{R^2}.$$

Therefore

$$\frac{\sin\alpha+\sin\beta+\sin\gamma}{\sin 2\alpha+\sin 2\beta+\sin 2\gamma} = \frac{R}{2r} \geqslant 1.$$

2.5 $\sqrt{\sin\alpha}+\sqrt{\sin\beta}+\sqrt{\sin\gamma} \leqslant 3\sqrt[4]{\tfrac{3}{4}}$.

PROOF. Since

$$(x+y+z)^2 \leqslant 3(x^2+y^2+z^2),$$

putting

$$x = \sqrt{\sin\alpha},\ y = \sqrt{\sin\beta},\ z = \sqrt{\sin\gamma},$$

we obtain

$$\sqrt{\sin\alpha}+\sqrt{\sin\beta}+\sqrt{\sin\gamma} \leqslant \sqrt{3(\sin\alpha+\sin\beta+\sin\gamma)}.$$

The given inequality follows from inequality 2.1.
I. Albu, Gaz. Mat. B 14 (1963), 177.

2.6 If $k \leqslant 1$, then

$$M_k(\sin\alpha, \sin\beta, \sin\gamma) \leqslant \tfrac{1}{2}\sqrt{3}.$$

Gaz. Mat. B 14 (1963), 681.

2.7 $\sin\alpha\sin\beta\sin\gamma \leqslant \tfrac{3}{8}\sqrt{3}$.

Equality holds if and only if the triangle is equilateral.
K. P. Wilkins, Amer. Math. Monthly 44 (1937), 579–583.

2.8 $0 < \sin\alpha \sin\beta \sin\gamma \leqslant \frac{3}{8}\sqrt{3}$ in each triangle.

$0 < \sin\alpha \sin\beta \sin\gamma \leqslant \frac{3}{8}\sqrt{3}$ in acute triangles.

$0 < \sin\alpha \sin\beta \sin\gamma < \frac{1}{3}\sqrt{3}$ in obtuse triangles.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.9 $1 < \sin\dfrac{\alpha}{2} + \sin\dfrac{\beta}{2} + \sin\dfrac{\gamma}{2} \leqslant \dfrac{3}{2}.$

Equality occurs if and only if $\alpha = \beta = \gamma$.
R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.
REMARK. The second inequality has also been proved in:
J. M. Child, Math. Gaz. 23 (1939), 138–143.

2.10 $\sin\alpha \sin\beta \sin\dfrac{\gamma}{2} \leqslant \frac{2}{9}\sqrt{3}.$

Equality holds if and only if $\alpha = \beta$, with $\cos\alpha = \frac{1}{3}\sqrt{3}$.
K. P. Wilkins, Amer. Math. Monthly 44 (1937), 579–583.

2.11 $\sin\dfrac{\alpha}{2} \sin\dfrac{\beta}{2} \sin\gamma \leqslant \frac{1}{54}(2\sqrt{13}-5)\sqrt{2\sqrt{13}+22}.$

Equality holds if and only if $\alpha = \beta$, where $\cos\alpha = \frac{1}{6}(\sqrt{13}-1)$.
K. P. Wilkins, Amer. Math. Monthly 44 (1937), 579–583.

2.12 $0 < \sin\dfrac{\alpha}{2} \sin\dfrac{\beta}{2} \sin\dfrac{\gamma}{2} \leqslant \dfrac{1}{8}.$

Equality holds if and only if the triangle is equilateral.

PROOF. Since

$$\sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

and as we have similar expressions for $\sin\beta/2$ and $\sin\gamma/2$, we obtain

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=\frac{(s-a)(s-b)(s-c)}{abc}=\frac{r}{4R}\leqslant\frac{1}{8},$$

where **5.1** has been used.

Equality holds if and only if the triangle is equilateral.

REMARK. This inequality was proved in 1897. See: T. Radó, Amer. Math. Monthly 39 (1932), 85–90.

V. Krylov, Matematika v škole 1938, No. 5–6, 134.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.13 $\sin\frac{\alpha}{4}\sin\frac{\beta}{4}\sin\frac{\gamma}{2}\leqslant\frac{1}{54}(5\sqrt{10}-14).$

Equality holds if and only if $\alpha=\beta$, when

$$\cos\frac{\alpha}{2}=\frac{1}{6}(\sqrt{10}+2).$$

K. P. Wilkins, Amer. Math. Monthly 44 (1937), 579–583.

2.14 $\frac{3}{4}\leqslant\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}<1.$ (1)

Equality holds only if the triangle is equilateral.

PROOF. Since

$$\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}=\tfrac{3}{2}-\tfrac{1}{2}(\cos\alpha+\cos\beta+\cos\gamma),$$

on the basis of **2.16**, we conclude that inequality (1) holds.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.15 $\sin\frac{\beta}{2}\sin\frac{\gamma}{2}+\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\leqslant\frac{3}{4}.$

Equality holds if and only if the triangle is equilateral.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

2.16 $1 < \cos\alpha+\cos\beta+\cos\gamma \leqslant \frac{3}{2}$. (1)

Equality holds if and only if the triangle is equilateral.

PROOF. Since $\alpha/2$, $\beta/2$, $\gamma/2 < \pi/2$, from

$$\cos\alpha+\cos\beta+\cos\gamma = 1+4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}, \qquad (2)$$

the first inequality in (1) immediately follows.

The second inequality in (1) follows from (2) on the basis of **2.12**.

REMARK. For the second inequality in (1) see: J. M. Child, Math. Gaz. 23 (1939), 138–143. Later a weaker inequality was established by A. N. Aheart, Problem E 1398, Amer. Math. Monthly 67(1960), 82.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.17 $\cos\alpha+\sqrt{2}(\cos\beta+\cos\gamma) \leqslant 2$.

Equality only if $\alpha = \pi/2$, $\beta = \gamma$.

PROOF. Since $\alpha+\beta+\gamma = \pi$, we have

$$\begin{aligned}\cos\alpha+\sqrt{2}(\cos\beta+\cos\gamma) &= \cos\alpha+2\sqrt{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}\\ &= \cos\alpha+2\sqrt{2}\sin\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2}\\ &\leqslant \cos\alpha+2\sqrt{2}\sin\frac{\alpha}{2}\\ &= 2-2\left(\sin\frac{\alpha}{2}-\frac{1}{2}\sqrt{2}\right)^2 \leqslant 2.\end{aligned}$$

This proof is due to R. P. Lučić.

2.18 If λ is real, then

$$\cos\alpha+\lambda(\cos\beta+\cos\gamma) \leqslant 1+\frac{\lambda^2}{2}.$$

PROOF. For any three real numbers β, γ, λ, the following inequality holds:

$$(\cos\beta+\cos\gamma-\lambda)^2 + (\sin\beta-\sin\gamma)^2 \geqslant 0.$$

After squaring and then dividing into groups, we get

$$-\cos(\beta+\gamma) + \lambda(\cos\beta+\cos\gamma) \leqslant 1 + \frac{\lambda^2}{2}.$$

Since $\alpha+\beta+\gamma = \pi$, we have

$$\cos\alpha+\lambda(\cos\beta+\cos\gamma) \leqslant 1 + \frac{\lambda^2}{2}.$$

Equality occurs if $0 < \lambda < 2$; $\cos\alpha = 1-\lambda^2/2$, $\cos\beta = \cos\gamma = \lambda/2$.

REMARK. For $\lambda = \sqrt{2}$, we obtain **2.17**.

Ž. Mitrović, Mat. Vesnik 4 (19) (1967), 341.

2.19 Let the lengths of the sides of a triangle be a, b, c, with $a \geqslant b \geqslant c$, and the opposite angles α, β, γ, respectively. Then, for the angles α_1, β_1, γ_1, of an arbitrary triangle,

$$bc+ca-ab < bc\cdot\cos\alpha_1+ca\cdot\cos\beta_1+ab\cdot\cos\gamma_1 \leqslant \tfrac{1}{2}(a^2+b^2+c^2). \tag{1}$$

Equality is valid only if $\alpha_1 = \alpha$, $\beta_1 = \beta$, $\gamma_1 = \gamma$.

REMARK. If $\alpha = \beta = \gamma$, inequalities (1) give **2.16**.
If $\alpha = \pi/2$ and $\beta = \gamma = \pi/4$, then (1) gives **2.17**.
If $\alpha = 2\pi/3$ and $\beta = \gamma = \pi/6$, then (1) reduces to

$$1 < \cos\alpha_1+\sqrt{3}(\cos\beta_1+\cos\gamma_1) \leqslant \tfrac{5}{2}.$$

Equality holds if and only if $\alpha_1 = 2\pi/3$, $\beta_1 = \gamma_1 = \pi/6$.

P. Szász, Monatsh. Math. 66 (1962), 174–178.

2.20 If x, y, z are real numbers such that $xyz > 0$, then

$$x\cdot\cos\alpha+y\cdot\cos\beta+z\cdot\cos\gamma \leqslant \frac{yz}{2x} + \frac{zx}{2y} + \frac{xy}{2z}. \tag{1}$$

If $xyz < 0$, then inequality (1) is reversed.

Equality is valid in both cases if and only if

$$\frac{1}{x} : \frac{1}{y} : \frac{1}{z} = \sin\alpha : \sin\beta : \sin\gamma.$$

PROOF. If x, y, z, α, β are real numbers, then

$$(xz\cdot\cos\alpha+yz\cdot\cos\beta-xy)^2+(xz\cdot\sin\alpha-yz\cdot\sin\beta)^2\geqslant 0, \quad (2)$$

i.e.,

$$y^2z^2+z^2x^2+x^2y^2+2xyz^2\cos(\alpha+\beta)-2xy^2z\cos\beta-2x^2yz\cos\alpha\geqslant 0. \quad (3)$$

Since $\alpha+\beta+\gamma=\pi$, (3) gives

$$2x^2yz\cos\alpha+2xy^2z\cos\beta+2xyz^2\cos\gamma\leqslant y^2z^2+z^2x^2+x^2y^2.$$

If $xyz > 0$, after division by $2xyz$, we get (1). If $xyz < 0$, we obtain the reversed inequality.

Equality in the considered inequalities holds if and only if

$$zx\cdot\cos\alpha+yz\cdot\cos\beta-xy=0,\quad zx\cdot\sin\alpha-yz\cdot\sin\beta=0,$$

i.e.,

$$\frac{1}{x} : \frac{1}{y} : \frac{1}{z} = \sin\alpha : \sin\beta : \sin\gamma.$$

REMARK. Inequality (1) in the case where x, y, z are positive numbers was established by D. F. Barrow. The above generalization and proof are due to R. R. Janić.

D. F. Barrow, Problem 3740, Amer. Math. Monthly 44 (1937), 252–254.

R. R. Janić, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181–196 (1967), 73–74.

2.21 $\frac{3}{4}\leqslant\cos^2\alpha+\cos^2\beta+\cos^2\gamma<3$ in each triangle.

$\frac{3}{4}\leqslant\cos^2\alpha+\cos^2\beta+\cos^2\gamma<1$ in acute triangles.

$1<\cos^2\alpha+\cos^2\beta+\cos^2\gamma<3$ in obtuse triangles.

Equality holds only for the equilateral triangle.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.22 $\cos\beta\cos\gamma+\cos\gamma\cos\alpha+\cos\alpha\cos\beta\leqslant\frac{3}{4}.$

There is strict inequality except when the triangle is equilateral.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

2.23 $\cos\alpha\cos\beta\cos\gamma \leqslant \frac{1}{8}$. (1)

Equality holds if and only if the triangle is equilateral.

PROOF. For acute triangle $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are positive, so that on the basis of arithmetic-geometric mean inequality, we have

$$\cos\alpha\cos\beta\cos\gamma \leqslant \left(\frac{\cos\alpha+\cos\beta+\cos\gamma}{3}\right)^3.$$

Using **2.16**, we conclude that inequality (1) is true.

For right and obtuse triangles inequality (1) is immediately established.

C. C. Popovici, Gaz. Mat. 31 (1925), 132.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

2.24 $-1 < \cos\alpha\cos\beta\cos\gamma \leqslant \frac{1}{8}$ in each triangle.

$0 \leqslant \cos\alpha\cos\beta\cos\gamma \leqslant \frac{1}{8}$ in acute triangles.

$-1 < \cos\alpha\cos\beta\cos\gamma < 0$ in obtuse triangles.

Equality on the left side holds for a rectangular triangle, and on the right side for an equilateral triangle.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.25 If $\alpha < \pi/2$ and $\beta < \pi/2$, then

$$\cos\alpha + \cos\beta > \sin\gamma.$$

If $\alpha > \pi/2$ or $\beta > \pi/2$, then reversed inequality is valid.

A. Pantazi, Gaz. Mat. 23 (1917/18), 144.

V. Cristescu, Gaz. Mat. 26 (1920/21), 88–89.

B. M. Barbalatt, Gaz. Mat. 30 (1924/25), 380–381.

2.26 $\cos\alpha\cos\beta\cos\gamma \leqslant \frac{1}{24}(\cos^2(\beta-\gamma)+\cos^2(\gamma-\alpha)+\cos^2(\alpha-\beta))$. (1)

PROOF. By $(x+y)^2 \geqslant 4xy$ we obtain

$$(\cos\alpha+2\cos\beta\cos\gamma)^2 \geqslant 8\cdot\cos\alpha\cos\beta\cos\gamma. \qquad (2)$$

Since

$$\cos\alpha = -\cos(\beta+\gamma),$$

(2) gives

$$(-\cos(\beta+\gamma) + 2\cos\beta\cos\gamma)^2 \geqslant 8\cdot\cos\alpha\cos\beta\cos\gamma,$$

i.e.

$$\cos^2(\beta-\gamma) \geqslant 8\cdot\cos\alpha\cos\beta\cos\gamma. \tag{3}$$

Similarly

$$\cos^2(\gamma-\alpha) \geqslant 8\cdot\cos\alpha\cos\beta\cos\gamma,\quad \cos^2(\alpha-\beta) \geqslant 8\cdot\cos\alpha\cos\beta\cos\gamma. \tag{4}$$

Adding (3) and (4) we get (1).

C. Cosnita and F. Turtoiu, Culegere de probleme de algebra, Bucuresti 1965, pp. 176–177.

2.27 $2 < \cos\frac{\alpha}{2} + \cos\frac{\beta}{2} + \cos\frac{\gamma}{2} \leqslant \frac{3}{2}\sqrt{3}.$

Equality only if the triangle is equilateral.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.28 $0 < \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} \leqslant \frac{3}{8}\sqrt{3}$ in each triangle.

$\frac{1}{2} < \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} \leqslant \frac{3}{8}\sqrt{3}$ in acute triangles.

$0 < \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} < \frac{3}{8}\sqrt{3}$ in obtuse triangles.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.29 $2 < \cos^2\frac{\alpha}{2} + \cos^2\frac{\beta}{2} + \cos^2\frac{\gamma}{2} \leqslant \frac{9}{4}.$

Equality holds if and only if the triangle is equilateral.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.30 $\mathrm{tg}\,\alpha + \mathrm{tg}\,\beta + \mathrm{tg}\,\gamma \geqslant 3\sqrt{3}$ in acute triangles.

$\mathrm{tg}\,\alpha + \mathrm{tg}\,\beta + \mathrm{tg}\,\gamma < 0$ in obtuse triangles.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.31 $\operatorname{tg}^2\alpha+\operatorname{tg}^2\beta+\operatorname{tg}^2\gamma > 0$ in each triangle.

$\operatorname{tg}^2\alpha+\operatorname{tg}^2\beta+\operatorname{tg}^2\gamma \geqslant 9$ in acute triangles.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.32 $\operatorname{tg}\alpha \operatorname{tg}\beta \operatorname{tg}\gamma \geqslant 3\sqrt{3}$ in acute triangles.

$\operatorname{tg}\alpha \operatorname{tg}\beta \operatorname{tg}\gamma < 0$ in obtuse triangles.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.33 $\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2} \geqslant \sqrt{3}.$

Equality holds if and only if the triangle is equilateral.

J. Karamata, Problem 119, Glasnik matematičko-fizički i astronomski 3 (1948) 223.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.34 $0 < \operatorname{tg}\frac{\alpha}{2} \operatorname{tg}\frac{\beta}{2} \operatorname{tg}\frac{\gamma}{2} \leqslant \frac{1}{9}\sqrt{3}.$

Equality holds if and only if the triangle is equilateral.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.35 $\operatorname{tg}^2\frac{\alpha}{2}+\operatorname{tg}^2\frac{\beta}{2}+\operatorname{tg}^2\frac{\gamma}{2} \geqslant 1.$ (1)

Equality holds if and only if the triangle is equilateral.

PROOF. Since

$$\frac{\gamma}{2}=\frac{\pi}{2}-\frac{\alpha+\beta}{2},$$

we obtain

$$\operatorname{tg}\frac{\gamma}{2}=\operatorname{ctg}\frac{\alpha+\beta}{2}=\frac{1-\operatorname{tg}\alpha/2 \operatorname{tg}\beta/2}{\operatorname{tg}\alpha/2+\operatorname{tg}\beta/2},$$

i.e.

$$\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}+\operatorname{tg}\frac{\gamma}{2}\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}=1.$$

By $\operatorname{tg} \alpha/2 = x$, $\operatorname{tg} \beta/2 = y$, $\operatorname{tg} \gamma/2 = z$ the last equality becomes

$$yz+zx+xy = 1. \tag{2}$$

From

$$2(x^2+y^2+z^2) - 2(yz+zx+xy) = (y-z)^2+(z-x)^2+(x-y)^2 \geqslant 0$$

and from (2) follows

$$x^2+y^2+z^2 \geqslant 1,$$

i.e. (1).

C. V. Durell and A. Robson, Advanced Trigonometry, London 1948, p. 277.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.36 $\operatorname{tg}^6 \dfrac{\alpha}{2} + \operatorname{tg}^6 \dfrac{\beta}{2} + \operatorname{tg}^6 \dfrac{\gamma}{2} \geqslant \dfrac{1}{9}.$

Equality holds if and only if the triangle is equilateral.

2.37 $\sqrt{\operatorname{tg}\dfrac{\beta}{2}\operatorname{tg}\dfrac{\gamma}{2}+5}+\sqrt{\operatorname{tg}\dfrac{\gamma}{2}\operatorname{tg}\dfrac{\alpha}{2}+5}+\sqrt{\operatorname{tg}\dfrac{\alpha}{2}\operatorname{tg}\dfrac{\beta}{2}+5} \leqslant 4\sqrt{3}.$

Ju. A. Izosimov, Matematika v škole 1953, No. 3, 94 and No. 6, 85.

2.38 $\operatorname{cotg}\alpha+\operatorname{cotg}\beta+\operatorname{cotg}\gamma \geqslant \sqrt{3},$

with equality holding if and only if the triangle is equilateral.

PROOF.

$$\operatorname{cotg}\alpha+\operatorname{cotg}\beta = \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} = \frac{2\sin\gamma}{\cos(\alpha-\beta)+\cos\gamma}$$

$$\geqslant \frac{2\sin\gamma}{1+\cos\gamma} = 2\operatorname{tg}\frac{\gamma}{2}.$$

Therefore

$$\operatorname{cotg}\alpha+\operatorname{cotg}\beta+\operatorname{cotg}\gamma \geqslant 2\cdot\operatorname{tg}\frac{\gamma}{2}+\operatorname{cotg}\gamma$$

$$= 2\cdot\operatorname{tg}\frac{\gamma}{2}+\frac{\operatorname{cotg}^2\gamma/2-1}{2\operatorname{cotg}\gamma/2} = \frac{\operatorname{cotg}^2\gamma/2+3}{2\operatorname{cotg}\gamma/2}$$

$$= \frac{1}{2}\left(\operatorname{cotg}\frac{\gamma}{2}+3\cdot\operatorname{tg}\frac{\gamma}{2}\right) \geqslant \sqrt{3}.$$

T. Varopoulos, Bull. Soc. Math. Grèce 15_1 (1934), 17.
R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.39 $\operatorname{cotg}^2\alpha+\operatorname{cotg}^2\beta+\operatorname{cotg}^2\gamma \geqslant 1.$

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.40 $\operatorname{cotg}\alpha\operatorname{cotg}\beta\operatorname{cotg}\gamma \leqslant \frac{1}{9}\sqrt{3}$ in acute triangles.

$\operatorname{cotg}\alpha\operatorname{cotg}\beta\operatorname{cotg}\gamma < 0$ in obtuse triangles.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.41 $\operatorname{cotg}\frac{\alpha}{2}+\operatorname{cotg}\frac{\beta}{2}+\operatorname{cotg}\frac{\gamma}{2} \geqslant 3\sqrt{3}.$

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.42 $\operatorname{cotg}\frac{\alpha}{2}\operatorname{cotg}\frac{\beta}{2}\operatorname{cotg}\frac{\gamma}{2} \geqslant 3\sqrt{3}.$

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.43 $\operatorname{cotg}^2\frac{\alpha}{2}+\operatorname{cotg}^2\frac{\beta}{2}+\operatorname{cotg}^2\frac{\gamma}{2} \geqslant 9.$ (1)

Equality holds if and only if the triangle is equilateral.

PROOF. By the harmonic-arithmetic mean inequality and by $\operatorname{tg}\alpha/2$, $\operatorname{tg}\beta/2$, $\operatorname{tg}\gamma/2 > 0$, we have

$$\left(\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}+\operatorname{tg}\frac{\gamma}{2}\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}\right)$$
$$\times\left(\operatorname{cotg}\frac{\beta}{2}\operatorname{cotg}\frac{\gamma}{2}+\operatorname{cotg}\frac{\gamma}{2}\operatorname{cotg}\frac{\alpha}{2}+\operatorname{cotg}\frac{\alpha}{2}\operatorname{cotg}\frac{\beta}{2}\right)\geqslant 9.$$

Since

$$\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}+\operatorname{tg}\frac{\gamma}{2}\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}=1,$$

we get

$$\operatorname{cotg}\frac{\beta}{2}\operatorname{cotg}\frac{\gamma}{2}+\operatorname{cotg}\frac{\gamma}{2}\operatorname{cotg}\frac{\alpha}{2}+\operatorname{cotg}\frac{\alpha}{2}\operatorname{cotg}\frac{\beta}{2}\geqslant 9. \quad (2)$$

By adding the inequality

$$\operatorname{cotg}^2\frac{\alpha}{2}+\operatorname{cotg}^2\frac{\beta}{2}\geqslant 2\operatorname{cotg}\frac{\alpha}{2}\operatorname{cotg}\frac{\beta}{2}$$

to two inequalities obtained by cyclic permutation of α, β, γ, we get

$$\operatorname{cotg}^2\frac{\alpha}{2}+\operatorname{cotg}^2\frac{\beta}{2}+\operatorname{cotg}^2\frac{\gamma}{2}$$
$$\geqslant\operatorname{cotg}\frac{\beta}{2}\operatorname{cotg}\frac{\gamma}{2}+\operatorname{cotg}\frac{\gamma}{2}\operatorname{cotg}\frac{\alpha}{2}+\operatorname{cotg}\frac{\alpha}{2}\operatorname{cotg}\frac{\beta}{2}$$

which together with (2) gives (1).

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

Ju. I. Gerasimov, Matematika v škole 1964, No. 3, 75.

2.44 $\operatorname{cotg}^2\frac{\alpha}{2}+\operatorname{cotg}^2\frac{\beta}{2}+\operatorname{cotg}^2\frac{\gamma}{2}$

$$\geqslant\left(\operatorname{cotg}\frac{\alpha}{2}+\operatorname{cotg}\frac{\beta}{2}+\operatorname{cotg}\frac{\gamma}{2}\right)(\operatorname{cotg}\alpha+\operatorname{cotg}\beta+\operatorname{cotg}\gamma). \quad (1)$$

Proof. Using the notation $\operatorname{cotg}\alpha/2=x$, $\operatorname{cotg}\beta/2=y$, $\operatorname{cotg}\gamma/2=z$, from

$$x+y+z=xyz$$

we have

$$(x+y+z)^2=xyz(x+y+z)$$

i.e.

$$x^2+y^2+z^2=(x^2-1)yz+(y^2-1)zx+(z^2-1)xy-(yz+zx+xy).$$

Since

$$x^2+y^2+z^2\geqslant yz+zx+xy$$

we get

$$2(x^2+y^2+z^2)\geqslant(x^2-1)yz+(y^2-1)zx+(z^2-1)xy$$

i.e.,

$$x^2+y^2+z^2\geqslant xyz\left(\frac{x^2-1}{2x}+\frac{y^2-1}{2y}+\frac{z^2-1}{2z}\right)$$
$$=(x+y+z)\left(\frac{x^2-1}{2x}+\frac{y^2-1}{2y}+\frac{z^2-1}{2z}\right). \quad (2)$$

By cotg $\alpha/2$, cotg $\beta/2$, cotg $\gamma/2$ substituting for x, y, z and using the formula

$$\frac{\operatorname{cotg}^2(t/2)-1}{2\operatorname{cotg} t/2}=\operatorname{cotg} t$$

from (2) we obtain (1).

C. Cosnita and F. Turtoiu, Culegere de probleme de algebra, Bucuresti 1965, p. 176.

2.45 $\sec\alpha+\sec\beta+\sec\gamma\geqslant 6.$

Equality holds if and only if the triangle is equilateral.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

2.46 $\sec^2\alpha+\sec^2\beta+\sec^2\gamma\geqslant 12$ in acute triangles.

$\sec^2\alpha+\sec^2\beta+\sec^2\gamma>3$ in obtuse triangles.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.47 $\sec\beta\sec\gamma+\sec\gamma\sec\alpha+\sec\alpha\sec\beta\geqslant 12.$

Equality only applies when the triangle is equilateral.

J. M. Child, Math. Gaz. 23 (1939), 136–143.

2.48 $\sec^2\frac{\alpha}{2}+\sec^2\frac{\beta}{2}+\sec^2\frac{\gamma}{2}\geqslant 4.$

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1958/57), 108–115.

2.49 $\operatorname{cosec}\alpha+\operatorname{cosec}\beta+\operatorname{cosec}\gamma\geqslant 2\sqrt{3},$

with equality only for equilateral triangles.

T. R. Curry, Problem E 1861, Amer. Math. Monthly 73 (1966), 199.

2.50 $\operatorname{cosec}^2\alpha+\operatorname{cosec}^2\beta+\operatorname{cosec}^2\gamma\geqslant 4.$

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.51 $\operatorname{cosec}\frac{\alpha}{2}+\operatorname{cosec}\frac{\beta}{2}+\operatorname{cosec}\frac{\gamma}{2}\geqslant 6,$

with equality holding if and only if the triangle is equilateral.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

M. S. Klamkin, Problem E 1361, Amer. Math. Monthly 66 (1959), 312 and 916–917.

2.52 $\operatorname{cosec}^2 \frac{\alpha}{2} + \operatorname{cosec}^2 \frac{\beta}{2} + \operatorname{cosec}^2 \frac{\gamma}{2} \geqslant 12.$

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.53 $\operatorname{cosec} \frac{\beta}{2} \operatorname{cosec} \frac{\gamma}{2} + \operatorname{cosec} \frac{\gamma}{2} \operatorname{cosec} \frac{\alpha}{2} + \operatorname{cosec} \frac{\alpha}{2} \operatorname{cosec} \frac{\beta}{2} \geqslant 12.$

Equality holds if and only if the triangle is equilateral.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

2.54 $$\operatorname{cosec} \alpha + \operatorname{cosec} \beta + \operatorname{cosec} \gamma \geqslant \tfrac{9}{4} \sec \frac{\alpha}{2} \sec \frac{\beta}{2} \sec \frac{\gamma}{2}. \qquad (1)$$

Proof. Putting $x = \sin \alpha$, $y = \sin \beta$, $z = \sin \gamma$ in

$$(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geqslant 9,$$

we obtain

$$\operatorname{cosec} \alpha + \operatorname{cosec} \beta + \operatorname{cosec} \gamma \geqslant \frac{9}{\sin \alpha + \sin \beta + \sin \gamma}. \qquad (2)$$

Since

$$\sin \alpha + \sin \beta + \sin \gamma = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

from (2), we get (1).

C. Ionescu-Tiu, Gaz. Mat. B 14 (1963), 225.

2.55 $$\begin{aligned} \operatorname{cosec} 2\alpha + \operatorname{cosec} 2\beta + \operatorname{cosec} 2\gamma &\geqslant \operatorname{cosec} \alpha + \operatorname{cosec} \beta + \operatorname{cosec} \gamma \\ &\geqslant \sec \frac{\alpha}{2} + \sec \frac{\beta}{2} + \sec \frac{\gamma}{2} \\ &\geqslant 2\sqrt{3}. \end{aligned}$$

2.56 $$\left(\sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2}\right)^2 \leqslant \cos^2 \frac{\alpha}{2} + \cos^2 \frac{\beta}{2} + \cos^2 \frac{\gamma}{2}. \qquad (1)$$

PROOF. One of the following relations holds for the angles of a triangle

$$\alpha \geqslant \frac{\pi}{3} \geqslant \beta \geqslant \gamma \text{ or } \gamma \geqslant \beta \geqslant \frac{\pi}{3} \geqslant \alpha.$$

In both cases

$$\left(\sin \frac{\beta}{2} - \frac{1}{2}\right)\left(\sin \frac{\gamma}{2} - \frac{1}{2}\right) \geqslant 0,$$

whence

$$4 \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \geqslant 2\left(\sin \frac{\beta}{2} + \sin \frac{\gamma}{2}\right) - 1.$$

Therefore

$$1+4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \geqslant 2\cdot\sin\frac{\alpha}{2}\left(\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\right)+1-\sin\frac{\alpha}{2}. \quad (2)$$

Since $\sin \beta/2 \sin \gamma/2$, for α constant, reaches its maximum when $\beta/2 = \gamma/2 = (\pi-\alpha)/4$, we have

$$\sin \frac{\beta}{2} \sin \frac{\gamma}{2} \leqslant \sin^2 \frac{\pi-\alpha}{4} = \frac{1-\sin \alpha/2}{2}.$$

By the last inequality, from (2) follows

$$\cos\alpha+\cos\beta+\cos\gamma = 1+4\cdot\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$$

$$\geqslant 2\left(\sin\frac{\beta}{2}\sin\frac{\gamma}{2}+\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right).$$

Therefore inequality (1) holds.

V. Thébault-L. Bankoff, Problem E 1272, Amer. Math. Monthly 67 (1960), 693–694.

2.57 $$\frac{\cos\beta/2\cos\gamma/2}{\sin\alpha/2}+\frac{\cos\gamma/2\cos\alpha/2}{\sin\beta/2}+\frac{\cos\alpha/2\cos\beta/2}{\sin\gamma/2} \geqslant \frac{9}{2}.$$

C. Ionescu-Tiu, Gaz. Mat. B 14 (1963), 225.

2.58 $$S=\frac{\sin\alpha/2}{\sin\beta/2\cos\gamma/2}+\frac{\sin\beta/2}{\sin\gamma/2\cos\alpha/2}+\frac{\sin\gamma/2}{\sin\alpha/2\cos\beta/2} \geqslant 2\sqrt{3}. \quad (1)$$

PROOF. By the arithmetic-geometric mean inequality, we have

$$S \geqslant 3\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\right)^{-1/3}. \tag{2}$$

Since (see: 2.28)

$$\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} \leqslant \frac{3}{8}\sqrt{3},$$

from (2) we get (1).

D. Nicolae, Gaz. Mat. B 14 (1963), 561.

2.59 $$\frac{1+\cos\alpha\cos\beta\cos\gamma}{\sin\alpha\sin\beta\sin\gamma} \geqslant \sqrt{3}.$$

Equality holds if and only if the triangle is equilateral.

H. W. Guggenheimer, Plane Geometry and its Groups, San Francisco, Cambridge, London, Amsterdam 1967, p. 189.

2.60 $$\sqrt{2}\left(\cos\frac{\alpha}{4}+\sin\frac{\alpha}{4}\right) < \cos\frac{\beta}{4}+\cos\frac{\gamma}{4}+\sin\frac{\beta}{4}+\sin\frac{\gamma}{4}.$$

PROOF.

$$\begin{aligned}
&\sqrt{2}\left(\cos\frac{\alpha}{4}+\sin\frac{\alpha}{4}\right)\\
&\quad=\sqrt{2}\left(\cos\frac{\pi-\beta-\gamma}{4}+\sin\frac{\pi-\beta-\gamma}{4}\right)\\
&\quad=2\cos\frac{\beta+\gamma}{4}<2\cos\frac{\beta-\gamma}{4}\\
&\quad=\cos\frac{\beta}{4}\cos\frac{\gamma}{4}+\sin\frac{\beta}{4}\sin\frac{\gamma}{4}+\cos\frac{\gamma}{4}\cos\frac{\beta}{4}+\sin\frac{\gamma}{4}\sin\frac{\beta}{4}\\
&\quad<\cos\frac{\beta}{4}+\cos\frac{\gamma}{4}+\sin\frac{\beta}{4}+\sin\frac{\gamma}{4}.
\end{aligned}$$

C. Ionescu-Tiu, Gaz. Mat. B 12 (1961), 304.

2.61 If α, β, γ are angles of a triangle measured in radians, then

$$\alpha\cos\beta+\sin\alpha\cos\gamma>0.$$

L. E. Bush, Amer. Math. Monthly 64 (1957), 24.

2.62 $2(\operatorname{cotg}\alpha+\operatorname{cotg}\beta+\operatorname{cotg}\gamma)\geqslant\operatorname{cosec}\alpha+\operatorname{cosec}\beta+\operatorname{cosec}\gamma.$

C. Cosnita and F. Turtoiu, Culegere de probleme de algebra, Bucuresti 1965, p. 176.

2.63 $\operatorname{tg}\beta\operatorname{tg}\gamma+\operatorname{tg}\gamma\operatorname{tg}\alpha+\operatorname{tg}\alpha\operatorname{tg}\beta\geqslant 3+\sec\alpha+\sec\beta+\sec\gamma.$

C. Cosnita and F. Turtoiu, Culegere de probleme de algebra, Bucuresti 1965, p. 176.

2.64 $$\frac{\sqrt{\operatorname{tg}\beta/2\operatorname{tg}\gamma/2}}{\cos\alpha/2}+\frac{\sqrt{\operatorname{tg}\gamma/2\operatorname{tg}\alpha/2}}{\cos\beta/2}+\frac{\sqrt{\operatorname{tg}\alpha/2\operatorname{tg}\beta/2}}{\cos\gamma/2}\leqslant 2.$$

Ju. A. Izosimov, Matematika v škole 1958, No. 5, 95.

2.65 If n is a natural number, then

$$\operatorname{cotg}^n\alpha+\operatorname{cotg}^n\beta+\operatorname{cotg}^n\gamma\geqslant 3\cdot 3^{-n/2}.$$

Equality holds if and only if the triangle is equilateral.

M. N. Kritikos, Actes du Congrès interbalkanique de mathématiciens, Athènes 1934, 157–158.

3. Inequalities for the angles and other elements of a triangle

3.1 If $a<\frac{1}{2}(b+c)$, then $\alpha<\frac{1}{2}(\beta+\gamma)$.

Proof. By virtue of the sine law, the inequality $a<\frac{1}{2}(b+c)$ gives

$$\sin\alpha<\tfrac{1}{2}(\sin\beta+\sin\gamma)=\sin\frac{\beta-\gamma}{2}\cos\frac{\beta-\gamma}{2}\leqslant\sin\frac{\beta+\gamma}{2}.$$

Since α and $\frac{1}{2}(\beta+\gamma)$ are acute angles, we infer that

$$\alpha<\tfrac{1}{2}(\beta-\gamma).$$

G. Pólya-A. Hess, Elem. Math. 13 (1958), 88.

3.2 $a\alpha+b\beta+c\gamma \geqslant \frac{1}{2}(b\gamma+c\alpha+a\beta+c\beta+a\gamma+b\alpha)$. (1)

PROOF. Let $a \geqslant b \geqslant c$. Then $\alpha \geqslant \beta \geqslant \gamma$. Multiplying $\alpha \geqslant \beta$ by $a-b \geqslant 0$, $\alpha \geqslant \gamma$ by $a-c \geqslant 0$ and $\beta \geqslant \gamma$ by $b-c \geqslant 0$, one obtains

$$a\alpha+b\beta \geqslant a\beta+b\alpha,\quad a\alpha+c\gamma \geqslant a\gamma+c\alpha,\quad b\beta+c\gamma \geqslant b\gamma+c\beta.$$

Adding these inequalities, we have

$$2(a\alpha+b\beta+c\gamma) \geqslant b\gamma+c\alpha+a\beta+c\beta+a\gamma+b\alpha,$$

i.e. (1).

3.3 $\dfrac{\pi}{3} \leqslant \dfrac{a\alpha+b\beta+c\gamma}{a+b+c} < \dfrac{\pi}{2}$.

Equality holds if and only if $a = b = c$.

PROOF. Let $a \geqslant b \geqslant c$. Then $\alpha \geqslant \beta \geqslant \gamma$ and consequently

$$(a-b)(\alpha-\beta) \geqslant 0,\quad (b-c)(\beta-\gamma) \geqslant 0,\quad (c-a)(\gamma-\alpha) \geqslant 0. \qquad (1)$$

Adding these inequalities, we obtain

$$(a-b)(\alpha-\beta) + (b-c)(\beta-\gamma) + (c-a)(\gamma-\alpha) \geqslant 0,$$

i.e.

$$2(a\alpha+b\beta+c\gamma) \geqslant (b+c)\alpha+(c+a)\beta+(a+b)\gamma.$$

Adding $a\alpha+b\beta+c\gamma$ to both sides of the last inequality, we get

$$3(a\alpha+b\beta+c\gamma) \geqslant (a+b+c)(\alpha+\beta+\gamma). \qquad (2)$$

Since $\alpha+\beta+\gamma = \pi$, from (2) follows

$$\frac{\pi}{3} \leqslant \frac{a\alpha+b\beta+c\gamma}{a+b+c},$$

with equality only if $a = b = c$, because equality in (1) holds only in this case.

Since a, b, c are the sides of a triangle, we have

$$a+b+c > 2a,\quad a+b+c > 2b,\quad a+b+c > 2c.$$

Multiplying these inequalities by α, β, γ respectively and adding together the inequalities thus obtained, we get

$$(a+b+c)(\alpha+\beta+\gamma) > 2(a\alpha+b\beta+c\gamma),$$

whence

$$\frac{a\alpha+b\beta+c\gamma}{a+b+c}<\frac{\pi}{2}.$$

REMARK. This proof follows the idea of J. Kürschak given in: G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, vol. II, Berlin 1925, p. 166 and 393.

3.4 If $\alpha \leqslant \beta \leqslant \gamma$, then

$$1°.\quad \frac{\pi}{3} \leqslant \frac{a\alpha+b\beta+c\gamma}{a+b+c} \leqslant \frac{\pi-\alpha}{2},$$

$$2°.\quad \frac{\pi}{3} \leqslant \frac{a\alpha+b\beta+c\gamma}{a+b+c} \leqslant \frac{\pi}{2}\left(1-\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}\right).$$

PROOF. 1°. Since $\alpha \leqslant \beta \leqslant \gamma$ and $(a+b-c)\alpha + (a-b+c)\alpha = 2a\alpha$, we have

$$(a-b+c)\beta + (a+b-c)\gamma \geqslant 2a\alpha,$$

i.e.,

$$2a\alpha+b\beta+c\gamma \leqslant (a+c)\beta + (a+b)\gamma.$$

Adding $b\beta+c\gamma$ to both sides of the last inequality, we get

$$2(a\alpha+b\beta+c\gamma) \leqslant (a+b+c)(\beta+\gamma)$$

whence

$$\frac{a\alpha+b\beta+c\gamma}{a+b+c} \leqslant \frac{\pi-\alpha}{2}.$$

2°. If $a \leqslant b \leqslant c$, then

$$\frac{a\alpha+b\beta+c\gamma}{a+b+c} \leqslant \frac{c(\alpha+\beta+\gamma)}{a+b+c} = \frac{c\pi}{2s}.$$

Since

$$\operatorname{tg}\frac{\alpha}{2} = \left(\frac{(s-b)(s-c)}{s(s-a)}\right)^{1/2}, \quad \operatorname{tg}\frac{\beta}{2} = \left(\frac{(s-c)(s-a)}{s(s-b)}\right)^{1/2},$$

we obtain

$$\frac{\pi}{2}\left(1-\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}\right) = \frac{\pi}{2}\left(1-\frac{s-c}{s}\right) = \frac{c\pi}{2s}.$$

Therefore

$$\frac{a\alpha+b\beta+c\gamma}{a+b+c} \leqslant \frac{\pi}{2}\left(1-\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}\right).$$

This proof is due to G. Kalajdžić.

D. Marković, Bull. Soc. Math. Phys. Serbie 4, No. 3–4 (1952), 71.

3.5 $$\frac{(b\gamma-c\beta)^2+(c\alpha-a\gamma)^2+(a\beta-b\alpha)^2}{(a+b+c)^2} < \frac{\pi^2}{4}.$$

This result is the best possible.

D. Blanuša, Problem 175, Glasnik matematičko-fizički i astronomski (2) 8 (1953), 160.

REMARK. A proof of this inequality, due to T. Leko, is very complicated and it would be desirable to have a simpler proof.

3.6 $$\frac{2bc\cos\alpha}{b+c} < b+c-a < \frac{2bc}{a}. \tag{1}$$

PROOF. Let $a \leqslant \max(b, c)$. Since $\max(b, c) < a+\min(b, c)$, we have

$$b+c-a = \min(b,c)+\max(b,c)-a < \min(b,c)+\min(b,c)+a-a$$
$$= 2\min(b, c) \leqslant \frac{2\min(b,c)\max(b,c)}{a} = \frac{2bc}{a}.$$

Let $a > \max(b, c)$. Then $a < 2\max(b, c)$ must hold, for otherwise $a \geqslant b+c$. Further, we have

$$b+c-a = \min(b,c)+\max(b,c)-a < \min(b,c)+a-a$$
$$= \min(b,c) < \frac{2\max(b,c)\min(b,c)}{a} = \frac{2bc}{a}. \tag{2}$$

Therefore we have proved the second inequality in (1).

Since

$$(b+c)(b+c-a)-2bc\cos\alpha = a\left(\frac{2bc}{a}-(b+c-a)\right),$$

using inequality (2), we obtain the first inequality in (1).

J. A. Kalman, Math. Gaz. 47 (1963), 224–225.

3.7 $$\left(1-\frac{a}{b+c}\right)\left(1-\frac{b}{c+a}\right)\left(1-\frac{c}{a+b}\right)\leqslant \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.$$

Equality occurs if and only if the triangle is equilateral.

Gaz. Mat. B 8 (1957), 47.

3.8 $$\operatorname{cotg}\alpha+\operatorname{cotg}\beta+\operatorname{cotg}\gamma\geqslant\frac{\sqrt{3}}{9}\cdot\frac{(a^2+b^2+c^2)(a+b+c)}{abc},$$

with equality holding if and only if the triangle is equilateral.

T. R. Curry, Problem E 1861, Amer. Math. Monthly 73 (1966), 199.

3.9 If $a\leqslant b\leqslant c$, then

$$g\leqslant\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\leqslant G,$$

where

$$g=\frac{2}{1+\operatorname{tg}\beta/2\operatorname{tg}\gamma/2}\geqslant 2\cos^2\frac{\gamma}{2},\quad G=\frac{2}{1+\operatorname{tg}\alpha/2\operatorname{tg}\beta/2}\leqslant 2\cos^2\frac{\alpha}{2}.$$

Equality occurs if and only if $a=b=c$.

Proof. Since $a\leqslant b\leqslant c$ and

$$\operatorname{tg}\frac{\alpha}{2}=\left(\frac{(s-b)(s-c)}{s(s-a)}\right)^{1/2},\quad \operatorname{tg}\frac{\beta}{2}=\left(\frac{(s-c)(s-a)}{s(s-b)}\right)^{1/2},$$

$$\operatorname{tg}\frac{\gamma}{2}=\left(\frac{(s-a)(s-b)}{s(s-c)}\right)^{1/2}$$

we have

$$g=\frac{2}{1+\operatorname{tg}\beta/2\operatorname{tg}\gamma/2}=\frac{2s}{2s-a}=\frac{a+b+c}{b+c}\leqslant\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b};$$

$$G=\frac{2}{1+\operatorname{tg}\alpha/2\operatorname{tg}\beta/2}=\frac{2s}{2s-c}=\frac{a+b+c}{a+b}\geqslant\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

This proof is due to G. Kalajdžić.

D. Marković, Bull. Soc. Math. Phys. Serbie 4, No. 3–4 (1952), 71.

3.10 If $k = c^t/(a^t+b^t)$, then

$$0 < k \leqslant 2^{t-1}\sin^t \frac{\gamma}{2} \qquad \text{for } t < 0,$$

$$2^{t-1}\sin^t \frac{\gamma}{2} \leqslant k < 1 \qquad \text{for } 0 < t \leqslant 1,$$

$$\min\left(1,\, 2^{t-1}\sin^t \frac{\gamma}{2}\right) \leqslant k \leqslant \max\left(2^{t-1}t^{-t/2},\, 2^{t-1}\sin^t \frac{\gamma}{2}\right) \qquad \text{for } 1 < t \leqslant 2,$$

$$\min\left(2^{t-1}t^{-t/2},\, 2^{t-1}\sin^t \frac{\gamma}{2}\right) \leqslant k \leqslant \max\left(1,\, 2^{t-1}\sin^t \frac{\gamma}{2}\right) \text{ for } t \geqslant 2.$$

REMARK. These inequalities are due to R. Ballieu, Simon Stevin 26 (1949), 129–134. The upper bound for the case $t \geqslant 2$ and t a positive integer is determined in: F. van der Blij, Simon Stevin 25 (1947), 231–235.

3.11 If $k = r/R$, then

$$-1+4k-k^2 \leqslant \cos\beta\cos\gamma+\cos\gamma\cos\alpha+\cos\alpha\cos\beta \leqslant k+k^2$$

$$-1+3k-\tfrac{3}{2}k^2 \leqslant \cos\alpha\cos\beta\cos\gamma \leqslant \tfrac{1}{2}k^2.$$

Equality occurs if and only if the triangle is equilateral.

W. J. Blundon, Problem E 1925, Amer. Math. Monthly 73 (1966), 1016.

3.12 If for a triangle $\alpha > \beta > \gamma$, then

$$2R\cos\alpha < R-d < 2R\cos\beta < R+d < 2R\cos\gamma,$$

where d is the distance from the circumcentre to the incentre.

O. Bottema, Euclides 39 (1963/64), 129–137.

3.13 $$a\sin\alpha+b\sin\beta+c\sin\gamma \geqslant \frac{2\sqrt{3}F}{R}.$$

Equality holds if and only if the triangle is equilateral.

PROOF. Since

$$\sin\alpha = a/(2R),\ \sin\beta = b/(2R),\ \sin\gamma = c/(2R),$$

we have

$$a\sin\alpha+b\sin\beta+c\sin\gamma=\frac{a^2+b^2+c^2}{2R}\geqslant\frac{2\sqrt{3}F}{R},$$

Weitzenböck's inequality **4.4** was used here.

3.14 $9r\leqslant a\sin\alpha+b\sin\beta+c\sin\gamma\leqslant\frac{9}{2}R,$

with equality holding if and only if the triangle is equilateral.

PROOF. Starting from

$a\sin\alpha=\dfrac{a^2}{2R}=\dfrac{2aF}{bc}=\dfrac{h_bh_c}{h_a}$ and similarly for $b\sin\beta$ and $c\sin\gamma$, we get

$$a\sin\alpha+b\sin\beta+c\sin\gamma=\frac{h_bh_c}{h_a}+\frac{h_ch_a}{h_b}+\frac{h_ah_b}{h_c}.$$

By means of the arithmetic-geometric mean inequality and **6.16**, we obtain

$$\frac{h_bh_c}{h_a}+\frac{h_ch_a}{h_b}+\frac{h_ah_b}{h_c}\geqslant 3\sqrt[3]{h_ah_bh_c}\geqslant 9r,$$

with equality if and only if $h_a=h_b=h_c$, i.e. if and only if the triangle is equilateral.

Therefore

$$a\sin\alpha+b\sin\beta+c\sin\gamma\geqslant 9r.$$

Using the sine law, we obtain

$$a\sin\alpha+b\sin\beta+c\sin\gamma=\frac{a^2+b^2+c^2}{2R}. \tag{1}$$

Since (see: **5.13**)

$$a^2+b^2+c^2\leqslant 9R^2,$$

from (1) we obtain

$$a\sin\alpha+b\sin\beta+c\sin\gamma\leqslant\tfrac{9}{2}R.$$

This result is due to R. R. Janić.

3.15 $\sin\alpha+\sin\beta+\sin\gamma\leqslant 2+(3\sqrt{3}-4)\cdot\dfrac{r}{R}\leqslant\frac{3}{2}\sqrt{3}.$

Equality holds if and only if the triangle is equilateral.

W. J. Blundon, Canad. Math. Bull. 8 (1965), 615–626.

4. Inequalities for the sides and the area of a triangle

4.1 $4F \leqslant \min(b^2+c^2, c^2+a^2, a^2+b^2)$.

PROOF. Let $a \geqslant b \geqslant c$. From

$$b^2+c^2 \geqslant 2bc \text{ and } 2bc \geqslant 2bc \sin \alpha,$$

follows

$$b^2+c^2 \geqslant 2bc \sin \alpha = 4F.$$

This concludes the proof.

4.2 $s^2 \geqslant 3F\sqrt{3}$.

Equality holds if and only if the triangle is equilateral.

PROOF. Since

$$[(s-a)(s-b)(s-c)]^{1/3} \leqslant \frac{(s-a)+(s-b)+(s-c)}{3} = \frac{s}{3},$$

i.e.

$$(s-a)(s-b)(s-c) \leqslant \frac{s^3}{27},$$

we obtain

$$F = [s(s-a)(s-b)(s-c)]^{1/2} \leqslant \frac{s^2}{3\sqrt{3}}.$$

H. Hadwiger, Jber. Deutsch. Math.-Verein. 49 (1939), 35–39 kursiv.

L. A. Santaló, Math. Notae 3 (1943), 65–73.

4.3 $s^2 \geqslant 3F\sqrt{3} + \frac{1}{2}Q$.

J. C. H. Gerretsen, Nieuw Tijdschr. Wisk. 41 (1953), 1–7.

4.4 $a^2+b^2+c^2 \geqslant 4F\sqrt{3}$.

Equality occurs if and only if the triangle is equilateral.

PROOF. By cosine law,

$$a^2+b^2+c^2 = 2(b^2+c^2)-2bc \cos \alpha.$$

Since

$$F = \tfrac{1}{2}bc \sin \alpha,$$

one obtains

$$\begin{aligned} a^2+b^2+c^2-4F\sqrt{3} &= 2(b^2+c^2)-2bc\cos\alpha-2bc\sqrt{3}\sin\alpha \\ &= 2(b^2+c^2)-4bc\left(\frac{1}{2}\cos\alpha-\frac{\sqrt{3}}{2}\sin\alpha\right) \\ &= 2(b^2+c^2)-4bc\cos\left(\frac{\pi}{3}-\alpha\right) \\ &\geqslant 2(b^2+c^2)-4bc \\ &= 2(b-c)^2. \end{aligned}$$

Equality holds if and only if $b = c$ and $\alpha = \pi/3$, i.e. if and only if the triangle is equilateral.

R. Weitzenböck, Math. Z. 5 (1919), 137–146.

P. Finsler and H. Hadwiger, Comment. Math. Helv. 10 (1937/38), 316–326.

L. A. Santaló, Math. Notae 3 (1943), 65–73.

F. Goldner, Problem 69, Elem. Math. 4 (1949), 120.

R.A., Elementa: matematik, fysik, kemi (Uppsala) 49 (1966), 250.

4.5 $bc+ca+ab \geqslant 4F\sqrt{3}.$ (1)

Proof. Since

$$(\sin\alpha)^{-1}+(\sin\beta)^{-1}+(\sin\gamma)^{-1} \geqslant 3\sqrt[3]{(\sin\alpha\sin\beta\sin\gamma)^{-1}}$$

and (see: **2.7**)

$$\sin\alpha\sin\beta\sin\gamma \leqslant \frac{3\sqrt{3}}{8},$$

we have

$$(\sin\alpha)^{-1}+(\sin\beta)^{-1}+(\sin\gamma)^{-1} \geqslant 2\sqrt{3}.$$

Using the last inequality and

$$bc+ca+ab = 2F[(\sin\alpha)^{-1}+(\sin\beta)^{-1}+(\sin\gamma)^{-1}],$$

we get (1).

REMARK. From (1) and

$$(bc+ca+ab)^2 \leqslant 3(b^2c^2+c^2a^2+a^2b^2) \leqslant (a^2+b^2+c^2)^2 \leqslant 3(a^4+b^4+c^4)$$

one obtains the inequalities: **4.12, 4.4, 4.10.**

V. O. Gordon, Matematika v škole, 1966, No. 1, 89.

4.6 $bc+ca+ab \geqslant 4F\sqrt{3}+\frac{1}{2}Q.$

J. C. H. Gerretsen, Nieuw Tijdschr. Wisk. 41 (1953), 1–7.

4.7 $4F\sqrt{3}+Q \leqslant a^2+b^2+c^2 \leqslant 4F\sqrt{3}+3Q.$

Equality holds if and only if the triangle is equilateral.

H. Hadwiger, Jber. Deutsch. Math.-Verein. 49 (1939), 35–39 kursiv.

REMARK. The first inequality was also proved in:

P. Finsler and H. Hadwiger, Comment. Math. Helv. 10 (1937/38), 316–326.

J. Karamata, Problem 119, Glasnik Mat.-Fiz. Astronom. 3 (1948), 223.

O. Kooi, Simon Stevin 32 (1958), 97–101.

J. Steinig, Elem. Math. 18 (1963), 127–131.

S. Zetel', Matematika v škole, 1965, No. 3, 66–69.

4.8 $4F\sqrt{3}+Q \leqslant a^2+b^2+c^2$

$$\leqslant 4F\left(\sec\frac{\alpha}{2}+\sec\frac{\beta}{2}+\sec\frac{\gamma}{2}-\sqrt{3}\right)+Q.$$

Equality occurs if and only if the triangle is equilateral.

S. T. Berkolaiko, Matematika v škole, 1967, No. 2, 72–74.

4.9 $12F\sqrt{3}+2Q \leqslant (a+b+c)^2 \leqslant 12F\sqrt{3}+8Q.$

Equality holds if and only if the triangle is equilateral.

H. Hadwiger, Jber. Deutsch. Math.-Verein. 49 (1939), 35–39 kursiv.

REMARK. The first inequality was also proved in:

P. Finsler and H. Hadwiger, Comment. Math. Helv. 10 (1937/38), 316–326.

4.10 $a^4+b^4+c^4 \geqslant 16F^2.$

Equality holds if and only if $a = b = c$.
F. Goldner, Problem 69, Elem. Math. 4 (1949), 120.

4.11 $a^4+b^4+c^4 \geqslant 16F^2+4FQ\sqrt{3}+\frac{1}{2}Q^2.$ (1)

Equality holds if and only if $a = b = c$.

PROOF. By squaring the Finsler-Hadwiger's inequality **4.7**, one obtains

$$a^4+b^4+c^4+2(b^2c^2+c^2a^2+a^2b^2) \geqslant 48F^2+8FQ\sqrt{3}+Q^2.$$

Since

$$2(b^2c^2+c^2a^2+a^2b^2) = 16F^2+a^4+b^4+c^4,$$

then

$$2(a^4+b^4+c^4)+16F^2 \geqslant 48F^2+8FQ\sqrt{3}+Q^2,$$

i.e. (1).
Matematika i fizika, Sofia, 6 (1965), 51–52.

4.12 $b^2c^2+c^2a^2+a^2b^2 \geqslant 16F^2.$ (1)

Equality holds if and only if the triangle is equilateral.

PROOF. On the basis of Heron's formula we have

$$2(b^2c^2+c^2a^2+a^2b^2) - (a^4+b^4+c^4) = 16F^2.$$

Since

$$(b^2-c^2)^2 + (c^2-a^2)^2 + (a^2-b^2)^2 \geqslant 0,$$

i.e.,

$$b^2c^2+c^2a^2+a^2b^2 \leqslant a^4+b^4+c^4,$$

we get (1).
F. Goldner, Problem 69, Elem. Math. 4 (1949), 120.

4.13 $4F\sqrt{3} \leqslant \dfrac{9abc}{a+b+c}.$

Equality occurs if and only if $a = b = c$.
T. R. Curry, Problem E 1861, Amer. Math. Monthly 73 (1966), 199.

4.14 $(abc)^2 \geqslant \left(\frac{4F}{\sqrt{3}}\right)^3.$ (1)

Equality holds if and only if $a = b = c$.

PROOF. From $abc = 4RF$ and from $a+b+c \leqslant 3R\sqrt{3}$ (see: 5.3), we have

$$\frac{4F}{\sqrt{3}} = \frac{abc}{R\sqrt{3}} \leqslant \frac{3abc}{2s} = \frac{3abc}{a+b+c} \leqslant (abc)^{2/3},$$

whence inequality (1) follows.

L. Carlitz-F. Leuenberger, Problem E 1454, Amer. Math. Monthly 68 (1961), 177 and 68 (1961), 805–806.

H. W. Guggenheimer, Problem E 1724, Amer. Math. Monthly 71 (1964), 911 and 72 (1965), 791–793.

4.15 If x, y, z are real numbers, then

$$a^x b^y c^z + b^x c^y a^z + c^x a^y b^z \geqslant 3 \cdot \frac{2^{x+y+z} F^{(x+y+z)/2}}{3^{(x+y+z)/4}}. \quad (1)$$

PROOF. By means of the arithmetic-geometric mean inequality, we obtain

$$a^x b^y c^z + b^x c^y a^z + c^x a^y b^z \geqslant 3(abc)^{(x+y+z)/3}.$$

From this inequality, on the basis of **4.14**, we get (1).

This inequality and proof are due to P. M. Vasić.

4.16 If x, y, z are real numbers, then

$$b^x c^y (b+c)^z + c^x a^y (c+a)^z + a^x b^y (a+b)^z \geqslant 3 \cdot 2^z \frac{2^{x+y+z} F^{(x+y+z)/2}}{3^{(x+y+z)/4}}.$$

(1)

PROOF. Since $a+b \geqslant 2\sqrt{ab}$, etc., we obtain

$$b^x c^y (b+c)^z + c^x a^y (c+a)^z + a^x b^y (a+b)^z$$
$$\geqslant 2^z (b^{x+(z/2)} c^{y+(z/2)} + c^{x+(z/2)} a^{y+(z/2)} + a^{x+(z/2)} b^{y+(z/2)})$$
$$\geqslant 3 \cdot 2^z \sqrt[3]{(abc)^{x+y+z}}.$$

From this inequality, on the basis of **4.14**, we get (1).

This inequality and proof are due to R. Ž. Djordjević.

4.17 If x, y, z are real numbers, then

$$a^x(a+b)^y(b+c)^z+b^x(b+c)^y(c+a)^z+c^x(c+a)^y(a+b)^z$$

$$\geqslant 3\cdot 2^x \frac{2^{2(y+z)}F^{(x+y+z)/2}}{3^{(x+y+z)/4}}.$$

This inequality is due to R. Ž. Djordjević.

4.18 If $a^2+b^2+c^2 = 2H$ and $bc+ca+ab = K$, then

$$\frac{(K-H)(3K-5H)}{12} \leqslant F^2 \leqslant \frac{(K-H)^2}{12}. \tag{1}$$

Equality in (1) holds if and only if the triangle is equilateral.
S. Beatty, Trans. Roy. Soc. Canada, III (3), 48 (1954), 1–5.

4.19 If $q = (Q/2)^{1/2}$, then

$$\frac{s(s+q)^2(s-2q)}{27} \leqslant F^2 \leqslant \frac{s(s-q)^2(s+2q)}{27}. \tag{1}$$

The first (second) equality sign in (1) holds for an isosceles triangle whose base is the smallest (largest) of the three sides; of course both equality signs apply when the triangle is equilateral, since then $q = 0$.
R. Frucht, Canad. J. Math. 9 (1957), 227–231.

4.20 $27(b^2+c^2-a^2)^2(c^2+a^2-b^2)^2(a^2+b^2-c^2)^2 \leqslant (4F)^6$.

Equality holds if and only if $a = b = c$.
F. Balitrand, Interméd. Math. 23 (1916), 86–87.

4.21 If $\lambda > 0$, then

$$F \leqslant \frac{\sqrt{3}}{4}\left(\frac{a^\lambda+b^\lambda+c^\lambda}{3}\right)^{2/\lambda}.$$

C. N. Mills-O. Dunkel, Problem 3207, Amer. Math. Monthly 34 (1927), 382–384.

5. Inequalities for the sides and the radii of a triangle

5.1 $2r \leqslant R.$

Equality holds if and only if the triangle is equilateral.

PROOF. This inequality is obtained from

$$OI^2 = R(R-2r) \geqslant 0.$$

L. Euleri, Novi commentarii academiae scientiarum Petropolitanae 11 (1765), 1767, 103–123.

L. Euleri, Opera Omnia, I 26, 1953, 139–157.

Ramus-E. Rouché, Nouv. Ann. Math. 10 (1851), 353.

5.2 If t and T are real numbers such $-\infty < t \leqslant T \leqslant 11$, then

$$2r \leqslant \frac{4s^2R^{-1}-2tr}{27-t} \leqslant \frac{4s^2R^{-1}-2Tr}{27-T} \leqslant R.$$

PROOF. Consider the function f, defined by

$$f(x) = \frac{4s^2R^{-1}-2rx}{27-x} \quad \text{for } x < 27.$$

From inequality (see 1.1 and 5.36)

$$(a+b+c)^2 \geqslant 3(bc+ca+ab) \geqslant 54Rr,$$

we have

$$f(x) \geqslant \frac{54r-2rx}{27-x} = 2r.$$

From inequalities (see 5.8)

$$4s^2 \leqslant 4(3r^2+4Rr+4R^2) \quad \text{and} \quad 2r \leqslant R,$$

we obtain

$$4s^2 \leqslant 16R^2+22Rr.$$

Since $f'(x) = \dfrac{4s^2R^{-1}-54r}{(27-x)^2} \geqslant 0$, we infer that f is a monotone non-decreasing function, so that $f(11) \leqslant R$.

I. Paasche-O. Reutter, Problem 516, Elem. Math. 20 (1965), 140 and 21 (1966), 139.

5.3 $a+b+c \leqslant 3R\sqrt{3}$.

Equality holds if and only if $a = b = c$.

S. Nakajima, Tôhoku Math. J. 25 (1925), 115–121.

A. Padoa, Period. Mat. (4) 5 (1925), 80–85.

REMARK. This inequality is equivalent to **2.1**.

5.4 $s \leqslant 2R+(3\sqrt{3}-4)r$.

Equality holds if and only if the triangle is equilateral.

W. J. Blundon, Canad. Math. Bull. 8 (1965), 615–626.

W. J. Blundon, Problem E 1935, Amer. Math. Monthly 73 (1966), 1122.

5.5 $9r(4R+r) \leqslant 3s^2 \leqslant (4R+r)^2$. (1)

PROOF. The roots of the equation

$$y^3-2sy^2+(s^2+r(4R+r))y-4sRr = 0 \qquad (2)$$

are a, b, c.

Taking $y = \dfrac{s(x-r)}{x}$, equation (2) becomes

$$x^3-(4R+r)x^2+s^2x-s^2r = 0,$$

with roots r_a, r_b, r_c.

In view of Rolle's theorem, both equations

$$3y^2-4sy+s^2+r(4R+r) = 0$$

and

$$3x^2-2(4R+r)x+s^2 = 0$$

have real roots, which implies

$$3s^2 \geqslant 9r(4R+r) \text{ and } (4R+r)^2 \geqslant 3s^2.$$

This proves (1).

G. Colombier-T. Doucet, Problem 1051, Nouv. Ann. Math. 31 (1872), 467.

5.6 $6r(4R+r) \leqslant 2s^2 \leqslant 2(2R+r)^2+R^2$.

Equalities hold if and only if the triangle is equilateral.

PROOF. Since $r = F/s$ and $R = abc/(4F)$, we have

$$r(4R+r) = \frac{abc}{s} + \frac{F^2}{s^2} = \tfrac{1}{3}s^2 - Q \leqslant \tfrac{1}{3}s^2,$$

and consequently

$$6r(4R+r) \leqslant 2s^2.$$

By the last inequality and the inequality $a^2+b^2+c^2 \leqslant 9R^2$ (see: **5.13**), we obtain

$$(2R+r)^2 + \frac{R^2}{2} = \frac{9R^2}{2} + r(4R+r) \geqslant \frac{a^2+b^2+c^2}{2} + r(4R+r) = s^2,$$

i.e.

$$2s^2 \leqslant 2(2R+r)^2 + R^2.$$

F. Leuenberger-L. Carlitz, Problem E 1481, Amer. Math. Monthly 68 (1961), 803 and 69 (1962), 312.

5.7 $2s^2(2R-r) \leqslant R(4R+r)^2$.

O. Kooi, Simon Stevin, 32 (1958), 97–101.

5.8 $r(16R-5r) \leqslant s^2 \leqslant 4R^2+4Rr+3r^2$.

Equalities hold if and only if the triangle is equilateral.

PROOF. Consider the triangle HIO. Then

$$OH^2 = 9R^2 - \sum a^2, \tag{1}$$

$$IO^2 = R^2 - 2rR, \tag{2}$$

$$IH^2 = 2r^2 - 4R^2 \cos\alpha \cos\beta \cos\gamma. \tag{3}$$

But

$$4R^2 \cos\alpha \cos\beta \cos\gamma = \tfrac{1}{4}(\textstyle\sum a)^2 - (2R+r)^2,$$

and thus

$$IH^2 = 3r^2 + 4rR + 4R^2 - \tfrac{1}{4}(\textstyle\sum a)^2.$$

This establishes the inequality

$$(\textstyle\sum a)^2 \leqslant 4(3r^2+4rR+4R^2),$$

with equality only in an equilateral triangle.

According to the theorem of Euler $OG : GH = 1 : 2$, we have

$$GI^2 = \tfrac{2}{3}IO^2+\tfrac{1}{3}IH^2+\tfrac{2}{9}OH^2. \tag{4}$$

By substituting (1), (2) and (3) in (4), we obtain

$$GI^2 = r^2-\tfrac{1}{12}(\sum a)^2+\tfrac{2}{9}\sum a^2,$$

which, in view of $(\sum a)^2 = 2\sum a^2+4r(4R+r)$, is equivalent to

$$36\cdot GI^2 = (\sum a)^2+20r^2-64rR,$$

whence

$$(\sum a)^2 \geqslant 4r(16R-5r).$$

Equality only when the triangle is equilateral.

J. Steinig, Elem. Math. 18 (1963), 127–131.

REMARK. In view of $R \geqslant 2r$ both inequalities strengthen 5.1.

5.9 Let $q(R, r)$ and $Q(R, r)$ be quadratic forms with real coefficients. Then the best possible inequalities of the form

$$q(R, r) \leqslant s^2 \leqslant Q(R, r),$$

with equalities only for the equilateral triangle, occur when

$$q(R, r) = 16Rr-5r^2$$

and

$$Q(R, r) = 4R^2+4Rr+3r^2.$$

W. J. Blundon, Canad. Math. Bull. 8 (1965), 615–626.

5.10 Let $f(R, r)$ and $F(R, r)$ be homogeneous real functions. Then the strongest possible inequalities of the form

$$f(R, r) \leqslant s^2 \leqslant F(R, r),$$

with equalities only for the triangle equilateral, occur when

$$f(R, r) = 2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr},$$

and

$$F(R, r) = 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}.$$

W. J. Blundon, Canad. Math. Bull. 8 (1965), 615–626.

5.11 $s^2 \geqslant 27r^2$.

Equality holds if and only if the triangle is equilateral.

PROOF. By the arithmetic-geometric mean inequality, we obtain

$$\frac{s}{3} = \frac{(s-a)+(s-b)+(s-c)}{3} \geqslant \sqrt[3]{(s-a)(s-b)(s-c)}.$$

Since $(s-a)(s-b)(s-c) = F^2/s = r^2s$, we have

$$\frac{s}{3} \geqslant \sqrt[3]{r^2s},$$

i.e.

$$s^2 \geqslant 27r^2.$$

N. A. Edwards, Problem 1273, Nouv. Ann. Math. 37 (1878), 475.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

5.12 $2s^2 \geqslant 27Rr$. (1)

PROOF. Since

$$(a+b+c)^3 \geqslant 27abc \text{ and } abc = 4RF = 4Rrs,$$

we have

$$8s^3 \geqslant 27 \cdot 4Rrs,$$

i.e. (1).

C. Cosnita and F. Turtoiu, Culegere de probleme de algebra, Bucuresti 1965, p. 177.

5.13 $36r^2 \leqslant a^2+b^2+c^2 \leqslant 9R^2$. (1)

Equalities hold if and only if $a = b = c$.

PROOF. Since

$$\left(\frac{a^2+b^2+c^2}{3}\right)^{1/2} \geqslant \frac{a+b+c}{3},$$

by virtue of $6r\sqrt{3} \leqslant a+b+c$ (see 5.11), one obtains

$$36r^2 \leqslant a^2+b^2+c^2.$$

The second inequality in (1) follows from

$$9R^2-(a^2+b^2+c^2) = OH^2 \geqslant 0.$$

J. Neuberg, Ed. Times, News Ser. 9 (1906), 51–52.
T. Kubota, Tôhoku Math. J. 25 (1925), 122–126.
J. Steinig, Elem. Math. 18 (1963), 127–131.

5.14 $24Rr-12r^2 \leqslant a^2+b^2+c^2 \leqslant 8R^2+4r^2.$

Equalities hold if and only if the triangle is equilateral.
J. C. Gerretsen, Nieuw Tijdschr. Wisk. 41 (1953), 1–7.
J. Steinig, Elem. Math. 18 (1963), 127–131.

5.15 $4R^2+16Rr-3r^2-4(R-2r)\sqrt{R^2-2Rr}$

$$\leqslant a^2+b^2+c^2$$

$$\leqslant 4R^2+16Rr-3r^2+4(R-2r)\sqrt{R^2-2Rr}.$$

W. J. Blundon, Canad. Math. Bull. 8 (1965), 615–626.

5.16 $36r^2 \leqslant bc+ca+ab \leqslant 9R^2.$

Equalities hold if and only if $a = b = c$.

PROOF. Multiplying the inequalities (see: **7.12**)

$$\frac{9r}{2F} \leqslant \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leqslant \frac{9R}{4F},$$

by $F = abc/(4R)$, we get

$$18Rr \leqslant bc+ca+ab \leqslant 9R^2.$$

By using $R \geqslant 2r$, inequalities (1) follow from (2).
F. Leuenberger, Elem. Math. 13 (1958), 121–126.

5.17 $4r(5R-r) \leqslant bc+ca+ab \leqslant 4(R+r)^2.$

Equalities hold if and only if the triangle is equilateral.
J. Steinig, Elem. Math. 18 (1963), 127–131.

5.18 $36r^2 \leqslant 4r(5R-r) \leqslant bc+ca+ab \leqslant 4(R+r)^2 \leqslant 9R^2.$

Equalities occur if and only if the triangle is equilateral
W. J. Blundon, Canad. Math. Bull. 8 (1965), 615–626.

REMARK. From 5.16, 5.17, and $R \geqslant 2r$ follows 5.18.

5.19 $$2R^2+14Rr-2(R-2r)\sqrt{R^2-2Rr} \leqslant bc+ca+ab \leqslant 2R^2+14Rr+2(R-2r)\sqrt{R^2-2Rr}.$$

Equalities hold if and only if the triangle is equilateral.
W. J. Blundon, Canad. Math. Bull. 8 (1965), 615–626.

5.20 $a(s-a)+b(s-b)+c(s-c) \leqslant 9Rr.$

S. G. Guba, Matematika v škole, 1966, No. 6, 67.

5.21 $abc \leqslant 8R^2r+(12\sqrt{3}-16)Rr^2.$

Equality holds if and only if the triangle is equilateral.
W. J. Blundon, Canad. Math. Bull. 8 (1965), 615–626.

5.22 $$\frac{\sqrt{3}}{R} \leqslant \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leqslant \frac{\sqrt{3}}{2r}.$$

F. Leuenberger, Elem. Math. 15 (1960), 77–79.

5.23 $$\frac{3\sqrt{3}}{2(R+r)} \leqslant \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leqslant \frac{\sqrt{3}}{2r}.$$

Equalities occur if and only if the triangle is equilateral.
J. Steinig, Elem. Math. 18 (1963), 127–131.

REMARK. This inequality improves 5.22.

5.24 $$\frac{1}{R^2} \leqslant \frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab} \leqslant \frac{1}{4r^2}. \quad (1)$$

PROOF. Since

$$\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{2s}{abc}=\frac{1}{2Rr},$$

inequalities (1) are equivalent to

$$\frac{1}{R^2} \leqslant \frac{1}{2Rr} \leqslant \frac{1}{4r^2}.$$

These inequalities are true because each of these can be reduced to $R \geqslant 2r$.

F. Leuenberger, Elem. Math. 13 (1958), 121–126.

5.25 $8r(R-2r) \leqslant Q \leqslant 8R(R-2r)$.

Equalities hold if and only if the triangle is equilateral.

J. C. H. Gerretsen, Nieuw Tijdschr. Wisk. 41 (1953), 1–7.

5.26 $4r^2 \leqslant \dfrac{abc}{a+b+c}$.

Equality occurs if and only if the triangle is equilateral.

M. S. Klamkin, Math. Teacher 60 (1967), 323–328.

5.27 $abc \leqslant (R\sqrt{3})^3$. (1)

Equality holds if and only if $a = b = c$.

PROOF. Since

$$\sqrt[3]{abc} \leqslant \frac{a+b+c}{3},$$

on the basis of **5.3**, we conclude that inequality (1) holds.

5.28 The inequality

$$M_k(a, b, c) \leqslant R\sqrt{3}$$

is valid for every triangle if and only if

$$k \leqslant \frac{\log 9 - \log 4}{\log 4 - \log 3}.$$

A. Makowski-J. Berkes, Elem. Math. 17 (1962), 40–41 and 18 (1963), 31–32.

5.29 $(a+b+c)\sqrt{3} \leqslant 2(r_a+r_b+r_c)$.

Equality occurs if and only if $a = b = c$.

J. C. H. Gerretsen, Nieuw Tijdschr. Wisk. 41 (1953), 1–7.

F. Leuenberger, Elem. Math. 16 (1961), 127–129.

5.30 $\frac{a^2}{r_b r_c} + \frac{b^2}{r_c r_a} + \frac{c^2}{r_a r_b} \geqslant 4.$

Equality holds if and only if the triangle is equilateral.
This inequality is due to R. R. Janić.

5.31 $(s-a)\sqrt{3} < 4R - r_a.$

E. Lemoine, Problem 578, J. Math. Elem. (4) 3 (1894), 263.

5.32 $(s-a)\left(\sqrt{3} + \frac{a^2+(b-c)^2}{2F}\right) \leqslant 4R - r_a.$ (1)

Equality holds if and only if $a = b = c$.

PROOF. From

$$4R+r = r_a+r_b+r_c, \quad r_a = \frac{rs}{s-a}, \quad r_b = \frac{rs}{s-b}, \quad r_c = \frac{rs}{s-c},$$

we get

$$\begin{aligned} 4R-r_a &= r_b+r_c-r \\ &= rs\left(\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}\right) \\ &= (s-a)\frac{s^2-bc}{F} \\ &= (s-a)\frac{a^2+b^2+c^2+2(-bc+ca+ab)}{4F}. \end{aligned}$$

On the basis of inequality **4.7**, we have

$$4R-r_a \geqslant (s-a)\frac{4F\sqrt{3}+2(a^2+b^2+c^2-2bc)}{4F},$$

i.e. (1).

Equality holds if and only if $a = b = c$, for in **4.7** we have equality only in that case.

R. R. Janić, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181–No. 196 (1967), 75–76.

5.33 $5R - r \geqslant s\sqrt{3}.$ (1)

Equalities hold if and only if the triangle is equilateral.

PROOF. Starting from

$$4R+r = r_a+r_b+r_c, \quad r_a = \frac{rs}{s-a}, \quad r_b = \frac{rs}{s-b}, \quad r_c = \frac{rs}{s-c}$$

and from $R \geqslant 2r$, we get

$$\begin{aligned} 5R-r &\geqslant r_a+r_b+r_c \\ &= rs\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right) \\ &= \frac{s}{F}(bc+ca+ab-s^2) \\ &= \frac{s}{4F}[2(bc+ca+ab)-(a^2+b^2+c^2)]. \end{aligned}$$

On the basis of 4.7, we have (1).

Equality holds only for $a = b = c$, because only in this case $R = 2r$ and equality in 4.7 holds.

R. R. Janić, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181–No. 196 (1967), 73–76.

5.34 $\quad s^2 \leqslant r_a^2+r_b^2+r_c^2.$ (1)

Equality occurs if and only if the triangle is equilateral.

PROOF. Since, by **2.35**,

$$\operatorname{tg}^2\frac{\alpha}{2}+\operatorname{tg}^2\frac{\beta}{2}+\operatorname{tg}^2\frac{\gamma}{2} \geqslant 1,$$

and

$$r_a = s\operatorname{tg}\frac{\alpha}{2}, \quad r_b = s\operatorname{tg}\frac{\beta}{2}, \quad r_c = s\operatorname{tg}\frac{\gamma}{2},$$

we obtain (1).

E. Bokov, Matematika v škole, 1956, No. 4, 95 and 1957, No. 1, 93.

5.35 $\quad 8r_ar_br_c \leqslant 3abc\sqrt{3}.$

Equality holds if and only if the triangle is equilateral.

F. Leuenberger, Elem. Math. 16 (1961), 127–129.

5.36 $54Rr \leqslant 3(bc+ca+ab) \leqslant 4(r_b r_c + r_c r_a + r_a r_b)$.

PROOF. The first of these inequalities was already proved in **5.16**. Since

$$4(r_b r_c + r_c r_a + r_a r_b) = (a+b+c)^2 \text{ and } 3(bc+ca+ab) \leqslant (a+b+c)^2,$$

we conclude that the second inequality also holds.

F. Leuenberger-J. Steinig, Elem. Math. 20 (1965), 89–90.

5.37 $32 r_a r_b r_c - 14r^2(r_a+r_b+r_c) \leqslant 9r(r_a+r_b+r_c)^2 + 9r^3$

holds for acute triangles.

Equality holds if and only if the triangle is equilateral.

S. Reich, Problem E 1930, Amer. Math. Monthly 73 (1966), 1017–1018.

5.38 $\frac{3}{2}R \leqslant \max(r_a, r_b, r_c)$.

P. Erdös, Mat. Lapok 1 (1949), 72.

5.39 If $r_a \leqslant r_b \leqslant r_c$, then

$$r_a \leqslant \tfrac{3}{2}R, \quad r_b < 2R, \quad \tfrac{3}{2}R \leqslant r_c < 4R.$$

H. J. Baron, Tôhoku Math. J. 48 (1941), 185–192.

5.40 $$\tfrac{4}{3}R < \frac{r_a+r_b+r_c}{3} \leqslant \tfrac{3}{2}R \leqslant \sqrt[4]{\frac{r_a^4+r_b^4+r_c^4}{3}}.$$

L. Fejes Tóth, Mat. Lapok 1 (1949), 72.

5.41 $9r \leqslant r_a+r_b+r_c \leqslant \frac{9}{2}R$.

M. S. Klamkin, Math. Teacher 60 (1967), 323–328.

5.42 If $G(x, y, z)$ denotes the geometric mean of x, y, z, then

$$G\left(r, r, \frac{R}{2}\right) \leqslant \tfrac{1}{3}G(r_a, r_b, r_c) \leqslant G\left(r, \frac{R}{2}, \frac{R}{2}\right).$$

Equalities hold if and only if the triangle is equilateral.

A. Makowski, Nieuw Arch. Wisk. (3) 12 (1964), 5–7.

5.43 $\frac{27}{4}R^2 \leqslant r_a^2+r_b^2+r_c^2$.

Equality holds if and only if the triangle is equilateral.

V. Thébault, Mat. Lapok, 3 (1952), 59–61.

A. Makowski, Nieuw Arch. Wisk. (3) 12 (1964), 5–7.

5.44 $\frac{3}{2}R \leqslant M_k(r_a, r_b, r_c) \quad (k \geqslant 2)$.

G. Hajós, Math. Lapok 1 (1950), 313.

L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, Berlin 1953, p. 27.

REMARK. This inequality is a generalization of inequality **5.40**.

5.45 $7R^2 \leqslant r^2+r_a^2+r_b^2+r_c^2$. (1)

PROOF. Since

$$r^2+r_a^2+r_b^2+r_c^2 = 16R^2-(a^2+b^2+c^2),$$

and on the basis of **5.13**, we have (1).

Ju. I. Gerasimov, Matematika v škole, 1967, No. 3, 84.

Comment by R. R. Janić. The following stronger inequality holds

$$8R^2-4r^2 \leqslant r^2+r_a^2+r_b^2+r_c^2, \tag{2}$$

because (see: **5.14**) $a^2+b^2+c^2 \leqslant 8R^2+4r^2$.

Inequality (2) is stronger than the inequalities **5.40** and **5.43**.

5.46 $$\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c} \leqslant \frac{abc}{r}. \tag{1}$$

Equality holds if and only if $a = b = c$.

J. Andersson, Problem 1779, Amer. Math. Monthly 59 (1952), 41.

REMARK. This inequality is equivalent to **1.9**.

5.47 $$\sqrt{\frac{a}{r_a}}+\sqrt{\frac{b}{r_b}}+\sqrt{\frac{c}{r_c}} \leqslant \frac{3}{2}\sqrt{\frac{s}{r}}. \tag{1}$$

PROOF. Since $r_a = rs/(s-a)$, etc., inequality (1) is equivalent to

$$\sqrt{a(s-a)}+\sqrt{b(s-b)}+\sqrt{c(s-c)} \leqslant \tfrac{3}{2}s.$$

By the arithmetic-geometric mean inequality, we get

$$\sqrt{a(s-a)}+\sqrt{b(s-b)}+\sqrt{c(s-c)}$$
$$\leqslant \frac{a+s-a}{2}+\frac{b+s-b}{2}+\frac{c+s-c}{2}=\tfrac{3}{2}s.$$

Gaz. Mat. B 15 (1964), 256.

6. Inequalities for the sides, the altitudes and the radii of a triangle

6.1 $2(h_a+h_b+h_c)\leqslant\sqrt{3}(a+b+c)$.

Equality holds if and only if the triangle is equilateral.

PROOF. First we have

$$(a+b+c)^2\geqslant 3(bc+ca+ab)=\frac{3abc}{2F}(h_a+h_b+h_c)=6R(h_a+h_b+h_c),$$

with equality only in the case of an equilateral triangle.

Since (see: 5.3) $3R\sqrt{3}\geqslant a+b+c$, with equality if and only if the triangle is equilateral, we obtain

$$\sqrt{3}(a+b+c)^2\geqslant 2(a+b+c)(h_a+h_b+h_c),$$

so that

$$\sqrt{3}(a+b+c)\geqslant 2(h_a+h_b+h_c),$$

with equality only in the case of an equilateral triangle.

A. Santaló, Math. Notae 3 (1943), 65–73.

F. Leuenberger-L. Carlitz, Problem E 1427, Amer. Math. Monthly 67 (1960), 692 and 68 (1961), 296–297.

6.2 $h_a+h_b+h_c\leqslant\sqrt{s}(\sqrt{s-a}+\sqrt{s-b}+\sqrt{s-c})\leqslant s\sqrt{3}$.

Equalities occur if and only if the triangle is equilateral.

L. Santaló, Math. Notae 3 (1943), 65–73.

6.3 $M_2(h_a, h_b, h_c)\leqslant\frac{\sqrt{3}}{2}M_2(a, b, c)$ and

$$M_{-2}(h_a, h_b, h_c)\leqslant\frac{\sqrt{3}}{2}M_{-2}(a, b, c).$$

Equality holds if and only if $a = b = c$.

PROOF. By means of the obvious inequality

$$h_a^2+h_b^2+h_c^2 \leqslant m_a^2+m_b^2+m_c^2$$

and by

$$m_a^2+m_b^2+m_c^2 = \tfrac{3}{4}(a^2+b^2+c^2)$$

one gets the first inequality.

The first and the second inequality are, however, equivalent (see: **6.6**).

L. A. Santaló, Math. Notae 3 (1943), 65–73.

J. Berkes, Elem. Math. 18 (1963), 31–32.

6.4 $3(bc+ca+ab) \geqslant 4(h_bh_c+h_ch_a+h_ah_b)$.

Equality holds if and only if the triangle is equilateral.
Math. Notae 3 (1943), 182.

6.5 $a^3+b^3+c^3 > \frac{8}{7}(h_a^3+h_b^3+h_c^3)$.

Gaz. Mat. B 9 (1958), 731.

6.6 $M_k(h_a, h_b, h_c) \leqslant \dfrac{\sqrt{3}}{2} M_k(a, b, c)$

$$\Leftrightarrow M_{-k}(h_a, h_b, h_c) \leqslant \frac{\sqrt{3}}{2} M_{-k}(a, b, c).$$

These inequalities hold for every number k if and only if $\beta < \pi/3$ and $\alpha \leqslant \beta \leqslant \gamma$.

A. Makowski, Elem. Math. 17 (1962), 40–41.

J. Steinig, Elem. Math. 20 (1965), 64–65.

6.7 $\dfrac{a^2}{h_b^2+h_c^2} + \dfrac{b^2}{h_c^2+h_a^2} + \dfrac{c^2}{h_a^2+h_b^2} \geqslant 2$.

V. O. Gordon, Matematika v škole, 1967, No. 3, 84.

6.8 $h_a+h_b+h_c \geqslant 9r$. (1)

Equality holds if and only if the triangle is equilateral.

By the arithmetic-geometric mean inequality, we get

$$\sqrt{a(s-a)}+\sqrt{b(s-b)}+\sqrt{c(s-c)}$$
$$\leqslant \frac{a+s-a}{2}+\frac{b+s-b}{2}+\frac{c+s-c}{2}=\tfrac{3}{2}s.$$

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F. Leuenberger-L. Carlitz, Problem E 1427, Amer. Math. Monthly 67 (1960), 692 and 68 (1961), 296–297.

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$$h_a^2+h_b^2+h_c^2 \leqslant m_a^2+m_b^2+m_c^2$$

and by

$$m_a^2+m_b^2+m_c^2 = \tfrac{3}{4}(a^2+b^2+c^2)$$

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A. Makowski, Elem. Math. 17 (1962), 40–41.

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6.7 $\dfrac{a^2}{h_b^2+h_c^2} + \dfrac{b^2}{h_c^2+h_a^2} + \dfrac{c^2}{h_a^2+h_b^2} \geqslant 2$.

V. O. Gordon, Matematika v škole, 1967. No. 3, 84.

6.8 $h_a+h_b+h_c \geqslant 9r$. (1)

Equality holds if and only if the triangle is equilateral.

PROOF. By

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$$

and by

$$\frac{h_a+h_b+h_c}{3}\geqslant\frac{3}{1/h_a+1/h_b+1/h_c},$$

we conclude that (1) is true.

From the proof we deduce that equality in (1) holds if and only if $h_a=h_b=h_c$, i.e. if and only if the triangle is equilateral.

S. I. Zetel', Zadači na maksimum i minimum, Moskva 1948, p. 64.

6.9 If $a\leqslant b\leqslant c$, then

$$h_a+h_b+h_c\leqslant\frac{3b(a^2+ac+c^2)}{4Rs}.$$

Equality holds if and only if the triangle is equilateral.

Ž. Živanović, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181–No. 196 (1967), 69–72.

6.10 $9r\leqslant 3(h_ah_bh_c)^{1/3}\leqslant h_a+h_b+h_c$

$$\leqslant 3r(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)^{1/2}\leqslant\tfrac{9}{2}R.$$

J. Berkes, Elem. Math. 14 (1959), 62–64.

6.11 $h_a+h_b+h_c\leqslant 3(R+r)$.

Equality holds if and only if the triangle is equilateral.

L. Carlitz, Problem E 1616, Amer. Math. Monthly 70 (1963), 758.

F. Leuenberger, Elem. Math. 19 (1964), 132–133.

6.12 $h_a+h_b+h_c\leqslant 2R+5r$.

F. Leuenberger, Elem. Math. 19 (1964), 132–133.

L. Bankoff-D. Woods, Math. Mag. 38 (1965), 240 and 39 (1966), 130.

6.13 $\frac{2r(5R-r)}{R} \leqslant h_a+h_b+h_c \leqslant \frac{2(R+r)^2}{R}.$

Equalities occur if and only if the triangle is equilateral.

J. Steinig, Elem. Math. 18 (1963), 127–131.

6.14 $h_a+h_b+h_c \leqslant w_a+w_b+w_c \leqslant 3(R+r) \leqslant r_a+r_b+r_c.$

L. Carlitz, Problem E 1616, Amer. Math. Monthly 70 (1963), 758 and 71 (1964), 558–559.

F. Leuenberger, Elem. Math. 18 (1963), 35–36.

F. Leuenberger, Elem. Math. 19 (1964), 132–133.

6.15 $2(h_bh_c+h_ch_a+h_ah_b) \leqslant 6F\sqrt{3} \leqslant 27Rr.$ (1)

Proof. Since

$$F = 4Rr \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2},$$

from

$$abc = 16R^2r \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2},$$

on the basis of 2.28, we get

$$abc \leqslant 6R^2r\sqrt{3}, \text{ i.e., } 2F \leqslant 3Rr\sqrt{3}. \qquad (2)$$

This proves the second inequality in (1).

If the equality $h_ah_bh_c = 2F^2/R$ is multiplied by $1/h_a$, $1/h_b$, $1/h_c$, respectively, and if the given values are added, we obtain

$$h_bh_c+h_ch_a+h_ah_b = \frac{2F^2}{Rr}.$$

By virtue of the last equality and of inequality (2), we conclude that the first inequality in (1) is true.

F. Leuenberger and J. Steinig, Elem. Math. 20 (1965), 89–90.

6.16 $h_ah_bh_c \geqslant 27r^3.$

Proof. Since $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$, the product $\frac{1}{h_a}\frac{1}{h_b}\frac{1}{h_c}$ has the maximum value if $\frac{1}{h_a}=\frac{1}{h_b}=\frac{1}{h_c}=\frac{1}{3r}$, whence it follows that the product $h_a h_b h_c$ attains the minimum value $27r^3$ if $h_a = h_b = h_c$.

Therefore $h_a h_b h_c \geqslant 27r^3$.

S. I. Zetel', Zadači na maksimum i minimum, Moskva 1948, p. 100.

6.17 $\sqrt{h_a} + \sqrt{h_b} + \sqrt{h_c} \leqslant \frac{3}{2}\sqrt{6R}$.

Equality holds if and only if the triangle is equilateral.

E. A. Bokov, Matematika v škole, 1967, No. 1, 83.

6.18 $h_b h_c + h_c h_a + h_a h_b \leqslant r_b r_c + r_c r_a + r_a r_b$. (1)

PROOF. From

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$$

we get

$$h_b h_c + h_c h_a + h_a h_b = \frac{h_a h_b h_c}{r_a r_b r_c} (r_b r_c + r_c r_a + r_a r_b). \tag{2}$$

Since (see: **6.27**)

$$h_a h_b h_c \leqslant r_a r_b r_c,$$

from (2) follows (1).

A. Makowski, Problem E 1675, Amer. Math. Monthly 71 (1964), 317 and 72 (1965), 187–188.

6.19 If $-1 < t < 0$, then

$$h_a^t + h_b^t + h_c^t \geqslant r_a^t + r_b^t + r_c^t. \tag{1}$$

If $t > 0$ or $t < -1$, then

$$h_a^t + h_b^t + h_c^t \leqslant r_a^t + r_b^t + r_c^t. \tag{2}$$

PROOF. Since $h_a = \dfrac{2F}{a}$, $r_a = \dfrac{2F}{b+c-a}$, etc., inequality (1) is equivalent to

$$a^{-t} + b^{-t} + c^{-t} \geqslant (b+c-a)^{-t} + (c+a-b)^{-t} + (a+b-c)^{-t} \tag{3}$$

for $-1 < t < 0$.

Since the function $f(x) = -x^{-t}$, for $x > 0$ and $-1 < t < 0$, is convex, Jensen's inequality can be applied, viz.

$$\frac{f(x_1)+f(x_2)}{2} \geqslant f\left(\frac{x_1+x_2}{2}\right) \tag{4}$$

for x_1 and $x_2 > 0$.

For $x_1 = c+a-b$, $x_2 = a+b-c$, inequality (4) becomes

$$(c+a-b)^{-t}+(a+b-c)^{-t} \leqslant 2a^{-t}. \tag{5}$$

Permuting a, b, c in (5) cyclically and adding the inequalities obtained we get inequality (3), i.e., (1).

Inequality (2) is equivalent to

$$a^{-t}+b^{-t}+c^{-t} \leqslant (b+c-a)^{-t}+(c+a-b)^{-t}+(a+b-c)^{-t},$$

for $t > 0$ or $t < -1$.

The proof of this inequality is analogous to the proof of (3).

A. Makowski, Elem. Math. 16 (1961), 60–61.

6.20 If x, y, z are non-negative real numbers, then

$$h_a^x h_b^y h_c^z + h_b^x h_c^y h_a^z + h_c^x h_a^y h_b^z \leqslant r_a^x r_b^y r_c^z + r_b^x r_c^y r_a^z + r_c^x r_a^y r_b^z.$$

Equality holds if and only if the triangle is equilateral.

J. I. Nassar, Problem E 1847, Amer. Math. Monthly 73 (1966), 82.

6.21 $$\frac{1}{h_a-2r}+\frac{1}{h_b-2r}+\frac{1}{h_c-2r} \geqslant \frac{3}{r}. \tag{1}$$

Proof. By

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$$

we obtain

$$\frac{h_a-2r}{h_a}+\frac{h_b-2r}{h_b}+\frac{h_c-2r}{h_c}=1.$$

From the last equality and from

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geqslant 9,$$

we have

$$\frac{h_a}{h_a-2r}+\frac{h_b}{h_b-2r}+\frac{h_c}{h_c-2r}\geqslant 9.$$

Since

$$\frac{2r}{h_a-2r}+\frac{2r}{h_b-2r}+\frac{2r}{h_c-2r}$$

$$=\frac{h_a-(h_a-2r)}{h_a-2r}+\frac{h_b-(h_b-2r)}{h_b-2r}+\frac{h_c-(h_c-2r)}{h_c-2r}\geqslant 9-3=6,$$

we obtain (1).

E. A. Bokov, Matematika v škole, 1966, No. 4, 77.

6.22 $$\frac{h_a+r}{h_a-r}+\frac{h_b+r}{h_b-r}+\frac{h_c+r}{h_c-r}\geqslant 6.$$

C. Cosnita and F. Turtoiu, Culegere de probleme de algebra, Bucuresti 1965, p. 17.

6.23 If $k>0$, then

$$M_k\left(\frac{h_a-r}{h_a+r_a},\ \frac{h_b-r}{h_b+r_b},\ \frac{h_c-r}{h_c+r_c}\right)\geqslant\sqrt[3]{\frac{4r^2}{27R^2}}.$$

If $k<0$, then

$$M_k\left(\frac{h_a-r}{h_a+r_c},\ \frac{h_b-r}{h_b+r_b},\ \frac{h_c-r}{h_c+r_c}\right)\leqslant\sqrt[3]{\frac{2r}{27R}}.$$

If $k=0$, then

$$\sqrt[3]{\frac{4r^2}{27R^2}}\leqslant M_k\left(\frac{h_a-r}{h_a+r_a},\ \frac{h_b-r}{h_b+r_b},\ \frac{h_c-r}{h_c+r_c}\right)\leqslant\sqrt[3]{\frac{2r}{27R}}.$$

Equalities hold if and only if the triangle is equilateral.

This result is due to R. Ž. Djordjević.

6.24 $$\frac{h_a-r_a}{h_a+r_a}+\frac{h_b-r_b}{h_b+r_b}+\frac{h_c-r_c}{h_c+r_c}\leqslant 0.$$

C. Cosnita and F. Turtoiu, Culegere de probleme de algebra, Bucuresti 1965, p. 17.

6.25 If $a \neq b \neq c \neq a$, then

$$M_k\left(\frac{h_a-r_b}{r_c-h_a},\ \frac{h_b-r_c}{r_a-h_b},\ \frac{h_c-r_a}{r_b-h_c}\right) > 1,$$

for $k > 0$,

$$M_k\left(\frac{h_a-r_b}{r_c-h_a},\ \frac{h_b-r_c}{r_a-h_b},\ \frac{h_c-r_a}{r_b-h_c}\right) < 1,$$

for $k < 0$,

$$M_k\left(\frac{h_a-r_b}{r_c-h_a},\ \frac{h_b-r_c}{r_a-h_b},\ \frac{h_c-r_a}{r_b-h_c}\right) = 1,$$

for $k = 0$.

This result is due to R. Ž. Djordjević.

6.26 $$\frac{r_a}{h_b+h_c} - \frac{r_b}{h_c+h_a} + \frac{r_c}{h_a+h_b} \geqslant \frac{3}{2}.$$

This inequality is due to R. R. Janić.

6.27 $$(h_a h_b h_c)^{1/3} \leqslant 3^{1/4} F^{1/2} \leqslant (r_a r_b r_c)^{1/3}. \tag{1}$$

Equalities hold if and only if the triangle is equilateral.

PROOF. On the basis of **7.9**,

$$2R^{-1} \leqslant 3^{3/4} F^{-1/2} \leqslant r^{-1}. \tag{2}$$

Commencing with

$$8F^3 = h_a h_b h_c abc = h_a h_b h_c . 4RF, \ r_a r_b r_c = F^2 r^{-1},$$

we obtain (1) from (2).

A. Makowski, Elem. Math. 16 (1961), 134.

6.28 $$\frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} \geqslant 3.$$

Equality occurs if and only if the triangle is equilateral.

PROOF. From

$$\frac{2}{h_a} = \frac{1}{r_b} + \frac{1}{r_c},$$

we get

$$\frac{r_a}{h_a} = \frac{1}{2}\left(\frac{r_a}{r_b} + \frac{r_a}{r_c}\right).$$

Similarly

$$\frac{r_b}{h_b} = \frac{1}{2}\left(\frac{r_b}{r_c} + \frac{r_b}{r_a}\right) \text{ and } \frac{r_c}{h_c} = \frac{1}{2}\left(\frac{r_c}{r_a} + \frac{r_c}{r_b}\right).$$

Therefore

$$\frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c} = \frac{1}{2}\left(\frac{r_a}{r_b}+\frac{r_b}{r_a}+\frac{r_b}{r_c}+\frac{r_c}{r_b}+\frac{r_c}{r_a}+\frac{r_a}{r_c}\right) \geqslant \tfrac{1}{2}\cdot 6 = 3.$$

Equality holds if and only if $r_a = r_b = r_c$, i.e. if and only if the triangle is equilateral.

H. Demir-D. C. B. Marsh, Problem E 1779, Amer. Math. Monthly 72 (1965), 420 and 73 (1966), 668.

REMARK 1. H. Guggenheimer (see: Amer. Math. Monthly 73 (1966), 668) proved the generalized inequality

$$\left(\frac{r_a}{h_a}\right)^n + \left(\frac{r_b}{h_b}\right)^n + \left(\frac{r_c}{h_c}\right)^n \geqslant 3,$$

for $n \geqslant 1$.

REMARK 2. T. Figiel (see: Amer. Math. Monthly 73 (1966), 668) observed that the inequality required is equivalent to the fact that the area of an orthic triangle (i.e. the pedal triangle for the orthocentre) is not greater than one-quarter of the area of a given acute-angled triangle.

6.29 $$\frac{3}{h_a^{-1}+h_b^{-1}+h_c^{-1}} = \frac{3}{r_a^{-1}+r_b^{-1}+r_c^{-1}} \leqslant \sqrt[4]{3F^2}.$$ (I)

Equality holds if and only if the triangle is equilateral.

PROOF. Starting from

$$h_a^{-1}+h_b^{-1}+h_c^{-1} = r_a^{-1}+r_b^{-1}+r_c^{-1} = r^{-1}$$

and using the second of inequalities in 7.9, we obtain (1).

A. Makowski, Elem. Math. 16 (1961), 134.

7. Inequalities for the sides, the area and the radii of a triangle

7.1 $F\sqrt{3} \leqslant (R+r)^2$.

Equality holds if and only if the triangle is equilateral.

PROOF. Since $a+b+c \leqslant 3R\sqrt{3}$ (see: **5.3**) by means of the arithmetic-geometric inequality, we get

$$R+r \geqslant \frac{1}{3}\left(\frac{s}{\sqrt{3}} + \frac{s}{\sqrt{3}} + 3r\right) \geqslant \sqrt[3]{s^2 r} = \sqrt[3]{sF}. \qquad (1)$$

By means of $s^2 \geqslant 3F\sqrt{3}$, following **4.2**, from (1) follows

$$(R+r)^2 \geqslant \sqrt[3]{s^2F^2} \geqslant \sqrt[3]{3F^3\sqrt{3}} = F\sqrt{3}.$$

L. Carlitz-F. Leuenberger, Problem E 1454, Amer. Math. Monthly 68 (1961), 177 and 68 (1961), 805–806.

H. W. Guggenheimer, Amer. Math. Monthly 72 (1965), 791–793.

7.2 $F\sqrt{3} \leqslant r(4R+r)$. (1)

Equality occurs if and only if the triangle is equilateral.

PROOF. Since (see: **5.8**)

$$F^2 \leqslant r^2(4R^2+4Rr+3r^2) = r^2[(4R^2+\tfrac{8}{3}Rr+\tfrac{1}{3}r^2)+\tfrac{1}{3}(4Rr+8r^2)],$$

using inequalities $4r^2 \leqslant 2Rr \leqslant R^2$, we obtain

$$F^2 \leqslant r^2[(4R^2+\tfrac{8}{3}Rr+\tfrac{1}{3}r^2)+\tfrac{4}{3}R^2] = \tfrac{1}{3}r^2(4R+r)^2,$$

i.e. (1).

J. C. H. Gerretsen, Nieuw Tijdschr. Wisk. 41 (1953), 1–7.

J. Steinig, Elem. Math. 18 (1963), 127–131.

7.3 $s^2 \leqslant 4R^2 - \dfrac{11}{3\sqrt{3}}F$.

Equality holds if and only if the triangle is equilateral.

S. Nakajima, Tôhoku Math. J. 25 (1925), 115–121.

PROOF. The sides a, b, c of a triangle are the roots of the equation

$$z^3 - \frac{2F}{r} z^2 + \left(\frac{F^2}{r^2} + 4Rr + r^2\right) z - 4FR = 0, \tag{2}$$

where F is the area of given triangle.

On the other hand, $b+c-a$, $c+a-b$ and $a+b-c$ are the roots of

$$u^3 - \frac{2F}{r} u^2 + 4r(4R+r)u - 8Fr = 0. \tag{3}$$

The condition that equation (2) and (3) have only real roots, leads to $R \geqslant 2r$ and inequality (1).

Ramus-É. Rouché, Nouv. Ann. Math. 10 (1851), 353.

REMARK 1. See: A. Laisant. Géométrie du triangle, Paris 1896, p. 112.

REMARK 2. See also **5.10**.

7.12 $$\frac{9r}{2F} \leqslant \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leqslant \frac{9R}{4F}. \tag{1}$$

Equalities occur if and only if the triangle is equilateral.

PROOF. From

$$h_a + h_b + h_c = 2F\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

and

$$9r \leqslant h_a + h_b + h_c \leqslant \frac{9R}{2},$$

on the basis of **6.8** and **6.10**, we can immediately conclude that inequalities (1) hold.

Equalities in (1) hold if and only if $a = b = c$.

F. Leuenberger, Elem. Math. 13 (1958), 121–126.

7.13 $F\sqrt{3} \leqslant 4Rr + r^2 \leqslant F\sqrt{3} + \frac{1}{2}Q.$

Equalities occur if and only if the triangle is equilateral.

J. C. H. Gerretsen. Nieuw Tijdschr. Wisk. 41 (1953), 1–7.

8. Inequalities for the medians, the angle-bisectors and other elements of a triangle

8.1 $\frac{3}{2}s < m_a+m_b+m_c < 2s.$

Proof. Let M denote the midpoint of BC and let A' be the point symmetrical to A with respect to M. Then

$$AA' = 2m_a < AB+BA' = b+c,$$

i.e.

$$2m_a < b+c.$$

Similarly,

$$2m_b < c+a, \; 2m_c < a+b.$$

By addition one obtains

$$m_a+m_b+m_c < 2s. \tag{1}$$

Let C'' be the intersection of BC and of the straight line passing through A and parallel to the median BN.

The sides of the triangle $AA'C''$ are $2m_a$, $2m_b$, $2m_c$, and its medians are $\frac{3}{2}c$, $\frac{3}{2}b$, $\frac{3}{2}a$.

By means of (1) we get

$$m_a+m_b+m_c > \frac{3}{2}s.$$

C. Sebastiano, Boll. Mat. (Firenze), 1938, 59.

L. A. Santaló, Math. Notae 3 (1943), 65–73.

8.2 $m_a+m_b+m_c \leqslant 4R+r.$ (1)

Proof. Let p_a, p_b, p_c be the distances of the circumcentre from the sides BC, CA, AB respectively. Then

$$m_a \leqslant R+p_a, \; m_b \leqslant R+p_b, \; m_c \leqslant R+p_c.$$

By adding these inequalities, we get

$$m_a+m_b+m_c \leqslant 3R+p_a+p_b+p_c. \tag{2}$$

Since

$$ap_a = R^2 \sin 2\alpha, \; bp_b = R^2 \sin 2\beta, \; cp_c = R^2 \sin 2\gamma,$$

we have

$$p_a+p_b+p_c=R(\cos\alpha+\cos\beta+\cos\gamma).$$

Since

$$\cos\alpha+\cos\beta+\cos\gamma=1+\frac{r}{R},$$

inequality (2) becomes (1).

F. Leuenberger, Elem. Math. 13 (1958), 121–126.

8.3 $9r\leqslant m_a+m_b+m_c\leqslant \frac{9}{2}R.$

E. G. Gotman, Matematika v škole, 1966, No. 5, 76.

REMARK. Second inequality is improved by **8.2**.

8.4 $\dfrac{\sqrt{3}}{27}(m_a+m_b+m_c)^2\geqslant F.$

Equality holds if and only if the triangle is equilateral.

L. A. Santaló, Math. Notae 3 (1943), 65–73.

8.5 $bc-\dfrac{a^2}{4}\leqslant m_a{}^2\leqslant bc+\dfrac{a^2}{4},\quad ca-\dfrac{b^2}{4}\leqslant m_b{}^2\leqslant ca+\dfrac{b^2}{4},$

$$ab-\frac{c^2}{4}\leqslant m_c{}^2\leqslant ab+\frac{c^2}{4}.$$

PROOF. Since

$$m_a{}^2=\frac{b^2+c^2}{2}-\frac{a^2}{4},$$

and by $b^2+c^2\geqslant 2bc$, we get

$$m_a{}^2\geqslant bc-\frac{a^2}{4}.$$

Equality (2) can be put in the form

$$m_a{}^2=\frac{b^2+c^2-a^2}{2}+\frac{a^2}{4},$$

whence, by

$$bc\geqslant\frac{b^2+c^2-a^2}{2}$$

we conclude that

$$m_a^2 \leqslant bc + \frac{a^2}{4}.$$

Analogously, one also proves the other inequalities of (1).
Gaz. Mat. B 7 (1956), 431.

8.6 $m_a^2 + m_b^2 + m_c^2 \geqslant 3F\sqrt{3}.$

Equality holds if and only if the triangle is equilateral.
L. A. Santaló, Math. Notae 3 (1943), 65–73.

REMARK. As

$$m_a^2 + m_b^2 + m_c^2 = \tfrac{3}{4}(a^2 + b^2 + c^2),$$

the inequality follows from 4.4.

8.7 If $\alpha < \gamma$, then $w_a > w_c$.

PROOF. One has

$$w_a^2 = \frac{4bcs(s-a)}{(b+c)^2}, \quad w_c^2 = \frac{4abs(s-c)}{(a+b)^2}$$

and therefore

$$\frac{(b+c)^2(a+b)^2}{2bs}(w_a^2 - w_c^2) = c(b+c-a)(a+b)^2 - a(a+b-c)(b+c)^2$$
$$= b^3(c-a) + b^2(c^2-a^2) + 3abc(c-a) + ca(c^2-a^2).$$

Hence $c-a > 0$ implies $w_a < w_c$.
This inequality is due to S. Moiseev and the proof to O. Bottema.

S. Moiseev, Matematika v škole, 1952, No. 5, 92 and 1953, No. 2, 89–90.

8.8 $w_a \leqslant \sqrt{s(s-a)} < \dfrac{2bc}{b+c} \leqslant \sqrt{bc}.$

H. W. Guggenheimer, Plane Geometry and its Groups, San Francisco, Cambridge, London, Amsterdam 1967, p. 178.

8.9 $s < w_a + w_b + w_c \leqslant \sqrt{s}(\sqrt{s-a} + \sqrt{s-b} + \sqrt{s-c}) \leqslant s\sqrt{3}.$

L. A. Santaló, Math. Notae 3 (1943), 65–73.

8.10 $w_a^2+w_b^2+w_c^2 \geqslant 3F\sqrt{3}.$ (1)

PROOF. Since

$$w_a^2 = bc - \frac{a^2bc}{(b+c)^2} \quad \text{and} \quad \frac{bc}{(b+c)^2} \leqslant \tfrac{1}{4},$$

we have

$$\sum w_a^2 \geqslant \sum ab - \tfrac{1}{4}\sum a^2. \qquad (2)$$

On the other hand, by means of **4.7**

$$\tfrac{1}{4}\sum a^2 \leqslant \tfrac{1}{2}\sum ab - F\sqrt{3},$$

so that (2) becomes

$$\sum w_a^2 \geqslant \tfrac{1}{2}\sum ab + F\sqrt{3}, \qquad (3)$$

and as $\sum ab \geqslant 4F\sqrt{3}$ (see: **4.5**) from (3) we get (1).

V. O. Gordon, Matematika v škole, 1966, No. 5, 76.

8.11 $16(w_a^4+w_b^4+w_c^4) \leqslant 9(a^4+b^4+c^4).$

J. Berkes, Elem. Math. 19 (1964), 138–139.

8.12 If λ is a real number, then

$$a^\lambda w_a + b^\lambda w_b + c^\lambda w_c \leqslant \sqrt{\tfrac{3}{2}abcs(a^{2(\lambda-1)}+b^{2(\lambda-1)}+c^{2(\lambda-1)})}.$$

Equality holds if and only if the triangle is equilateral.

This result is due to R. Ž. Djordjević.

8.13 If λ is a real number, then

$$a^\lambda w_a^2 + b^\lambda w_b^2 + c^\lambda w_c^2 \leqslant \tfrac{1}{2}abcs(a^{\lambda-2}+b^{\lambda-2}+c^{\lambda-2}).$$

Equality holds if and only if the triangle is equilateral.

This inequality is due to R. Ž. Djordjević.

8.14 $w_a w_b w_c \leqslant rs^2.$

Equality holds if and only if the triangle is equilateral.

L. Carlitz, Problem E 1628, Amer. Math. Monthly 70 (1963), 891 and 71 (1964), 687.

8.15 $(w_b w_c)^2+(w_c w_a)^2+(w_a w_b)^2 \leqslant rs^2(4R-r).$

Equality holds if and only if the triangle is equilateral.

L. Carlitz, Problem E 1628, Amer. Math. Monthly 70 (1963), 891 and 71 (1964), 687.

8.16 $w_b w_c + w_c w_a + w_a w_b \leq r_b r_c + r_c r_a + r_a r_b.$

PROOF. Since (see: 8.8),

$$w_a \leq \sqrt{s(s-a)}, \quad w_b \leq \sqrt{s(s-b)},$$

we have

$$w_a w_b \leq s\sqrt{(s-a)(s-b)} \leq s \cdot \frac{c}{2}.$$

Adding $w_a w_b \leq \frac{1}{2} cs$ and the two other inequalities for $w_b w_c$ and $w_c w_a$, we obtain

$$w_b w_c + w_c w_a + w_a w_b \leq s^2. \tag{1}$$

On the other hand, from $r_a = F/(s-a)$ etc., we get

$$r_b r_c + r_c r_a + r_a r_b = s^2. \tag{2}$$

From (1) and (2) follows the inequality which was to be proved.

A. Makowski and J. Schopp, Problem E 1675, Amer. Math. Monthly 72 (1965), 187–188.

REMARK. The following stronger inequality exists

$$w_a^2 + w_b^2 + w_c^2 \leq r_b r_c + r_c r_a + r_a r_b.$$

8.17 $r_a w_a + r_b w_b + r_c w_c$

$$\leq s\left(\sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)} + \sqrt{(s-a)(s-b)}\right)$$

$$\leq s^2$$

$$\leq F\sqrt{1 - \frac{38}{27}\left(\frac{s}{r}\right)^2 + \left(\frac{8}{27}\right)^2\left(\frac{s}{r}\right)^4}.$$

Equalities hold if and only if the triangle is equilateral.

PROOF. From the formulas

$$r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}} \quad \text{and} \quad w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)}, \text{ etc.,}$$

by applying the arithmetic-geometric mean inequality twice we obtain

$$r_a w_a + r_b w_b + r_c w_c$$
$$\leqslant s[\sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)} + \sqrt{(s-a)(s-b)}] \leqslant s^2.$$

Equalities hold if and only if the triangle is equilateral.

It remains to be proved that

$$s^2 \leqslant F\sqrt{1 - \frac{38}{27}\left(\frac{s}{r}\right)^2 + \left(\frac{8}{27}\right)^2\left(\frac{s}{r}\right)^4},$$

or equivalently $(x-27)(64x-27) \geqslant 0$, with $x = (s/r)^2$. This follows immediately from the inequality $(s/r)^2 \geqslant 27$ (see: **5.6**), in which equality holds if and only if the triangle is equilateral.

H. Guggenheimer-J. Steinig, Problem 520, Elem. Math. 21 (1966), 20.

8.18 $\left(\frac{r_a}{w_a}\right)^\lambda + \left(\frac{r_b}{w_b}\right)^\lambda + \left(\frac{r_c}{w_c}\right)^\lambda \geqslant 3,$

for $\lambda > 0$.

Equality holds if and only if the triangle is equilateral.

Proof. Since

$$r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}} \text{ and } w_a \leqslant \sqrt{s(s-a)}, \text{ etc.,}$$

we have

$$\left(\frac{r_a}{w_a}\right)^\lambda + \left(\frac{r_b}{w_b}\right)^\lambda + \left(\frac{r_c}{w_c}\right)^\lambda$$
$$= \frac{\sqrt{(s-b)^\lambda(s-c)^\lambda}}{(s-a)^\lambda} + \frac{\sqrt{(s-c)^\lambda(s-a)^\lambda}}{(s-b)^\lambda} + \frac{\sqrt{(s-a)^\lambda(s-b)^\lambda}}{(s-c)^\lambda}.$$

The result desired now follows from the geometric-arithmetic mean inequality.

H. Guggenheimer, Amer. Math. Monthly 73 (1966), 668.

8.19 $w_a^2 + w_b^2 + w_c^2 \leqslant s^2 \leqslant m_a^2 + m_b^2 + m_c^2.$

Equalities occur if and only if the triangle is equilateral.

PROOF. Starting with

$$w_a^2 \leqslant s(s-a), \quad w_b^2 \leqslant s(s-b), \quad w_c^2 \leqslant s(s-c),$$

we have

$$w_a^2+w_b^2+w_c^2 \leqslant s(3s-a-b-c) = s^2,$$

with equality if and only if the triangle is equilateral.

The inequality

$$(b-c)^2+(c-a)^2+(a-b)^2 \geqslant 0,$$

yields

$$s^2 \leqslant \tfrac{3}{4}(a^2+b^2+c^2).$$

Since

$$\frac{b^2+c^2}{2}-\frac{a^2}{4} = m_a^2, \text{ etc.,}$$

inequality (1) becomes

$$s^2 \leqslant m_a^2+m_b^2+m_c^2.$$

F. Ryžkov, Matematika v škole, 1939, No. 1, 80; No. 4,
Z. A. Skopec, Matematika v škole, 1963, No. 3, 89.
L. Carlitz-S. Philipp, Problem E 1628, Amer. Math. Mont 70 (1963), 891 and 71 (1964), 687.
F. Leuenberger, Elem. Math. 18 (1963), 35–36.

8.20 $w_a+w_b+w_c \leqslant m_a+m_b+m_c \leqslant r_a+r_b+r_c.$

Equalities hold if and only if the triangle is equilateral.
F. Leuenberger, Elem. Math. 16 (1961), 127–129.

8.21 $w_a w_b w_c \leqslant r_a r_b r_c \leqslant m_a m_b m_c.$

Equalities occur if and only if the triangle is equilateral.
F. Leuenberger, Elem. Math. 16 (1961), 127–129.

9. Inequalities related to two triangles inscribed one in the other

9.1 Consider a triangle ABC divided into four smaller triangles, a central one DEF inscribed in ABC and three others on the three sides of DEF. Then

$$\text{area } DEF \geqslant \min(\text{area } AEF, \text{area } BDF, \text{area } CED). \qquad (1)$$

Equality holds if and only if the points D, E, F are the midpoints of the sides of ABC.

Proof. Let F_1, F_2, F_3 $(0 < F_1 \leqslant F_2 \leqslant F_3)$ be the areas of the corner triangles and F_0 the area of the central triangle. Then we will prove that

$$F_0 \geqslant \sqrt{F_1 F_2},$$

with equality if and only if D, E, F are the midpoints of the sides of the triangle ABC. This result is slightly stronger than inequality (1).

Let BC, CA, AB be divided at D, E, F respectively in the ratio $x:x', y:y', z:z'$ with $x+x' = y+y' = z+z' = 1$. Let ABC also be of unit area. Then the corner triangles are of areas $y'z$, $z'x$, $x'y$. Also

$$F_0 = 1-y'z-z'x-x'y = xyz+x'y'z'.$$

If $F_3 < \frac{1}{4}$ then $F_1 \leqslant F_2 \leqslant F_3$ and $F_1+F_2+F_3+F_0 = 1$ imply

$$F_0 \geqslant \sqrt{F_1 F_2}.$$

If $F_3 \geqslant \frac{1}{4}$ then

$$F_0 = xyz+x'y'z' \geqslant 2\sqrt{xyzx'y'z'} = 2\sqrt{F_1 F_2 F_3} \geqslant \sqrt{F_1 F_2}.$$

Equality is obtained if and only if $xyz = x'y'z'$ and $F_3 = \frac{1}{4}$, which occurs if and only if $F_1 = F_2 = F_3 = F_0 = \frac{1}{4}$ and $x = y = z = \frac{1}{2}$.

This proof is due to P. H. Diananda.

REMARK. Inequality (1) was proposed by H. Debrunner in 1956 (see: Problem 260, Elem. Math. 11 (1956), 20). According to J. Rainwater this inequality may be attributed to P. Erdős (see: J. Rainwater, Problem 4908, Amer. Math. Monthly 67 (1960), 479).

Proof of inequality (1) was first given by A. Bager in Elem. Math. 12 (1957), 43. In the Amer. Math. Monthly 68 (1961), 368, a proof of P. H. Diananda was published.

Inequality (1) was also investigated by:

E. Morgantini, Rend. Sem. Mat. Univ. Padova 30 (1960), 245–247.

G. C. Citterio, Period. Mat. (4) 40 (1962), 45–50.

B. I. Baidaff, Bol. Mat. 35, No. 17–20 (1962), 65–66.

F. E. G.-Rodeja, Gac. Mat. 15 (1963), 23–24.

N. J. Torres, Gac. Mat. 15 (1963), 127–128.

H. T. Croft, Math. Gaz. 49 (1965), 45–49.

E. Szekeres, Elem. Math. 22 (1967), 17–18.

9.2 Let DEF be defined as in 9.1, then

$$\text{per } DEF \geq \min(\text{per } AEF, \text{per } BDF, \text{per } CED), \qquad (1)$$

where per denotes the perimeter.

Equality holds if and only if the points D, E, F are the mid-points of the sides of ABC.

PROOF. To prove this it will convenient to regard DEF as given, and to define angular coordinates φ, ψ, θ as the angles which the sides of ABC make with the corresponding sides of DEF. More precisely, a line through D and parallel to FE would have to be rotated anticlockwise i.e. in the sense DEF, through an angle φ in order to become coincident with BC; for a clockwise rotation we would assign a negative value to φ. Similarly ψ and θ are defined by means of anticlockwise rotations about E and F.

Let D, E, F be the angles of triangle DEF. Then it follows that angle $AFE = E+\theta$ and angle $AEF = F-\psi$. The only restrictions on the range of values for φ, ψ, θ are that DEF must remain inside ABC, so that all angles like $E-\theta$, $F-\psi$ must be positive, and the sum of any pair of angles in the same triangle must be less than two right angles.

Let s_0, s_1, s_2, s_3 denote the perimeters of the triangles DEF, AEF, BDF, CED respectively. We prove that $s_0 \geqslant \min(s_1, s_2, s_3)$. The sine rule, applied to triangle DEF, gives

$$\frac{DF+DE}{FE} = \frac{\sin E+\sin F}{\sin(E+F)} = \frac{\cos\frac{1}{2}(E-F)}{\cos\frac{1}{2}(E+F)},$$

and similarly, in triangle AEF, we have

$$\frac{AF-AE}{FE} = \frac{\cos\frac{1}{2}(E-F+\theta+\psi)}{\cos\frac{1}{2}(E+F+\theta-\psi)}.$$

Since all the denominators are positive, s_0-s_1 has the same sign as

$$\cos\tfrac{1}{2}(E-F)\cos\tfrac{1}{2}(E-F+\theta-\psi)-\cos\tfrac{1}{2}(E+F)\cos\tfrac{1}{2}(E-F+\theta+\psi)$$

$$= \tfrac{1}{2}\cos\left(E+\frac{\theta-\psi}{2}\right)+\tfrac{1}{2}\cos\left(F+\frac{\theta-\psi}{2}\right)$$

$$-\tfrac{1}{2}\cos\left(E+\frac{\theta+\psi}{2}\right)-\tfrac{1}{2}\cos\left(F-\frac{\theta+\psi}{2}\right)$$

$$= \sin\frac{\psi}{2}\sin\left(E+\frac{\theta}{2}\right)-\sin\frac{\theta}{2}\sin\left(F-\frac{\psi}{2}\right). \qquad (2)$$

There are now three possibilities:

1°. φ, ψ, θ are all zero. Then the sides of ABC are parallel to those of DEF, D is the midpoint of BC, etc., and $s_0=s_1=s_2=s_3$.

2°. φ, ψ, θ do not all have the same sign, or not more than two are zero. Then there must be a pair for which, in the cyclic order $\varphi\psi\theta\varphi$, negative (or zero) follows positive: if, for instance, $\psi \geqslant 0$, $\theta < 0$ (or $\psi > 0$, $\theta \leqslant 0$), then $s_0 > s_1$.

3°. φ, ψ, θ are all of the same sign and different from zero. Then the product $\sin\psi/2 \sin\theta/2$ is positive, and the expression (2), having the sign of s_0-s_1, can be divided by this product to give

$$\sin E\cot g\frac{\theta}{2}+\cos E-\sin F\cot g\frac{\psi}{2}+\cos F,$$

and dividing further by $\sin E \sin F$ and bearing in mind that $\cos E+\cos F>0$, we see that there is some number $k_1>0$ such that

$$k_1(s_0-s_1) > \operatorname{cosec} F \operatorname{cotg}\frac{\theta}{2} - \operatorname{cosec} E \operatorname{cotg}\frac{\psi}{2}.$$

Adding similar results for s_0-s_2 and s_0-s_3, we have

$$k_1(s_0-s_1)+k_2(s_0-s_2)+k_3(s_0-s_3) > 0,$$

so that s_0 is greater than at least one of the s_i.

This proof is due to L. A. G. Dresel.

REMARK. Inequality (1) appeared as problem 4964 in the Amer. Math. Monthly 68 (1961), 384 (E. Trost and A. Bager). However, before that Debrunner in 1956 and Oppenheim in 1960 posed the question of validity of inequality (1). A proof of (1) was given by L. A. G. Dresel, Nabla (Bull. of the Malayan Math. Soc.), 8 (1961), 97–99. A proof of R. Breusch was published in the Amer. Math. Monthly 69 (1962), 672.

Inequality (1) was also proved by:

T. H. Croft, Math. Gaz. 49 (1965), 45–49.

W. A. Zalgaller, Matematyka, Warsaw, 19 (1966), 49–53.

E. Szekeres. Elem. Math. 22 (1967), 17–18.

9.3 Let r_0, r_1, r_2, r_3 denote the inradii of DEF, AEF, BDF, CED respectively, where DEF is defined in **9.1**. Then

$$r_0 \geqslant \min(r_1, r_2, r_3).$$

Equality holds if and only if D, E, F are the midpoints of the sides of ABC.

PROOF. With the notations of **9.2**, we have

$$r_0\left(\operatorname{cotg}\frac{E}{2}+\operatorname{cotg}\frac{F}{2}\right) = EF = r_1\left(\operatorname{cotg}\frac{E+\theta}{2}+\operatorname{cotg}\frac{F-\psi}{2}\right).$$

Hence r_0-r_1 has the same sign as

$$\operatorname{cotg}\frac{E+\theta}{2}+\operatorname{cotg}\frac{F-\psi}{2}-\operatorname{cotg}\frac{E}{2}-\operatorname{cotg}\frac{F}{2}$$

$$=\sin\frac{\psi}{2}\operatorname{cosec}\frac{F}{2}\operatorname{cosec}\frac{F-\psi}{2}-\sin\frac{\theta}{2}\operatorname{cosec}\frac{E}{2}\operatorname{cosec}\frac{E+\theta}{2}.$$

The argument is now the same as before in cases 1° and 2° (see: **9.2**). In case 3°, where φ, ψ, θ are either all positive or negative, we find that r_0-r_1 has the same sign as

$$\operatorname{cosec}\frac{F}{2}\operatorname{cosec}\frac{\theta}{2}\sin\frac{E+\theta}{2}-\operatorname{cosec}\frac{E}{2}\operatorname{cosec}\frac{\psi}{2}\sin\frac{F-\psi}{2}$$

$$=\left(\sin\frac{E}{2}\operatorname{cotg}\frac{\theta}{2}+\cos\frac{E}{2}\right)\operatorname{cosec}\frac{F}{2}$$

$$-\left(\sin\frac{F}{2}\operatorname{cotg}\frac{\psi}{2}-\cos\frac{E}{2}\right)\operatorname{cosec}\frac{E}{2}$$

$$>\left(\operatorname{cosec}^2\frac{F}{2}\operatorname{cotg}\frac{\theta}{2}-\operatorname{cosec}^2\frac{E}{2}\operatorname{cotg}\frac{\psi}{2}\right)\sin\frac{E}{2}\sin\frac{F}{2}.$$

Since the terms within brackets, when added to similar terms. cancel each other out, we see that there are positive numbers k_i such that

$$k_1(r_0-r_1)+k_2(r_0-r_2)+k_3(r_0-r_3)>0,$$

so that r_0 is greater than at least one of the r_i.

We note that in case 3° our proof does not tell us whether one and the same i gives $s_0>s_i$ and $r_0>r_i$.

L. A. G. Dresel, Nabla (Bull. of the Malayan Math. Soc.), 8 (1961), 97–99.

9.4 If ABC and DEF are acute triangles, where DEF is defined in **9.1**, then

$$\min(R_1, R_2, R_3)\leqslant R_0\leqslant\max(R_1, R_2, R_3),$$

where R_1, R_2, R_3 are the circumradii of the three corner triangles and R_0 the circumradius of the central triangle DEF.

Equalities occur if and only if DEF is similar to ABC.

Proof. We have

$$FE=2R_1\sin\alpha=2R_0\sin D.$$

Therefore

$$\frac{R_0}{R_1}=\frac{\sin\alpha}{\sin D}.$$

Similarly

$$\frac{R_0}{R_2} = \frac{\sin\beta}{\sin E}, \quad \frac{R_0}{R_3} = \frac{\sin\gamma}{\sin F}.$$

Since $\alpha+\beta+\gamma = \pi = D+E+F$, it follows that at least one of the ratios $\alpha:D$, $\beta:E$, $\gamma:F$ is $\geqslant 1$, and at least one of these ratios is $\leqslant 1$. If all the angles are acute, the same is true of the ratios of the sides, and hence of the ratios $R_0:R_1$, $R_0:R_2$, $R_0:R_3$. Therefore

$$\min(R_1, R_2, R_3) \leqslant R_0 \leqslant \max(R_1, R_2, R_3),$$

and equalities hold if and only if ABC and DEF are similar.

A. Oppenheim-L. A. G. Dresel, Nabla 7 (1960), 175 and 8 (1961), 72.

9.5 Let P be a point inside a circle concentric with the circumscribed circle and with radius $< R\sqrt{5}$. If D, E, F denote the orthogonal projections of P on the sides of this triangle then

$$4 \cdot \text{area } DEF \leqslant F. \tag{1}$$

Equality holds if and only if P is the circumcentre of the triangle or if P lies on a circle with centre O and radius $R\sqrt{2}$.

Proof. If $PO = p$ then according to a theorem which was given by Gergonne (1823):

$$\text{area } DEF = \tfrac{1}{2}(R^2-p^2)\sin\alpha\sin\beta\sin\gamma.$$

For $p = R$ the area DEF is zero; for $p > R$ area is negative. If we consider positive areas only, then

$$\text{area } DEF = \frac{|R^2-p^2|}{4R^2} F.$$

From this follows (1).

9.6 Among all the triangles inscribed in a given acute-angled triangle, the triangle that has for its vertices the feet of the altitudes of the given triangle has the least perimeter.

Remark. In 1775 in the Acta Eruditorum, J. F. de Fuschis a Fagnano proved this theorem using differential calculus. Other proofs, which were geometric in nature, appeared later, a most ingenious and elegant one being the proof given by H. A. Schwarz. Several proofs of Fagnano's theorem, including Schwarz's proof are given in: N. A. Court, Scripta Math. 17 (1951), 147–150 and 18 (1952), 95–96. In 1930, L. Fejér proved the theorem in a very simple way.

9.7 Let D, E, F be the feet of the altitudes of an acute-angled triangle ABC. If p demotes the perimeter of the triangle DEF, then

$$s \geqslant p.$$

Equality holds only for the equilateral triangle.

Proof. If D, E, F be the feet of the altitudes of the triangle ABC, then $AF = b\cdot\cos\alpha$, $AE = c\cdot\cos\alpha$, so that

$$EF^2 = AF^2 + AE^2 - 2\cdot AE\cdot AF\cdot\cos\alpha = a^2\cos^2\alpha,$$

i.e.,

$$EF = a\cdot\cos\alpha.$$

Similarly

$$FD = b\cdot\cos\beta \text{ and } DE = c\cdot\cos\gamma.$$

Therefore we have

$$p = EF+FD+DE = a\cdot\cos\alpha + b\cdot\cos\beta + c\cdot\cos\gamma. \tag{1}$$

Since

$$a = 2R\sin\alpha,\ b = 2R\sin\beta,\ c = 2R\sin\gamma, \tag{2}$$

(1) becomes

$$\begin{aligned} p &= R(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) \\ &= 4R\sin\alpha\sin\beta\sin\gamma, \end{aligned}$$

or, on the basis of (2)

$$p = 4R\cdot\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R} = \frac{2}{R}\cdot\frac{abc}{4R} = \frac{2F}{R},$$

i.e.,

$$p = \frac{2rs}{R}. \tag{3}$$

Using $R \geqslant 2r$, from (3), we obtain $s \geqslant p$.

This proof is due to R. R. Janić.

A. Zirakzadeh, Math. Mag. 39 (1966), 96–99.

L. Carlitz, Math. Mag. 39 (1966), 289.

9.8 Let D, E, F be the points in which the bisectors of angles α, β, γ, meet the sides of a triangle ABC. Then

$$4 \cdot \text{area } DEF \leqslant F.$$

Equality holds only for the equilateral triangle.

V. Gridasov, Matematika i fizika, Sofia, 6 (1965), 52–53.

9.9 Let p denote the perimeter of the triangle whose vertices are the points of contact with the incircle of the triangle ABC. Then

$$p \geqslant 6r \sqrt[3]{\frac{s}{4R}}.$$

PROOF. Let D, E, F be the points of contact with the incircle of the triangle ABC. Then

$$EF = 2(s-a) \sin\frac{\alpha}{2}, \quad FD = 2(s-b) \sin\frac{\beta}{2}, \quad DE = 2(s-c) \sin\frac{\gamma}{2}.$$

By means of the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned} EF+FD+DE &= 2\left[(s-a)\sin\frac{\alpha}{2} + (s-b)\sin\frac{\beta}{2} + (s-c)\sin\frac{\gamma}{2}\right] \\ &\geqslant 6\sqrt[3]{(s-a)(s-b)(s-c)\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}} \\ &= 6r\sqrt[3]{\frac{s}{4R}}. \end{aligned}$$

Ž. Živanović, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181–No. 196 (1967), 69–72.

9.10 If D, E, F are the points on the sides of the triangle ABC, such that the sides of the triangle DEF are parallel to the altitudes of the triangle ABC, then

$$\text{area } DEF \geqslant \frac{16}{81R^4} F^3.$$

Ž. Živanović, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181–No. 196 (1967), 69–72.

9.11 If D, E, F are the points on the sides BC, CA, AB respectively of the triangle ABC, such that $BD = CE = AF = x$ ($x \leqslant \min(a, b, c)$), then

$$\text{area } DEF \geqslant \frac{a^2b(2c-b)+b^2c(2a-c)+c^2a(2b-a)}{32Rs}.$$

Ž. Živanović, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181–No. 196 (1967), 69–72.

O. Bottema, id. No. 200–No. 209 (1967), 11–12.

9.12 Let DEF be defined as in **9.1** and let

$$2a' = b+c,\ 2b' = c+a,\ 2c' = a+b,$$

$$p = AE+AF,\ q = BF+BD,\ r = CD+CE,$$

$$3t = a+b+c.$$

If $t \leqslant p \leqslant a$, $t \leqslant q \leqslant b$ and $t \leqslant r \leqslant c$, then

$$\text{per } DEF \geqslant s,$$

and equality holds if and only if the points D, E, F are the midpoints of the sides of the triangle ABC.

B. Bollobás, Can. J. Math. 19 (1967), 523–528.

9.13 P is a point inside the triangle ABC; the intersection of AP, BP and CP and the opposite side is D, E, and F respectively. The area of DEF is F_1. Then

$$AD \cdot BE \cdot CF \geqslant 4F_1 s.$$

PROOF. If x, y, z are the barycentric coordinates of P with respect to ABC and $x_1 = x/a$, $y_1 = y/b$, $z_1 = z/c$ the distance-coordinates, then

$$BD = \frac{az}{y+z}, \quad DC = \frac{ay}{y+z}$$

and therefore

$$a \cdot AD^2 = \frac{c^2y^2+2bcyz\cos\alpha+b^2z^2}{(y+z)^2}.$$

If F_1 is the area of DEF, then

$$F_1 = F\left[1 - \sum \frac{xy}{(x+z)(y+z)}\right] = \frac{2Fxyz}{(y+z)(z+x)(x+y)}.$$

Hence

$$k^2 = \frac{AD^2BE^2CF^2}{F_1^2} = \frac{1}{4F^2x^2y^2z^2} \prod (c^2y^2+2bcyz\cdot\cos\alpha+b^2z^2)$$

$$= \frac{a^2b^2c^2}{4F^2x_1^2y_1^2z_1^2} \prod (y_1^2+2y_1z_1\cos\alpha+z_1^2)$$

$$= \frac{a^2b^2c^2}{4F^2} \prod \left(\frac{y_1}{z_1}+\frac{z_1}{y_1}+2\cos\alpha\right).$$

As $x_1, y_1, z_1 > 0$, we have $y_1/z_1 + z_1/y_1 \geqslant 2$ and thus

$$k^2 \geqslant \frac{a^2b^2c^2}{4F^2} \prod (2+2\cos\alpha) = \frac{16a^2b^2c^2}{F^2} \prod \frac{s(s-a)}{bc} = 16s^2$$

and thus $k \geqslant 4s$.

Equality holds for the incentre.

O. Bottema, Nieuw Arch. Wisk. 14 (1966), 268.

9.14 If D, E, F the points defined in 9.13, then

$$\frac{AD}{AP}+\frac{BE}{BP}+\frac{CF}{CP} \geqslant \frac{9}{2}.$$

Ž. Živanović, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181–No. 196 (1967), 69–72.

9.15 If D, E, F are the points of contact of the incircle of the triangle ABC and its sides, then

$$\left(\frac{a}{EF}\right)^2+\left(\frac{b}{FD}\right)^2+\left(\frac{c}{DE}\right)^2 \geqslant 12.$$

Equality holds if and only if the points D, E, F are the mid-points of the sides, i.e., if the triangle ABC is equilateral.

E. A. Bokov, Matematika v škole 1954, No. 5, 76.

9.16 If D, E, F are the feet of the altitudes of a triangle, then

$$\left(\frac{EF}{a}\right)^2+\left(\frac{FD}{b}\right)^2+\left(\frac{DE}{c}\right)^2 \geqslant \frac{3}{4}. \tag{1}$$

Equality holds if and only if the triangle is equilateral.

PROOF. Starting from (see: **2.21**)

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma \geqslant \tfrac{3}{4}$$

and putting

$$EF = a\,|\cos\alpha|,\ FD = b\,|\cos\beta|,\ DE = c\,|\cos\gamma|,$$

we get (1).

Ju. I. Gerasimov, Matematika v škole 1965, No. 1, 79.

9.17 If F_1 be the area of the equilateral triangle inscribed in ABC, then

$$F_1 \geqslant \frac{2F^2\sqrt{3}}{(a^2+b^2+c^2+4F\sqrt{3})}.$$

Equality holds if and only if ABC is equilateral.

E. W. Hobson, Problem 4, Treatise on Plane and Advanced Trigonometry, 1957, p. 211.

10. Inequalities involving elements of two triangles

10.1 If A_1, B_1, C_1 are the second points of intersection of the angle-bisectors and the circumcircle of the triangle ABC, then

$$\text{area } A_1B_1C_1 \geqslant \text{area } ABC,$$

where equality occurs only for the equilateral triangle.

M. S. Klamkin, Math. Teacher 60 (1967), 323–328.

10.2 If A_1, B_1, C_1 are the second points of intersection of the medians and the circumcircle of the triangle ABC, then

$$\text{area } A_1B_1C_1 \geqslant \text{area } ABC.$$

Equality holds if and only if the triangle is equilateral.

This result is due to R. R. Janić.

10.3 If F_1 is the area of a triangle with sides $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, then

$$4F_1^2 \geqslant \sqrt{3}\cdot F.$$

P. Finsler and H. Hadwiger, Comment. Math. Helv. 10 (1937/38), 316–326.

10.4 Let P be a point inside the equilateral triangle ABC, and let D, E, F be the points situated symmetrically to P with respect to the sides BC, CA, AB. Then

$$\text{area } DEF \leqslant \text{area } ABC.$$

Equality holds if and only if P is the centre of the triangle ABC.

Proof. Since

$$\text{area } ABC = \frac{\sqrt{3}}{3}(r_1+r_2+r_3)^2,$$

$$\text{area } DEF = \sqrt{3}(r_2r_3+r_3r_1+r_1r_2),$$

we obtain

$$\text{area } ABC - \text{area } DEF = \frac{\sqrt{3}}{3}[(r_2-r_3)^2+(r_3-r_1)^2+(r_1-r_2)^2] \geqslant 0.$$

Ž. Živanović, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181–No. 196 (1967), 69–72.

10.5 If a, b, c and a', b', c' are the sides of two triangles ABC and $A'B'C'$ inscribed in the same circle such that $AA' \parallel BC$, $BB' \parallel CA$, $CC' \parallel AB$, then

$$\frac{a'+b'+c'}{a+b+c} \leqslant 1, \quad \frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c} \leqslant 3.$$

Equalities hold only if the triangle ABC is equilateral.

These inequalities are due to H. Demir.

10.6 Let the triangles $A_1A_2A_3$ and $B_1B_2B_3$ be inscribed in a circle of radius R. Let T_1 and T_2 be the centroids of these triangles respectively. Then

$$T_1T_2 < \tfrac{1}{3}(4R + \min A_iB_j) \quad (i, j = 1, 2, 3).$$

E. Jucović, Problem 353, Elem. Math. 15 (1960), 85–87.

10.7 Let the triangles $A_1A_2A_3$ and $B_1B_2B_3$, with the orthocentres H_1 and H_2 respectively, be inscribed in a circle of radius R. Then

$$H_1H_2 < 4R + \min A_iB_j \quad (i, j = 1, 2, 3).$$

E. Jucović, Problem 352, Elem. Math. 15 (1960), 85–87.

10.8 Let $A_1B_1C_1$ be a triangle with sides a_1, b_1, c_1 and let $A_2B_2C_2$ be another triangle with sides a_2, b_2, c_2. Let F_1 and F_2 be their areas respectively. Then

$$a_1^2(-a_2^2 - b_2^2 + c_2^2) + b_1^2(a_2^2 - b_2^2 + c_2^2) + c_1^2(a_2^2 + b_2^2 - c_2^2) \geqslant 16F_1F_2,$$

with equality if and only if the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar.

Proof. Let A_3 be the point such that the triangles $A_3B_1C_1$ and $A_2B_2C_2$ are directly similar. By virtue of the equalities

$$\frac{B_1C_1}{a_2} = \frac{C_1A_3}{b_2} = \frac{A_3B_1}{c_2},$$

implied from the assumed similarity, applying the law of cosines to the triangle $A_3C_1A_1$, we get

$$\begin{aligned} a_2^2(A_1A_3)^2 &= a_2^2b_1^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 \cos(C_1 - C_2) \\ &= a_2^2b_1^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 \cos C_1 \cos C_2 \\ &\qquad - 2a_1a_2b_1b_2 \sin C_1 \sin C_2 \geqslant 0. \end{aligned}$$

Since

$$a_1a_2b_1b_2 \sin C_1 \sin C_2 = 4F_1F_2,$$

we obtain

$$a_2^2b_1^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 \cos C_1 \cos C_2 \geqslant 8F_1F_2,$$

i.e.,

$$a_2^2b_1^2+a_1^2b_2^2-\tfrac{1}{2}(b_1^2+a_1^2-c_1^2)(b_2^2+a_2^2-c_2^2) \geqslant 8F_1F_2.$$

The last inequality is equivalent to the one we had to prove.

REMARK. D. Pedoe has proposed this inequality as a problem E 15?? in the Amer. Math. Monthly 70 (1963), 92 and 1012.

D. Pedoe, Proc. Cambridge Philos. Soc. 38 (1942), 357–358.

D. Pedoe, Math. Gaz. 26 (1942), 202–208.

10.9 Let the triangle ABC be divided by the straight line BD in two triangles. If r_1, r_2, r are the radii of incircles of triangles ABD, CBD, ABC, then

$$r_1+r_2 > r.$$

PROOF. Since

$$2s = AB+BC+CA > AB+BD+DA = 2s_1,$$
$$2s = AB+BC+CA > CB+BD+DC = 2s_2,$$

we have

$$r_1+r_2 = \frac{F_1}{s_1}+\frac{F_2}{s_2} > \frac{F_1}{s}+\frac{F_2}{s} = \frac{F}{s} = r,$$

where F_1, F_2, F are the areas of the triangles ABD, BCD, ABC, respectively.

10.10 Let a circle be inscribed in a triangle ABC and let a triangle $A_1B_1C_1$ be inscribed in the same circle. If s is the semi-perimeter of the triangle ABC and s_1 that of $A_1B_1C_1$, then

$$s \geqslant 2s_1.$$

Equality occurs if and only if both triangles are equilateral.

10.11 $4F \leqslant \text{area}\, I_aI_bI_c \leqslant \left(\dfrac{R}{r}\right)^2 F.$

Equalities hold if and only if the triangle ABC is equilateral.

M. S. Klamkin, Math. Teacher 60 (1967), 323–328.

10.12 Suppose that $A_iB_iC_i$ $(i = 1, 2)$ are triangles with sides a_i, b_i, c_i, area F_i and altitudes p_i, q_i, r_i. Define numbers a_3, b_3, c_3

by $a_3 = (a_1^2+a_2^2)^{1/2}$, etc. Then

1°. a_3, b_3, c_3 are the sides of a triangle;

2°. $p_3^2 \geqslant p_1^2+p_2^2$, $q_3^2 \geqslant q_1^2+q_2^2$, $r_3^2 \geqslant r_1^2+r_2^2$,

equality occurring in all three if and only if the original two triangles are similar;

3°. $F_3 \geqslant F_1+F_2$,

with equality if and only if the triangles are similar;

4°. $F_3^2 \geqslant 4 \cdot F_1F_2$,

with equality if and only if the triangles are congruent.

PROOF. We give the Nolan's proof without any change. Taking

$$(ux+vy)^2 \leqslant (u^2+v^2)(x^2+y^2), \tag{1}$$

with equality if and only if $vx = uy$.

1°. $a_3^2 = a_1^2+a_2^2 < (b_1+c_1)^2+(b_2+c_2)^2$
$$= b_3^2+c_3^2+2(b_1c_1+b_2c_2).$$

But $b_1c_1+b_2c_2 \leqslant (b_1^2+b_2^2)^{1/2}(c_1^2+c_2^2)^{1/2} = b_3c_3$ by (1).

Therefore $a_3^2 < (b_3+c_3)^2$, i.e. $a_3 < b_3+c_3$, the symmetry of which gives the result.

2°. The cosine formula gives

$$c_3a_3 \cos \beta_3 = c_1a_1 \cos \beta_1 + c_2a_2 \cos \beta_2.$$

Squaring, applying (1), and dividing by $a_3^2 = a_1^2+a_2^2$, we obtain

$$c_3^2 \cos^2 \beta_3 \leqslant c_1^2 \cos^2 \beta_1 + c_2^2 \cos^2 \beta_2.$$

Subtraction gives

$$c_3^2 \sin^2 \beta_3 \geqslant c_1^2 \sin^2 \beta_1 + c_2^2 \sin^2 \beta_2,$$

which is $p_3^2 \geqslant p_1^2+p_2^2$ as required. Similarly for q_3^2 and r_3^2. Equality holds for p_3^2 if and only if $a_1/a_2 = c_1 \cos \beta_1/c_2 \cos \beta_2$, similarly for r_3^2 if and only if $c_1/c_2 = a_1 \cos \beta_1/a_2 \cos \beta_2$, giving

$$\cos \beta_1 = \cos \beta_2, \ a_1/a_2 = c_1/c_2.$$

Therefore for equality for one altitude, similarity is sufficient but not necessary (e.g. both triangles isosceles is sufficient). Equality for two or three altitudes holds if and only if the original triangles are similar.

3°. $$2(F_1+F_2) = p_1a_1+p_2a_2 \leqslant (p_1^2+p_2^2)^{1/2}(a_1^2+a_2^2)^{1/2} \leqslant p_3a_3 = 2\cdot F_3,$$

as required, equality occurring if and only if $c_1 \cos\beta_1/c_2 \cos\beta_2 = a_1/a_2$ as in 2°, and $a_1/a_2 = p_1/p_2 = c_1 \sin\beta_1/c_2 \sin\beta_2$, giving $\operatorname{tg}\beta_1 = \operatorname{tg}\beta_2$, etc., i.e. the triangles are similar.

4°. $$F_3^2 \geqslant (F_1+F_2)^2 = (F_1-F_2)^2+4F_1F_2 \geqslant 4F_1F_2,$$

with equality if and only if the triangles are similar, as in 3°, and $F_1 = F_2$, i.e. the triangles are congruent.

A. Oppenheim-R. P. Nolan, Problem 5092, Amer. Math. Monthly 70 (1963), 444 and 71 (1964), 444.

10.13 For two triangles with sides a, b, c and m_a, m_b, m_c, whose angles are α, β, γ and $\alpha_m, \beta_m, \gamma_m$ respectively, the following implications are valid:

$$a > b > c \Rightarrow m_a < m_b < m_c;$$

$$\alpha > \beta > \gamma \Rightarrow \alpha_m < \beta_m < \gamma_m;$$

$$\alpha > \alpha_m \Rightarrow \gamma < \gamma_m;$$

$$\alpha > \beta_m \Rightarrow \beta < \gamma_m;$$

$$\beta > \alpha_m \Rightarrow \gamma < \beta_m.$$

L. Toscano, Archimede 8 (1956), 278–279.

11. Special triangles

11.1 In each acute triangle two angles exist whose difference is less than or equal to $\pi/6$.

11.2 For each acute triangle,

$$-\operatorname{cotg} 2\alpha-\operatorname{cotg} 2\beta-\operatorname{cotg} 2\gamma \geqslant \operatorname{cotg} \alpha+\operatorname{cotg} \beta+\operatorname{cotg} \gamma$$
$$\geqslant \operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}$$
$$\geqslant \operatorname{tg}\frac{\pi-\alpha}{4}+\operatorname{tg}\frac{\pi-\beta}{4}+\operatorname{tg}\frac{\pi-\gamma}{4}$$
$$\geqslant \sqrt{3}.$$

Equalities hold if and only if the triangle is equilateral.

C. Ionescu-Tiu, Gaz. Mat. B 14 (1963), 560.

11.3 A necessary and sufficient condition for a triangle to be acute-angled is

$$\operatorname{tg} \alpha \cdot \operatorname{tg} \beta > 1.$$

G. Piskarev, Matematika v škole, 1952, No. 5, 93 and 1963, No. 3, 88.

11.4 For each acute triangle,

$$\operatorname{tg} \alpha(\operatorname{cotg} \beta+\operatorname{cotg} \gamma)+\operatorname{tg} \beta(\operatorname{cotg} \gamma+\operatorname{cotg} \alpha)$$
$$+\operatorname{tg} \gamma(\operatorname{cotg} \alpha+\operatorname{cotg} \beta) \geqslant 6. \qquad (1)$$

PROOF. According to our assumption,

$$\operatorname{tg} \alpha>0, \quad \operatorname{tg} \beta>0, \quad \operatorname{tg} \gamma>0.$$

Using

$$x+\frac{1}{x} \geqslant 2 \quad (x>0)$$

for $x=\operatorname{tg} \alpha / \operatorname{tg} \beta$, etc., and then adding these inequalities, we obtain

$$\frac{\operatorname{tg} \alpha}{\operatorname{tg} \beta}+\frac{\operatorname{tg} \beta}{\operatorname{tg} \alpha}+\frac{\operatorname{tg} \beta}{\operatorname{tg} \gamma}+\frac{\operatorname{tg} \gamma}{\operatorname{tg} \beta}+\frac{\operatorname{tg} \alpha}{\operatorname{tg} \gamma}+\frac{\operatorname{tg} \gamma}{\operatorname{tg} \alpha} \geqslant 6,$$

whence (1).

11.5 For each acute triangle

$$\operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma \geqslant 3\sqrt{3}. \qquad (1)$$

Equality holds if and only if the triangle is equilateral.

PROOF. Starting from the identity

$$\operatorname{tg}\alpha\ \operatorname{tg}\beta\ \operatorname{tg}\gamma = \operatorname{tg}\alpha+\operatorname{tg}\beta+\operatorname{tg}\gamma,$$

which holds for any triangle, we obtain

$$\operatorname{tg}\alpha \operatorname{tg}\beta \operatorname{tg}\gamma = \operatorname{tg}\alpha+\operatorname{tg}\beta+\operatorname{tg}\gamma \geqslant 3\sqrt[3]{\operatorname{tg}\alpha\cdot\operatorname{tg}\beta\cdot\operatorname{tg}\gamma},$$

whence

$$\operatorname{tg}^3\alpha \operatorname{tg}^3\beta \operatorname{tg}^3\gamma \geqslant 27\operatorname{tg}\alpha \operatorname{tg}\beta \operatorname{tg}\gamma,$$

i.e. (1).

N. Dzigava, Matematika v škole, 1949, No. 4, 60.

11.6 For each acute triangle,

$$\operatorname{tg}\alpha+\operatorname{tg}\beta+\operatorname{tg}\gamma \geqslant 3\sqrt{3}.$$

Equality holds if and only if the triangle is equilateral.

Math. Notae 6 (1946), 196.

11.7 If n is a positive integer, then, for each acute triangle,

$$\operatorname{tg}^n\alpha+\operatorname{tg}^n\beta+\operatorname{tg}^n\gamma > 3+\tfrac{3}{2}n.$$

Ju. I. Gerasimov, Matematika v škole, 1964, No. 1, 82–83.

11.8 If n denotes a non-negative real number, then, for each acute triangle

$$\operatorname{tg}^n\alpha+\operatorname{tg}^n\beta+\operatorname{tg}^n\gamma \geqslant 3\cdot 3^{n/2}. \qquad (1)$$

Equality holds if and only if the triangle is equilateral.

PROOF. On the basis of **11.5**, we have

$$\left(\frac{\operatorname{tg}^n\alpha+\operatorname{tg}^n\beta+\operatorname{tg}^n\gamma}{3}\right)^{1/n} \geqslant (\operatorname{tg}\alpha \operatorname{tg}\beta \operatorname{tg}\gamma)^{1/3} \geqslant 3^{1/2},$$

whence follows (1).

M. N. Kritikos, Actes du Congrès interbalkanique de mathématiciens, Athènes, 2–9 september 1934, 157–158.

REMARK. Inequality (1) in the paper mentioned was proved only for n an odd, positive integer.

11.9 $\sin\alpha+\sin\beta+\sin\gamma+\operatorname{tg}\alpha+\operatorname{tg}\beta+\operatorname{tg}\gamma > 2\pi$ holds for each acute triangle.

Ju. I. Gerasimov, Matematika v škole, 1965, No. 2, 63.

11.10 For each acute triangle

$$1+\frac{\sin 2\alpha+\sin 2\beta+\sin 2\gamma}{\sin 4\alpha+\sin 4\beta+\sin 4\gamma}\leqslant 0. \tag{1}$$

PROOF. By virtue of

$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$$

and

$$\sin 4\alpha+\sin 4\beta+\sin 4\gamma = -4\sin 2\alpha\sin 2\beta\sin 2\gamma,$$

inequality (1) becomes

$$1-\frac{\sin\alpha\sin\beta\sin\gamma}{\sin 2\alpha\sin 2\beta\sin 2\gamma}\leqslant 0,$$

i.e.

$$\sin 2\alpha\sin 2\beta\sin 2\gamma\leqslant\sin\alpha\sin\beta\sin\gamma.$$

The last inequality is true by virtue of **2.23.**

11.11 $\sum r_i(H)\leqslant\sum r_i(O)$ holds for each acute triangle.

F. Leuenberger, Problem 444, Elem. Math. 18 (1963), 18.

11.12 For each acute triangle,

$$2\sum r_i(H)\leqslant 6r\leqslant 2\sum r_i(G)\leqslant 2\sum r_i(O)=\sum R_i(H)\leqslant 3R.$$

L. Bernstein-J. Steinig, Elem. Math. 19 (1964), 870.

11.13 For each acute triangle

$$6r\leqslant\sum R_i(I)\leqslant\sum R_i(H)\leqslant 3R.$$

L. Bernstein-J. Steinig, Elem. Math. 19 (1964), 870.

11.14 If D, E, F are the feet of the altitudes of an acute triangle then

$$AI+BI+CI\geqslant 2(DH+EH+FH).$$

Equality holds if and only if the triangle is equilateral.

L. Bankoff, Problem E 1564, Amer. Math. Monthly 70 (1963), 210 and 70 (1963), 1101.

11.15 If t_a, t_b, t_c are the lengths of transversals drawn from the vertices A, B, C of an acute-angled triangle to the opposite sides, then

$$\frac{1}{2} < \min_{t_a, t_b, t_c} \left\{ \frac{t_a^2+t_b^2+t_c^2}{a^2+b^2+c^2} \right\} \leqslant \frac{3}{4},$$

$$1 \leqslant \max_{t_a, t_b, t_c} \left\{ \frac{t_a^2+t_b^2+t_c^2}{a^2+b^2+c^2} \right\} < \frac{3}{2}.$$

Equalities hold if and only if the triangle is equilateral.

A. S. B. Holland, Elem. Math. 22 (1967), 49–55.

11.16 If a triangle is non-obtuse, then

$$R+r \leqslant \max(h_a, h_b, h_c).$$

REMARK. In fact, this is an inequality of P. Erdös. A proof can be found in: Matematika v škole, 1962, No. 6, 87–88.

11.17 If $c = \min(a, b, c)$, then, for a non-obtuse triangle, we have

$$\frac{R}{r} \leqslant \frac{a+b}{c}.$$

Matematika v škole, 1962, No. 6, 87–88.

11.18 For an acute or right-angled triangle

$$4(\cos^2\beta\cos^2\gamma+\cos^2\gamma\cos^2\alpha+\cos^2\alpha\cos^2\beta) \leqslant \cos^2\alpha+\cos^2\beta+\cos^2\gamma,$$

$$4(\cos^2\beta\cos^2\gamma+\cos^2\gamma\cos^2\alpha+\cos^2\alpha\cos^2\beta) + 16\cos^2\alpha\cos^2\beta\cos^2\gamma \leqslant 1.$$

Equalities occur when the triangle is equilateral or right-angled isosceles and in no other case.

A. Oppenheim, Problem E 1838, Amer. Math. Monthly 72 (1965), 1129.

11.19 If n is an integer greater than 2, then for any right triangle,

$$a^n+b^n < c^n.$$

PROOF. Since $a < c$, $b < c$, and $c^2 = a^2+b^2$, we o

$$\begin{aligned} c^n &= c^{n-2}c^2 = c^{n-2}(a^2+b^2) \\ &= a^2c^{n-2}+b^2c^{n-2} \\ &> a^2a^{n-2}+b^2b^{n-2} \\ &= a^n+b^n. \end{aligned}$$

11.20 For any right triangle,

$$a-b \leqslant c\sqrt{2}.$$

PROOF. Since

$$c^2 = a^2+b^2 = \frac{(a+b)^2}{2} + \frac{(a-b)^2}{2},$$

we conclude

$$2c^2 \geqslant (a+b)^2, \text{ i.e. } a+b \leqslant c\sqrt{2}.$$

Equality in (1) holds only for $a = b$.

Matematika v škole, 1965, No. 5, 76.

11.21 For any right triangle,

$$\cos^2 \frac{\alpha-\beta}{2} \geqslant \frac{2ab}{c^2}.$$

PROOF. Suppose that $\alpha \neq \beta$. Then

$$\tfrac{1}{2}(\sin\alpha+\sin\beta) > \sqrt{\sin\alpha\sin\beta},$$

i.e.

$$\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} > \sqrt{\sin\alpha\sin\beta}.$$

Since

$$\sin\frac{\alpha+\beta}{2} = \frac{\sqrt{2}}{2}, \quad \sin\alpha = \frac{a}{c}, \quad \sin\beta = \frac{b}{c},$$

we get

$$\frac{\sqrt{2}}{2}\cos\frac{\alpha-\beta}{2} > \sqrt{\frac{ab}{c^2}}.$$

Equality in (1) holds only for $\alpha = \beta$.

N. Dzigava, Matematika v škole, 1949, No. 1, 62
No. 4, 59.

11.22 For any right triangle

$$\sqrt{2}-1 \leqslant \frac{r}{h} < \frac{1}{2},$$

where h is the altitude of the hypotenuse.

Equality holds only for the isosceles triangle.

Proof. Since

$$F = \tfrac{1}{2}(a+b+c)r = \tfrac{1}{2}ch,$$

we get

$$\frac{r}{h} = \frac{c}{a+b+c}.$$

Since $a+b > c$, we have

$$\frac{r}{h} < \frac{c}{c+c} = \tfrac{1}{2}.$$

Furthermore by 11.21,

$$\frac{r}{h} \geqslant \frac{c}{c\sqrt{2}+c} = \sqrt{2}-1.$$

11.23 For any right triangle,

$$R+r \geqslant \sqrt{2F}.$$

Matematika v škole, 1965, No. 5, 77.

11.24 If $\alpha > \pi/2$, then

$$|\cos\alpha| \leqslant \frac{a^6}{54b^3c^3}.$$

Proof. For $\alpha > \pi/2$ we have $\cos\alpha < 0$, so that

$$\cos\alpha = -|\cos\alpha|, \quad a^2 = b^2+c^2+2bc|\cos\alpha|.$$

On the basis of the arithmetic-geometric mean inequa get

$$\frac{a^2}{3} = \frac{b^2+c^2+2bc|\cos\alpha|}{3} \geqslant \sqrt[3]{b^2c^2\cdot 2bc|\cos\alpha|},$$

whence (1) follows.

H. Hamzin, Matematika v škole, 1949, No. 5, 62 and 1950, No. 2, 54.

11.25 If t_a, t_b, t_c are the lengths of transversals drawn from the vertices A, B, C of an obtuse triangle, then

$$\frac{1}{3} < \frac{t_a^2+t_b^2+t_c^2}{a^2+b^2+c^2} < \frac{3+\sqrt{3}}{3}.$$

A. S. B. Holland, Elem. Math. 22 (1967), 49–55.

11.26 A triangle is acute, right or obtuse, depending upon whether the expression

$$(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)$$

is positive, zero or negative.

C. Ciamberlini, Rassegna di Matematica e Fisica (Roma) 5 (1925), 241–244.

11.27 Depending upon the fact whether a triangle is acute, right or obtuse, one of the following statements hold:

1°. $s > 2R+r$ or $s = 2R+r$ or $s < 2R+r$.

2°. $\sum \sin^2 \alpha > 2$ or $\sum \sin^2 \alpha = 2$ or $\sum \sin^2 \alpha < 2$.

3°. $\sum a^2 > 8R^2$ or $\sum a^2 = 8R^2$ or $\sum a^2 < 8R^2$.

4°. $R > 3 \cdot OG$ or $R = 3 \cdot OG$ or $R < 3 \cdot OG$.

5°. $\prod (s-r_a) > 0$ or $\prod (s-r_a) = 0$ or $\prod (s-r_a) < 0$.

C. Ciamberlini, Boll. Un. Mat. Ital. (2) 5 (1943), 37–41.

11.28 Whether a triangle is acute, right or obtuse, we have

$$h_a+h_b+h_c \gtreqless 2R+4r+\frac{r^2}{R},$$

respectively.

We were informed of this result by A. Bager.

12. Inequalities for the distances of a point to the vertices and the sides of a triangle

12.1 $6r \leqslant AI+BI+CI \leqslant 3R.$

Equalities hold if and only if the triangle is equilateral.

L. Carlitz, Math. Mag. 36 (1963), 264.

12.2 $AI+BI+CI \leqslant 2(R+r) = AH+BH+CH.$

PROOF. According to **2.56**, we have

$$\left(\sum \sin \frac{\alpha}{2}\right)^2 \leqslant \sum \cos^2 \frac{\alpha}{2}.$$

Hence

$$2 \sum \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \leqslant \sum \cos^2 \frac{\alpha}{2} - \sum \sin^2 \frac{\alpha}{2}$$

$$= \sum \cos \alpha = 4\left(\prod \sin \frac{\alpha}{2}\right) + 1.$$

Then

$$2 \sum \operatorname{cosec} \frac{\alpha}{2} \leqslant 4 + \prod \operatorname{cosec} \frac{\alpha}{2} = 4 + \frac{4R}{r},$$

or

$$AI+BI+CI \leqslant 2(R+r).$$

G. Daniellson, Elementa: matematika, fysika, kemi 36 (1953), 135.

L. Bankoff, Problem E 1397, Amer. Math. Monthly 67 (1960), 82 and 67 (1960), 695.

12.3 Let x, y, z be the distances of the circumcentre of a triangle to its sides. Then

$$w_a+w_b+w_c \leqslant 3(x+y+z).$$

Equality holds if and only if the triangle is equilateral.

F. Leuenberger, Problem E 1573, Amer. Math. Monthly 70 (1963), 331 and 71 (1964), 93.

12.4 If ABC is an equilateral triangle and P a point outside its plane, $PA = p$, $PB = q$, $PC = r$ then

$$(-p+q+r)(p-q+r)(p+q-r) > 0. \tag{1}$$

REMARK. This is a generalization of Pompeiu's theorem. An elegant geometric proof was given by G. R. Veldkamp, Nieuw Tijdschr. Wisk. 44 (1956), 1–4. An analytic one by O. Bottema, ibid. 44 (1956), 183–184, who proved furthermore

$$a^2F^2 \geqslant 12V^2.$$

Here a represents the sides of ABC, F the area of the triangle the sides of which are PA, PB, PC and V the volume of the tetrahedron $PABC$. In (1) the equality sign holds if P is on the sphere trough A, B, C with its centre at the centre of ABC.

12.5 For any acute triangle

$$9\,\frac{AG^2+BG^2+CG^2}{AO^2+BO^2+CO^2} - \frac{AH\cdot BH\cdot CH}{AI\cdot BI\cdot CI} \leqslant 8.$$

Equality holds if and only if the triangle is equilateral.

S. Reich, Problem E 1887, Amer. Math. Monthly 73 (1966), 538.

12.6 Let P be a point in the interior of the triangle ABC. If the straight lines AP, BP, CP intersect the sides BC, CA, AB in D, E, F respectively, then

$$PD+PE+PF < \max(a, b, c).$$

PROOF. Assume that $a > b > c$. Let $PX \| AB$, $PY \| AC$, $XK \| PF$, $YL \| PE$, where X and Y are points on BC, K a point on AB and L a point on AC. Since a is greater than AD, CF, BE by virtue of the similarity of the triangles PXY and ABC, BXK and BCF, and finally CYL and CBE, we get, respectively

$$XY > PD,\ BX > XK = PF,\ YC > YL = PE.$$

Hence,

$$a = BX+XY+YC > PD+PE+PF.$$

P. Erdös, Problem 3746, Amer. Math. Monthly 42 (1935), 454 and 44 (1937), 400.

12.7 If the point P is inside a triangle ABC, then

$$s < AP+BP+CP < 2s.$$

H. W. Guggenheimer, Plane Geometry and Its Groups, San Francisco, Cambridge, London, Amsterdam 1967, p. 178.

12.8 If $a = \max(a, b, c)$, then

$$R_1+R_2-R_3 > b+c.$$

E. G. Gotman, Matematika v škole, 1966, No. 1, 89.

12.9 $\min(h_a, h_b, h_c) \leqslant r_1+r_2+r_3 \leqslant \max(h_a, h_b, h_c)$.

S. I. Zetel', Zadači na maksimum i minimum, Moskva 1948, pp. 21–22.

REMARK. A consequence of these inequalities is

$$\min(h_a, h_b, h_c) \leqslant 3r \leqslant \max(h_a, h_b, h_c).$$

12.10 $(h_a-r_1)(h_b-r_2)(h_c-r_3) \geqslant 8r_1r_2r_3$.

Equality holds iff and only if the triangle is equilateral and P is its centre.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

12.11 $h_ah_bh_c \geqslant 27r_1r_2r_3$.

Equality holds iff and only if the triangle is equilateral and P is its centre.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

12.12 $\dfrac{h_a}{r_1}+\dfrac{h_b}{r_2}+\dfrac{h_c}{r_3} \geqslant 9$.

S. I. Zetel', Zadači na maksimum i minimum, Moskva 1948, p. 63.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

12.13 $R_1+R_2+R_3 \geqslant 2(r_1+r_2+r_3)$.

Equality holds iff and only if the triangle is equilateral and P is its centre.

Proof. Let L, M, N be the feet of the perpendiculars from P to BC, CA, AB respectively. Then

$$MN = (r_2^2+r_3^2+2r_2r_3\cos\alpha)^{1/2}, \quad MN = R_1 \sin\alpha.$$

We now have, in turn,

$$\begin{aligned} R_1+R_2+R_3 &= \sum \frac{(r_2^2+r_3^2+2r_2r_3\cos\alpha)^{1/2}}{\sin\alpha} \\ &= \sum \frac{[(r_2\sin\gamma+r_3\sin\beta)^2+(r_2\cos\gamma-r_3\cos\beta)^2]^{1/2}}{\sin\alpha} \\ &\geqslant \sum \frac{(r_2\sin\gamma+r_3\sin\beta)}{\sin\alpha} \\ &= \sum r_1\left(\frac{\sin\beta}{\sin\gamma}+\frac{\sin\gamma}{\sin\beta}\right) \\ &\geqslant 2(r_1+r_2+r_3). \end{aligned}$$

P. Erdős–L. J. Mordell, Problem 3740, Amer. Math. Monthly 42 (1935), 396 and 44 (1937), 252–254.

G. R. Veldkamp, Nieuw Tijdschr. Wisk. 45 (1957/58), 193–196.

L. J. Mordell, Math. Gaz. 46 (1962), 213–215.

12.14 Let P be any point in the plane of the triangle ABC. Then

$$R_1+R_2+R_3 \geqslant 6r.$$

M. Schreiber, Aufgabe 196, Jber. Deutsch. Math.-Verein. 45 (1935), 63 kursiv.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

L. Bankoff, Math. Mag. 39 (1966), 69.

12.15 In any non-equilateral triangle T, the line through the incentre I and perpendicular to the line joining I with the circumcentre O divides T and its boundary in two regions. For all points in that region which contains the vertex opposite the triangle's smallest side, we have

$$R_1+R_2+R_3 \geqslant 2(r_1+r_2+r_3) \geqslant 6r.$$

In the second region,

$$R_1+R_2+R_3 \geqslant 6r \geqslant 2(r_1+r_2+r_3),$$

while on the dividing line,

$$R_1+R_2+R_3 \geqslant 2(r_1+r_2+r_3) = 6r.$$

J. Steinig, Acta Math. Acad. Sci. Hungar. 16 (1965), 19–22.

12.16 $$R_1+R_2+R_3 \geqslant r_1\left(\frac{b}{c}+\frac{c}{b}\right)+r_2\left(\frac{a}{c}+\frac{c}{a}\right)+r_3\left(\frac{a}{b}+\frac{b}{a}\right) \geqslant 2(r_1+r_2+r_3).$$

Short and elegant proofs were given by Veldkamp and by Eggleston.

Proof. Let P' be the reflection of P in the internal bisector of angle BAC. Then P' is distant R_1 from A, r_3 from the side AC, and r_2 from side AB. The triangles ABP', ACP' do not meet except along their common side AP'. We have

$$\begin{aligned}\tfrac{1}{2}aR_1 &\geqslant \text{area } ABP'C\\ &= \text{area } ABP' + \text{area } ACP'\\ &= \tfrac{1}{2}r_2c+\tfrac{1}{2}r_3b.\end{aligned}$$

Hence

$$R_1 \geqslant r_2\left(\frac{c}{a}\right)+r_3\left(\frac{b}{a}\right),$$

and equality holds if and only if AP' is perpendicular to BC, that is, if

$$\measuredangle BAP = \measuredangle ABC + \measuredangle BAC - \frac{\pi}{2}.$$

By reflecting P in the internal bisectors of angles CBA and BCA we obtain

$$R_2 \geqslant r_1\left(\frac{c}{b}\right)+r_3\left(\frac{a}{b}\right),\quad R_3 \geqslant r_1\left(\frac{b}{c}\right)+r_2\left(\frac{a}{c}\right).$$

Equality holds in the latter of these inequalities if and only if

$$\measuredangle PBA = \measuredangle ABC + \measuredangle BAC - \frac{\pi}{2}.$$

Thus if equality occurs in the first and third inequality then $AP = BP$, and equality occurs in all three if and only if P is the circumcentre of a triangle ABC. Adding these inequalities we obtain

$$R_1+R_2+R_3 \geqslant r_1\left(\frac{b}{c}+\frac{c}{b}\right)+r_2\left(\frac{c}{a}+\frac{a}{c}\right)+r_3\left(\frac{a}{b}+\frac{b}{a}\right).$$

Since $b/c + c/b \geqslant 2$, etc., we have

$$r_1\left(\frac{b}{c}+\frac{c}{b}\right)+r_2\left(\frac{c}{a}+\frac{a}{c}\right)+\left(\frac{a}{b}+\frac{b}{a}\right) \geqslant 2(r_1+r_2+r_3)$$

and equality holds if and only if $a = b = c$ and P is the centre of the triangle ABC.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

D. K. Kazarinoff, Michigan Math. J. 4 (1957), 97–98.

G. R. Veldkamp, Nieuw Tijdschr. Wisk. 45 (1957/58), 193–196.

H. G. Eggleston, Math. Gaz. 42 (1958), 54–55.

Z. A. Skopec, Matematičeskoe prosveščenie 5 (1960), 151–152.

12.17 Let P be a point within the triangle ABC and let D, E, F be its orthogonal projections on sides BC, CA, AB respectively, then

$$R_1+R_2+R_3 \geqslant r_1\left(\frac{FD\cdot PC}{DE\cdot PB}+\frac{DE\cdot PB}{FD\cdot PC}\right)+r_2\left(\frac{DE\cdot PA}{FE\cdot PC}+\frac{FE\cdot PC}{DE\cdot PA}\right)$$

$$+r_3\left(\frac{FD\cdot PA}{FE\cdot PB}+\frac{FE\cdot PB}{FD\cdot PA}\right)$$

$$\geqslant 2(r_1+r_2+r_3).$$

L. Bankoff, Amer. Math. Monthly 65 (1958), 521.

REMARK. This theorem and 12.16 interpolate an expression 12.13.

12.18 $R_1+R_2+R_3 \geqslant 2\sqrt{F\sqrt{3}}.$ (1)

Equality occurs if and only if the triangle is equilateral and P is its centre.

PROOF. Let $BT \| CS \| PA$ and $BT = CS = PA$, $CR \| AT \| PB$ and $CR = AT = PB$, $AS \| BR \| PC$ and $AS = BR = PC$. Then the hexagon $ATBRCS$ has the area $2F$ and the perimeter $2(R_1+R_2+R_3)$. Between all hexagons of given area the regular hexagon has the smallest perimeter. Therefore

$$R_1+R_2+R_3 \geqslant 2\sqrt{F\sqrt{3}}.$$

The hexagon $ATBRCS$ will be regular only if P is the centre of an equilateral triangle.

U. T. Bödewadt, Jber. Deutsch. Math.-Verein. 46 (1936), 7 kursiv.

REMARK 1. Inequality also holds when the point lies anywhere in the plane of the triangle considered.

REMARK 2. Inequality (1) is stronger than **12.14**, because (see: **5.11**)

$$6r \leqslant 2\sqrt{F\sqrt{3}}.$$

12.19 $aR_1+bR_2+cR_3 \geqslant 2(ar_1+br_2+cr_3).$

G. Steensholt, Amer. Math. Monthly 63 (1956), 571–572.

REMARK. J. Schopp has proved a similar result for n-dimensional simplex. See Amer. Math. Monthly 66 (1959), 896–897.

12.20 $R_1 \sin \dfrac{\alpha}{2} + R_2 \sin \dfrac{\beta}{2} + R_3 \sin \dfrac{\gamma}{2} \geqslant r_1+r_2+r_3.$ (1)

Equality occurs if and only if P is the incentre of the triangle.

PROOF. Since

$$r_1 = R_3 \sin \measuredangle PCB \quad \text{and} \quad r_2 = R_3 \sin \measuredangle PCA,$$

we have

$$r_1+r_2 = R_3(\sin \measuredangle PCB+\sin \measuredangle PCA) \leqslant 2R_3 \sin \frac{\gamma}{2}. \qquad (2)$$

By virtue of two analogous relations valid for r_2+r_3 and r_3+r_1, we obtain the inequality (1).

Equality in (2) holds if and only if $\sphericalangle PCB = \sphericalangle PCA$, i.e., only if P is a point of the bisector of the angle ACB. Therefore in (1) equality holds if and only if P is the incentre of the triangle.

L. Carlitz, Elem. Math. 21 (1966), 115.

12.21 $R_2R_3+R_3R_1+R_1R_2 \geqslant 4(r_2r_3+r_3r_1+r_1r_2)$.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

12.22 $R_2R_3+R_3R_1+R_1R_2$

$$\geqslant (r_1+r_2)(r_3+r_1)+(r_2+r_3)(r_1+r_2)+(r_3+r_1)(r_2+r_3).$$

Equality holds only for the equilateral triangle, the point P being its centre.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

A. Oppenheim, Ann. Math. Pura Appl. (4) 53 (1961), 157–164.

12.23 $R_1^{-1}+R_2^{-1}+R_3^{-1} \leqslant 2^{-1}(r_1^{-1}+r_2^{-1}+r_3^{-1})$.

Equality holds if and only if the triangle is equilateral and if P is its centre.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

12.24 $R_1^t+R_2^t+R_3^t < 2(r_1^t+r_2^t+r_3^t)$,

for $t < -1$.

$$R_1^t+R_2^t+R_3^t \leqslant 2^t(r_1^t+r_2^t+r_3^t),$$

for $-1 \leqslant t \leqslant 0$.

$$R_1^t+R_2^t+R_3^t \geqslant 2^t(r_1^t+r_2^t+r_3^t),$$

for $0 \leqslant t \leqslant 1$.

$$R_1^t+R_2^t+R_3^t > 2(r_1^t+r_2^t+r_3^t),$$

for $t > 1$.

Equality occurs only for the equilateral triangle, the point P being its centre.

A. Florian, Elem. Math. 13 (1959), 55–58.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

12.25 $R_1R_2R_3 \geqslant 8r_1r_2r_3$.

J. M. Child, Math. Gaz. 23 (1939), 138–143.
J. Berkes, Elem. Math. 12 (1957), 121–123.
A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

12.26 $R_1R_2R_3S \geqslant r_1r_2r_3$,

with $S = \sin\alpha/2 \sin\beta/2 \sin\gamma/2$.
Equality holds if and only if P coincides with I.
O. Bottema, Nieuw Tijdschr. Wisk. 53 (1965), 50–51.

Remark. Since $S \leqslant \frac{1}{8}$ (see: **2.12**), this inequality improves **12.25**.

12.27 $R_1R_2R_3 \geqslant (r_2+r_3)(r_3+r_1)(r_1+r_2)$.

Equality holds if and only if the triangle is equilateral and P being its centre.
A. Oppenheim, Problem E 1433, Amer. Math. Monthly 67 (1960), 502 and 68 (1961), 380.
A. Oppenheim, Ann. Math. Pura Appl. (4) 53 (1961), 157–163.

12.28 $8SR_1R_2R_3 \geqslant (r_2+r_3)(r_3+r_1)(r_1+r_2)$, (1)

with $S = \sin\alpha/2 \sin\beta/2 \sin\gamma/2$.
Equality occurs if and only if P is incentre of the triangle.

Proof. Applying the law of sines and the law of cosines to the triangle AMN, where M and N are defined in **12.13**, we obtain

$$R_1^2 \sin^2\alpha = r_2^2+r_3^2+2r_2r_3\cos\alpha$$

$$= (r_2+r_3)^2\cos^2\frac{\alpha}{2} + (r_2-r_3)^2\sin^2\frac{\alpha}{2}$$

$$\geqslant (r_2+r_3)^2\cos^2\frac{\alpha}{2}$$

and so

$$2R_1\sin\frac{\alpha}{2} \geqslant r_2+r_3,$$

with equality only when $r_2 = r_3$. Hence (1).

In (1) equality holds if and only if $r_1 = r_2 = r_3$.

L. J. Mordell, Math. Gaz. 46 (1962), 213–215.

REMARK. Since $S \leqslant \frac{1}{8}$ (see: 2.12), this inequality improves 12.27.

12.29 $$r_1 r_2 r_3 \leqslant \frac{2}{27}\,\frac{F^2}{R}. \tag{1}$$

Equality holds if and only if point P is the centroid of the triangle.

PROOF. By

$$ar_1+br_2+cr_3 = 2F$$

and by arithmetic-geometric mean inequality, we have

$$\frac{ar_1}{2F}\cdot\frac{br_2}{2F}\cdot\frac{cr_3}{2F} \leqslant \frac{\left(\dfrac{ar_1}{2F}+\dfrac{br_2}{2F}+\dfrac{cr_3}{2F}\right)^3}{27} = \frac{1}{27},$$

where equality holds if and only if $ar_1 = br_2 = cr_3$, i.e.,

$$\text{area } PBC = \text{area } PCA = \text{area } PAB,$$

which is equivalent to the condition that P is the centroid of the triangle.

Since $abc = 4FR$, we obtain (1).

L. Carlitz-J. H. Tyrrell, Math. Mag. 37 (1964), 279.

12.30 $R_2R_3+R_3R_1+R_1R_2 \geqslant 2(r_1R_1+r_2R_2+r_3R_3)$.

Equality holds if and only if the triangle is equilateral and the point is its centre.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

12.31 $$r_1R_1+r_2R_2+r_3R_3$$
$$\geqslant\left(\frac{b}{c}+\frac{c}{b}\right)r_2r_3+\left(\frac{c}{a}+\frac{a}{c}\right)r_3r_1+\left(\frac{a}{b}+\frac{b}{a}\right)r_1r_2$$
$$\geqslant 2(r_2r_3+r_3r_1+r_1r_2).$$

Equality holds if and only if the triangle is equilateral and if P is its centre or if P is one of the vertices of the triangle.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

12.32 $$\frac{1}{r_2r_3}+\frac{1}{r_3r_1}+\frac{1}{r_1r_2}\geqslant 2\left(\frac{1}{r_1R_1}+\frac{1}{r_2R_2}+\frac{1}{r_3R_3}\right).$$

Equality holds if and only if the triangle is equilateral and if P is its centre.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

12.33 If $0<t<1$, then

$$(r_1R_1)^t+(r_2R_2)^t+(r_3R_3)^t\geqslant 2^t[(r_2r_3)^t+(r_3r_1)^t+(r_1r_2)^t].$$

Equality holds if and only if the triangle is equilateral and if P is its centre or if P is one of the vertices of the triangle.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

12.34 $$\frac{1}{r_1R_1}+\frac{1}{r_2R_2}+\frac{1}{r_3R_3}\geqslant 2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right).$$

Equality holds if and only if the triangle is equilateral and if P is its centre.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

12.35 $$\frac{1}{r_2r_3}+\frac{1}{r_3r_1}+\frac{1}{r_1r_2}\geqslant 4\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right).$$

Equality holds if and only if the triangle is equilateral and if P is its centre.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

12.36 $$\frac{R_1}{R_1+r_1}+\frac{R_2}{R_2+r_2}+\frac{R_3}{R_3+r_3}\geqslant 2.$$

J. Berkes, Elem. Math. 12 (1957), 121–123.

12.37 If x, y, z are arbitrary positive numbers, then

$$xR_1+yR_2+zR_3 \geqslant 2\left(\frac{yz(R_2+R_3)}{yR_2+zR_3}w_1+\frac{zx(R_3+R_1)}{zR_3+xR_1}w_2+\frac{xy(R_1+R_2)}{xR_1+yR_2}w_3\right).$$

Equality occurs if and only if $xR_1=yR_2=zR_3$ and the angles BPC, CPA, APB are $2\pi/3$.

A. Oppenheim, Ann. Math. Pura Appl. (4) 53 (1961), 157–163.

12.38 Let P be a point inside the triangle ABC. Let D, E, F be the points of intersection of the lines AP, BP, CP with the sides BC, CA, AB respectively. Then

$$\frac{AP}{PD}+\frac{BP}{PE}+\frac{CP}{PF}\geqslant 6.$$

Equality holds if and only if the point P is the centroid of the triangle.

PROOF. Let S_1, S_2, S_3 represent the areas of triangles PBC, PCA, PAB. Since the triangles ABC and PBC have the common side BC,

$$\frac{AD}{PD}=\frac{S_1+S_2+S_3}{S_1},$$

whence

$$\frac{AD-PD}{PD}=\frac{S_2+S_3}{S_1}, \text{ i.e. } \frac{PA}{PD}=\frac{S_2}{S_1}+\frac{S_3}{S_1}.$$

Analogously, one obtains

$$\frac{PB}{PE}=\frac{S_3}{S_2}+\frac{S_1}{S_2}, \quad \frac{PC}{PF}=\frac{S_1}{S_3}+\frac{S_2}{S_3}.$$

By adding together these equalities, we get

$$\frac{AP}{PD}+\frac{BP}{PE}+\frac{CP}{PF}$$
$$=\left(\frac{S_1}{S_2}+\frac{S_2}{S_1}\right)+\left(\frac{S_2}{S_3}+\frac{S_3}{S_2}\right)+\left(\frac{S_3}{S_1}+\frac{S_1}{S_3}\right)\geqslant 2+2+2=6.$$

O. J. Ramler, Problem E 1043, Amer. Math. Monthly 59 (1952), 697 and 60 (1953), 421.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

12.39 With the notations of **12.38**, we have

$$\frac{AP}{PD}\cdot\frac{BP}{PE}\cdot\frac{CP}{PF}\geqslant 8.$$

Equality holds if and only if the point P is the centroid of the triangle.

PROOF. By virtue of 12.38, we obtain

$$\frac{AP}{PD}\cdot\frac{BP}{PE}\cdot\frac{CP}{PF}=\frac{(S_2+S_3)(S_3+S_1)(S_1+S_2)}{S_1S_2S_3}\geqslant\frac{8S_1S_2S_3}{S_1S_2S_3}=8.$$

O. J. Ramler, Problem 1043, Amer. Math. Monthly 59 (1952), 697 and 60 (1953), 421.

Ch. W. Trigg Math. Mag. 36 (1963), 244.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

12.40 If P is a point inside the triangle ABC, and A', B', C' are the intersections of AP, BP, CP and the sides BC, CA, AB, respectively, then

$$\frac{AA'}{AP}+\frac{BB'}{BP}+\frac{CC'}{CP}\geqslant\frac{9}{2}.$$

Ž. Živanović Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181–No. 196 (1967), 69–72.

12.41 Let ABC be any triangle and let P be any point inside it; let AP, BP, CP produced meet the opposite sides of the triangle at A', B', C' respectively. Then at least one of the inequalities

$$\frac{AA'}{BC}\geqslant\frac{\sqrt{3}}{2},\quad\frac{BB'}{CA}\geqslant\frac{\sqrt{3}}{2},\quad\frac{CC'}{AB}\geqslant\frac{\sqrt{3}}{2}\tag{1}$$

holds.

This statement of G. N. Watson is included in the following theorem due to T. W. Chaundy:

That one of the above inequalities which involves the shortest of the three sides of the triangle is true.

Or in the following theorem of Watson:

That one of the above inequalities which involves the longest of the three transversals AA', BB', CC' is true.

$\sqrt{3}/2$ in (1) is the best possible constant.

G. N. Watson Quart. J. Math. (Oxford Ser.) 11 (1940), 273–276.

A. Rosenblatt offers an elementary proof of the theorem due to Watson (see Actas. Acad. Sci. Lima 4 (1941), 156–161 and 213–220).

REMARK. In fact, the priority of the theorem is due to J. J. L. Hinrichsen, who proposed in the Amer. Math. Monthly 39 (1932), 549, the following problem:

Given any triangle with three line segments concurrent in a point interior to the triangle and joining each vertex to its opposite side. The length of the longest of these three lines cannot be less than $\frac{1}{2}\sqrt{3}$ times the length of the opposite side of the triangle.

J. J. L. Hinrichsen, Problem 3576, Amer. Math. Monthly 39 (1932), 549 and 41 (1934), 193–194.

In the same Journal, 41 (1934), 193–196 a solution of this problem was published by F. Underwood and the editors of the Amer. Math. Monthly.

12.42 If $a \leqslant b \leqslant c$, then

$$R_1+R_2+R_3 > \frac{s}{c}(r_1'+r_2'+r_3').$$

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

12.43 $R_1+R_2+R_3 \geqslant 2(\sqrt{r_2'r_3'}+\sqrt{r_3'r_1'}+\sqrt{r_1'r_2'})$.

Equality occurs if and only if the triangle is equilateral and if P is its centre.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

12.44 $R_1^2+R_2^2+R_3^2 \geqslant 2(r_2'r_3'+r_3'r_1'+r_1'r_2')+6\sqrt[3]{(r_1'r_2'r_3')^2}$.

Equality holds if and only if the triangle is equilateral and if P is its centre.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

12.45 $R_2R_3+R_3R_1+R_1R_2 \geqslant r_2'r_3'+r_3'r_1'+r_1'r_2'+9\sqrt[3]{(r_1'r_2'r_3')^2}$.

Equality holds if and only if the triangle is equilateral and if P is its centre.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

12.46 $2(\sqrt{R_2R_3}+\sqrt{R_3R_1}+\sqrt{R_1R_2})$

$$\leqslant R_1+R_2+R_3+2(r_1'+r_2'+r_3').$$

Equality holds if and only if P is centroid of the triangle.
L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

12.47 $$\frac{R_2R_3}{r_2'r_3'}+\frac{R_3R_1}{r_3'r_1'}+\frac{R_1R_2}{r_1'r_2'} \geqslant 12.$$

Equality occurs if and only if the triangle is equilateral and P is its centre.
L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

12.48 $R_1+R_2+R_3 \geqslant 2(w_1+w_2+w_3)$.

Equality holds if and only if the triangle is equilateral and if P is its centre.
D. F. Barrow, Problem 3740, Amer. Math. Monthly 44 (1937), 252–254.

12.49 $R_2R_3+R_3R_1+R_1R_2 \geqslant 2(R_1w_1+R_2w_2+R_3w_3)$.

A. Oppenheim, Ann. Math. Pura Appl. (4) 53 (1961), 157–163.

12.50 $R_2R_3+R_3R_1+R_1R_2 \geqslant 4(w_2w_3+w_3w_1+w_1w_2)$.

Equality holds if and only if the triangle is equilateral and if P is its centre.
L. Carlitz, Math. Gaz. 47 (1964), 181–182.

12.51 $R_1R_2R_3 \geqslant 8w_1w_2w_3$.

Equality holds if and only if the triangle is equilateral and if P is its centre.
L. Carlitz, Math. Gaz. 47 (1964), 181–182.

12.52 $R_1R_2R_3 \geqslant (w_2+w_3)(w_3+w_1)(w_1+w_2)$.

Equality occurs if and only if the triangle is equilateral and if P is its centre.
A. Oppenheim, Ann. Math. Pura Appl. (4) 53 (1961), 157–163.

12.53 $R_1^2+R_2^2+R_3^2 \geqslant \frac{1}{3}(a^2+b^2+c^2)$.

Equality holds if and only if P coincides with G.
Encyklopädie der Mathematischen Wissenschaften, Bnd. III', Heft 6, p. 1118, and Heft 7, p. 1185.

R. Sturm, Maxima und minima in der elementaren Geometrie, Leipzig, Berlin 1910, p. 71.

T. Lalesco, La géométrie du triangle, Paris 1937, p. 41.

12.54 $$r_1^2+r_2^2+r_3^2 \geqslant \frac{4F^2}{a^2+b^2+c^2}.$$

Equality holds if P coincides with Lemoine's point.

Encyklopädie der Mathematischen Wissenschaften, Bnd. III', Heft 6, p. 1118, and Heft 7, p. 1185.

T. Lalesco, La géométrie du triangle, Paris 1937, p. 41.

O. Bottema, Hoofdstukken uit de elementaire meetkunde, Den Haag 1944, p. 91.

12.55 If no angle $\geqslant \frac{2}{3}\pi$, then

$$R_1+R_2+R_3 \geqslant \left(\frac{a^2+b^2+c^2}{2}+2F\sqrt{3}\right)^{1/2}.$$

If $\alpha \geqslant \frac{2}{3}\pi$, then

$$R_1+R_2+R_3 \geqslant b+c.$$

Equality holds if P coincides with Torricelli's point.

Encyklopädie der Mathematischen Wissenschaften, Bnd. III', Heft 6, p. 1118, and Heft 7, p. 1185.

T. Lalesco, La géométrie du triangle, Paris 1937, p. 41.

12.56 $$a_1R_1+b_1R_2+c_1R_3$$

$$\geqslant \left[\frac{a^2(-a_1^2+b_1^2+c_1^2)+b^2(a_1^2-b_1^2+c_1^2)+c^2(a_1^2+b_1^2-c_1^2)}{2}+8FF_1\right]^{1/2},$$

where a_1, b_1, c_1, F_1 are sides and area of a second triangle $A_1B_1C_1$.

The triangles ABC and $A_1B_1C_1$ are interchangeable.

O. Bottema, Hoofdstukken uit de elementaire meetkunde, Den Haag 1944, p. 97–99.

13. Necessary and sufficient conditions for the existence of a triangle

13.1 A necessary and sufficient condition for a triangle to exist is

$$\max(a, b, c) < \tfrac{1}{2}(a+b+c).$$

13.2 If $a = \max(a, b, c)$, then a necessary but insufficient condition for a, b, c to be the sides of a triangle is

$$\tfrac{1}{2} a^2 < b^2+c^2 < 2a^2.$$

13.3 Let $f(x)$ be any non-negative, non-decreasing, subadditive function on the domain $x > 0$; for instance $f(x) = \sqrt[n]{x}$ (n a natural number).

If a, b, c form a triangle, then $f(a)$, $f(b)$, $f(c)$ form a triangle.

PROOF. From

$$a \leqslant b+c,$$

we have

$$f(a) \leqslant f(b+c) \leqslant f(b)+f(c).$$

Cyclic permutation of a, b, c then yields the additional inequalities

$$f(b) \leqslant f(c)+f(a), \quad f(c) \leqslant f(a)+f(b),$$

therefore $f(a)$, $f(b)$, $f(c)$ form a triangle.

This proof is due to J. L. Brown, Jr. Amer. Mathem. Monthly 67 (1960), 82–83.

REMARK. The theorem is a generalization of Problem E 1366, Amer. Math. Monthly 66 (1959) 423, and 67 (1960), 82–83, proposed by V. E. Hoggatt, Jr., where the case $f(x) = \sqrt{x}$ is considered.

13.4 Let p, q be real numbers such that $p+q = 1$. Then a triangle with the sides a, b, c exists if and only if

$$pa^2+qb^2 > pqc^2 \text{ for all } p, q. \tag{1}$$

PROOF. A triangle with sides a, b, c, exists if and only if

$$a+b-c > 0, \; b+c-a > 0, \; c+a-b > 0. \tag{2}$$

R. Sturm, Maxima und minima in der elementaren Geometrie, Leipzig, Berlin 1910, p. 71.

T. Lalesco, La géométrie du triangle, Paris 1937, p. 41.

12.54 $$r_1^2+r_2^2+r_3^2 \geqslant \frac{4F^2}{a^2+b^2+c^2}.$$

Equality holds if P coincides with Lemoine's point.

Encyklopädie der Mathematischen Wissenschaften, Bnd. III', Heft 6, p. 1118, and Heft 7, p. 1165.

T. Lalesco, La géométrie du triangle, Paris 1937, p. 41.

O. Bottema, Hoofdstukken uit de elementaire meetkunde, Den Haag 1944, p. 91.

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If $\alpha \geqslant \frac{2}{3}\pi$, then

$$R_1+R_2+R_3 \geqslant b+c.$$

Equality holds if P coincides with Torricelli's point.

Encyklopädie der Mathematischen Wissenschaften, Bnd. III', Heft 6, p. 1118, and Heft 7, p. 1165.

T. Lalesco, La géométrie du triangle, Paris 1937, p. 41.

12.56 $a_1R_1+b_1R_2+c_1R_3$

$$\geqslant \left[\frac{a^2(-a_1^2+b_1^2+c_1^2)+b^2(a_1^2-b_1^2+c_1^2)+c^2(a_1^2+b_1^2-c_1^2)}{2} + 8FF_1\right]^{1/2},$$

where a_1, b_1, c_1, F_1 are sides and area of a second triangle $A_1B_1C_1$.

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13.4 Let p, q be real numbers such that $p+q = 1$. Then a triangle with the sides a, b, c exists if and only if

$$pa^2+qb^2 > pqc^2 \text{ for all } p, q. \tag{1}$$

PROOF. A triangle with sides a, b, c, exists if and only if

$$a+b-c > 0, \; b+c-a > 0, \; c+a-b > 0. \tag{2}$$

We shall prove that (2) is equivalent to (1). Let

$$K = pa^2+qb^2-pqc^2.$$

Since $q = 1-p$, we have

$$K = pa^2+(1-p)b^2-p(1-p)c^2$$
$$= c^2p^2+(a^2-b^2-c^2)p+b^2,$$

where a, b, c are fixed and p is a variable. Therefore K is a quadratic trinomial in respect to p. In order to have $K > 0$, it is necessary that

$$D = (a^2-b^2-c^2)^2-4b^2c^2 < 0,$$

i.e.

$$(a+b+c)(b+c-a)(c+a-b)(a+b-c) > 0. \qquad (3)$$

If the triangle exists, then (2) holds, so that $D < 0$. Therefore if a, b, c are the sides of a triangle, then (1) is valid.

Inversely, if $D < 0$, then (3) holds, i.e. a, b, c are sides of a triangle.

V. B. Lidskiĭ, L. V. Ovsjannikov, A. N. Tulaikov and M. I. Šabunin, Zadači po elementarnoĭ matematike, Moskva 1962, p. 21.

13.5 Let α, β, γ be the angles of a triangle. Then a triangle with sides $\cos^2\alpha/2$, $\cos^2\beta/2$, $\cos^2\gamma/2$ exists.

T. P. Čerepanova, Matematika v škole 1966, No. 6, 66.

13.6 Let α, β, γ be the angles of a triangle. Then a triangle whose sides are $\cos\alpha/2$, $\cos\beta/2$, $\cos\gamma/2$ exists.

Matematika v škole 1963, No. 3, 89.

REMARK. **13.6** follows from **13.5** by means of **13.3**.

13.7 G. Petrov has given a series of the necessary and sufficient conditions which must be satisfied by certain elements of a triangle:

1°. $0 < w_c < \dfrac{2ab}{a+b}$.

2°. $a+2b-2m_a > 0$, $a-2b+2m_a > 0$, $-a+2b+2m_a > 0$.

3°. $2r < h_b \leqslant a$.

4°. $8Rw_a \geqslant a^2+4w_a^2$ for $a \leqslant 2w_a$,

$2R > a$ for $a > 2w_a$.

5°. $a < b+c$, $2m_a < b+c$, $(b+c)^2 \leqslant a^2+4m_a^2$.

6°. $2R \geqslant a$, $2R \geqslant \dfrac{a^2+4h_a^2}{4h_a}$.

7°. $r > 0$, $4r^6+4(2a^2+2b^2+7ab)r^4$

$$+4(a^4-a^3b-a^2b^2-ab^3+b^4)r^2-a^2b^2(a-b)^2 \leqslant 0.$$

G. Petrov, Časopis pro peštovani matematiky 77 (1952), 77–92.

13.8 A necessary and sufficient condition for the existence of a triangle with the elements R, r and s, is

$$s^4-2(2R^2+10Rr-r^2)s^2+r(4R+r)^3 \leqslant 0.$$

R. Sondat, Nouv. Ann. Math. (3) 10 (1891), 43*–47*.

14. Miscellaneous inequalities for the elements of a triangle

14.1 If λ_1, λ_2, λ_3 are real numbers, then

$$(\lambda_1+\lambda_2+\lambda_3)^2R^2 \geqslant \lambda_2\lambda_3a^2+\lambda_3\lambda_1b^2+\lambda_1\lambda_2c^2. \tag{1}$$

REMARK 1. Taking for λ_1, λ_2, λ_3 special values we deduce from (1) the following inequalities:

4.4 for $\lambda_1 = a^2$, $\lambda_2 = b^2$, $\lambda_3 = c^2$.

4.7 for $\lambda_1 = a(4s^2-a^2-b^2-c^2-2sa)$,

$\lambda_2 = b(4s^2-a^2-b^2-c^2-2sb)$,

$\lambda_3 = c(4s^2-a^2-b^2-c^2-2sc)$.

5.1 for $\lambda_1 = a$, $\lambda_2 = b$, $\lambda_3 = c$.

5.13 for $\lambda_1 = 2s-3a$, $\lambda_2 = 2s-3b$, $\lambda_3 = 2s-3c$ and for

$\lambda_1 = \lambda_2 = \lambda_3 = 1$.

5.25 for $\lambda_1 = 2s-a$, $\lambda_2 = 2s-b$, $\lambda_3 = 2s-c$.

5.7 for $\lambda_1 = a(s-a)$, $\lambda_2 = b(s-b)$, $\lambda_3 = c(s-c)$.

REMARK 2. O. Bottema has given the following remarks:

1°. $(\lambda_1+\lambda_2+\lambda_3)^2R^2-(\lambda_2\lambda_3a^2+\lambda_3\lambda_1b^2+\lambda_1\lambda_2c^2)$ has a geometrical meaning: it is equal to $2\lambda^2GO^2$ ($\lambda \equiv \lambda_1+\lambda_2+\lambda_3 \neq 0$) in which G is the centroid of the points A, B, C if the weights λ_1, λ_2, λ_3 are given to them. From this the inequality (1) follows.

2°. Equality holds in (1) if and only if G and O coincide. If λ_1, λ_2, λ_3 are given a triangle for which this occurs exists if and only if a triangle exists, the sides of which are $|\lambda_1|$, $|\lambda_2|$, $|\lambda_3|$.

O. Kooi, Simon Stevin 32 (1958), 97–101.

O. Bottema, Simon Stevin 33 (1959), 97–100.

14.2 $2F \leqslant a^2-ab+b^2$, $4F \leqslant a^2+b^2$, $8F \leqslant (a+b)^2$ hold for each triangle.

Equalities hold only for the right isosceles triangle.

Z. A. Skopec and V. A. Žarov, Zadači i teoremi po geometrii, Moskva 1962, pp. 81–82.

14.3 If $s^2 = \lambda ab$, where $\lambda > 1$, then

$$s < \lambda a, \quad s < \lambda b, \quad c < (\lambda-1)a, \quad c < (\lambda-1)b.$$

W. O. Pennell and A. L. Epstein, Problem E 1068, Amer. Math. Monthly 61 (1954), 49.

14.4 If $a \geqslant b \geqslant c$, then

1°. $\frac{2}{3}s \leqslant a < s$, $\frac{1}{2}s < b < s$, $0 < c \leqslant \frac{2}{3}s$.

2°. $\pi/3 \leqslant \alpha < \pi$, $0 < \beta < \pi/2$, $0 < \gamma \leqslant \pi/3$.

3°. $0 < h_a \leqslant \frac{1}{3}s\sqrt{3}$, $0 < h_b \leqslant \frac{1}{3}s\sqrt{3}$, $\frac{1}{3}s\sqrt{3} \leqslant h_c < s$.

4°. $0 < w_a \leqslant \frac{1}{3}s\sqrt{3}$, $0 < w_b < \frac{2}{3}s$, $\frac{1}{3}s\sqrt{3} \leqslant w_c < s$.

5°. $0 < m_a < \frac{1}{3}s\sqrt{3}$, $\frac{1}{2}s < m_b < \frac{3}{4}s$, $\frac{1}{3}s\sqrt{3} \leqslant m_c \leqslant s$.

6°. $\frac{2}{9}s\sqrt{3} \leqslant R < +\infty$.

7°. $0 < r \leqslant \frac{1}{9}s\sqrt{3}$.

V. M. Černov, Matematika v škole, 1958 No. 3, 91–95.

14.5 If $a \geqslant b \geqslant c$, then

$$a + h_a \geqslant b + h_b \geqslant c + h_c.$$

A. V. Aljaev, Matematika v škole, 1963, No. 3, 88 and 1964, No. 2, 74.

14.6 If $b = \text{med}(a, b, c)$, then

$$\frac{s^2}{9} + r^2 \gtrless 2Rr,$$

for $c + a \lesseqgtr 2b$.

S. G. Guba, Matematika v škole, 1965, No. 5, 69 and 1966, No. 4, 77–78.

14.7 If a is the basis of an isosceles triangle and w the angle-bisector of an angle on the basis, then

$$\tfrac{2}{3}a < w < a\sqrt{2}.$$

Matematika v škole, 1963, No. 2, 88.

14.8 $a \geqslant 2h_a \operatorname{tg} \alpha/2.$

Matematika v škole, 1965, No. 4, 72.

14.9 If a triangle is one of the angles 4φ, then

$$\frac{F}{s^2} \leqslant \frac{2 \operatorname{tg} \varphi (1 - \operatorname{tg} \varphi)}{(1 + \operatorname{tg} \varphi)^2}.$$

O. Bottema, Wisk. Opgaven 20 (19[illegible]), 4.

14.10 Let ρ_1, ρ_2, ρ_3 be the radii of the circles which touch two sides and the circumcircle of a given triangle. Then

$$4r \leqslant \rho_1 + \rho_2 + \rho_3 \leqslant 2R.$$

Remark. The first inequality is due to K. V. Vetrov and the second to S. T. Berkolaiko. See: K. V. Vetrov, Matematika v škole, 1966, No. 3, 59 and S. T. Berkolaiko, Matematika v škole, 1967, No. 2, 74.

14.11 If A_1, B_1, C_1 are points on the sides BC, CA, AB of a triangle, respectively, then

$$\frac{1}{2} < \frac{AA_1+BB_1+CC_1}{a+b+c} < \frac{3}{2}.$$

Gaz. Mat. B 8 (1957), 163.

14.12 Let λ be a positive number. Let X and Y denote given points on the sides of a triangle ABC, such that

$$\frac{\sphericalangle XAB}{\sphericalangle CAB} = \frac{\sphericalangle YBA}{\sphericalangle CBA} = \lambda.$$

Then the following equivalences hold:

$$AX > BY \Leftrightarrow AC > BC,$$

for $0 < \lambda < 1$,

$$CY > CX \Leftrightarrow AC > BC,$$

for $0 < \lambda < 1$,

$$AY > BX \Leftrightarrow AC > BC,$$

for $0 < \lambda < \frac{1}{2}$.

C. Mak, Problem E 1626 Amer. Math. Monthly 70 (1963), 891 and 71 (1964), 684–686.

14.13 Let P be a point inside a triangle ABC. Let A', B', C' be the intersections of the lines AP, BP, CP with BC, CA, AB respectively. Then, if $AA' \geqslant BB'$ and $AA' \geqslant CC'$, we have

$$AA' \geqslant PA'+PB'+PC',$$

with equality only for $AA' = BB' = CC'$.

14.14 $OH \geqslant IH\sqrt{2}$.

Equality holds if and only if the triangle is equilateral.

S. G. Guba, Matematika v škole, 1966, No. 3, 60.

14.15
$$\frac{\cos^2\beta+\cos^2\gamma}{\cos\beta+\cos\gamma} + \frac{\cos^2\gamma+\cos^2\alpha}{\cos\gamma+\cos\alpha} + \frac{\cos^2\alpha+\cos^2\beta}{\cos\alpha+\cos\beta} \geqslant 1+\frac{r}{R}. \qquad (1)$$

PROOF. By adding together the relations

$$x^2+y^2 \geqslant 2xy \text{ and } x^2+y^2 = x^2+y^2,$$

we get

$$2(x^2+y^2) \geqslant (x+y)^2.$$

Whence, for $x+y > 0$,

$$\frac{x^2+y^2}{x+y} \geqslant \frac{x+y}{2},$$

i.e.

$$\frac{y^2+z^2}{y+z} + \frac{z^2+x^2}{z+x} + \frac{x^2+y^2}{x+y} \geqslant x+y+z. \tag{2}$$

Since $\cos\beta+\cos\gamma > 0$, etc., for $x = \cos\alpha$, $y = \cos\beta$, $z = \cos\gamma$, from (2) follows (1), because $\cos\alpha+\cos\beta+\cos\gamma = 1+r/R$.

Gaz. Mat. B 13 (1962), 133–134.

14.16 If m_b and m_c are perpendicular to each other, then

$$\operatorname{cotg}\beta+\operatorname{cotg}\gamma \geqslant \tfrac{2}{3}.$$

S. Reich, Problem E 1839, Amer. Math. Monthly 72 (1965), 1129.

14.17 Let P be a point on the median m_a of a triangle ABC. Let BP and CP meet the sides AC and AB in the points E and F respectively. Then, if $AB > AC$, we have $BE > CF$.

K. Tan and N. Harrell, Math. Mag. 38 (1965), 57–58.

14.18 Let a square with the side a be inscribed in a triangle. Then

$$r\sqrt{2} < a < 2r.$$

14.19 If a square lies inside a triangle, the area of the square does not exceed half the area of the triangle.

D. J. Newman, Problem E 1425, Amer. Math. Monthly 67 (1960), 593 and 68 (1961), 180.

14.20 Let P be a fixed point in a plane and consider in the plane the set of triangles ABC with

$$PA^2+PB^2+PC^2 = 3c^2,$$

where e is a constant. Then $BC \cdot CA \cdot AB$ is maximum if and only if ABC is an equilateral triangle with its vertices on the circle with centre P and radius e.

A. C. Aitken, Proc. Edinburgh Math. Soc. (2) 13 (1962/63) 173–174.

14.21 If D, E, F are the intersections of AI, BI, CI with the sides BC, CA, AB respectively, then

$$AI > ID, \ BI > IE, \ CI > IF.$$

PROOF. Since the angle-bisectors are cevians, we have

$$\frac{AI}{ID} = \frac{AF}{FB} + \frac{AE}{EC} = \frac{b}{a} + \frac{c}{a} = \frac{b+c}{a} > 1.$$

Tien-Hsung Lin, Math. Mag. 38 (1965), 158–159.
Ch. W. Trigg, Math. Mag. 40 (1967), 28.

14.22 Let $d_a \leqslant d_b \leqslant d_c$, where d_a, d_b, d_c are the distances of the circumcentre of a triangle to its excentres. Then

$$R < d_a < 2R, \ R < d_b < R\sqrt{5}, \ 2R < d_c < 3R.$$

H. J. Baron, Tôhoku Math. J. 48 (1941), 185–192.

14.23 Let the incircle of a right triangle touch its hypotenuse at N and one of the sides of triangle at M. Then

$$MN \leqslant \frac{2\sqrt{3}}{9} c,$$

where c is the hypotenuse.

E. A. Bokov, Matematika v škole, 1963, No. 4, 86 and 1964, No. 4, 78.

14.24 If R_1, R_2, R_3 are the radii of the circles inscribed in the sectors AOB, BOC, COA, respectively, then

$$\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \geqslant \frac{3+2\sqrt{3}}{R}.$$

Ju. I. Gerasimov, Matematika v škole, 1967, No 3, 84.

14.25 Let T and T' denote Brocard's points in a triangle, then

$$2 \cdot TT' \leqslant R.$$

T. Lalesco, La géométrie du triangle, Paris 1937, p. 120.
V. G. Cavallaro, Mathesis 53 (1939), 155–160.

14.26 Let M and N be two points one unit apart. With M and N as centres and with unit radii draw arcs ANB and AMB. Let Q be any point on arc AMB and P_1 and P_2 any points on arc ANB such that N is the midpoint of arc P_1P_2. Then

$$QP_1+QP_2 \leqslant 2 \leqslant QP_1^2+QP_2^2.$$

A. W. Goodman, Amer. Math. Monthly 58 (1951), 564.

14.27 If d denotes the distance between the circumcentre and the incentre of a triangle, then

$$(R-d)(3R+d)^3 \leqslant 4R^2s^2 \leqslant (R+d)(3R-d)^3.$$

Equalities hold if and only if the triangle is equilateral.
O. Bottema, Nieuw Arch. Wisk. 13 (1965), 246.

14.28 In any triangle ABC

$$\omega \leqslant \frac{\pi}{6}$$

in which ω denotes Brocard's angle.
Equality holds only for the equilateral triangle.

Proof. Inequality follows from

$$\operatorname{cotg} \omega = \operatorname{cotg} \alpha + \operatorname{cotg} \beta + \operatorname{cotg} \gamma$$

by means of **2.38**.
T. Lalesco, La géométrie du triangle, Paris 1937, p. 120.

14.29 Let P be an interior point of the triangle ABC. Let L, M, N be the points in which the lines AP, BP, CP intersect the sides BC, CA, AB respectively. Then

$$8 \cdot AM \cdot BN \cdot CL \leqslant abc.$$

Equality holds if and only if P is the centroid of the triangle.

S. I. Zetel', Novaja geometrija treugol'nika, Moskva 1962, pp. 12–13.

14.30 Let P be an interior point of the triangle ABC. The lines through the midpoints of the segments AP, BC, CP parallel to BC, CA, AB, respectively form, with the sides of ABC, the triangles with areas F_A, F_B, F_C. Then

$$F_A+F_B+F_C \geqslant \tfrac{1}{3}F.$$

Equality holds if and only if P is the centroid of the triangle ABC.

S. I. Zetel', Novaja geometrija treugol'nika, Moskva 1962, pp. 32–33.

14.31 If ABC and $A'B'C'$ are any two equilateral triangles in a plane, their vertices being taken in the same sense of rotation, of the three lines AA', BB', CC', the sum of any two is not less than the third.

The same theorem holds for any pair of directly similar triangles. There is equality if and only if the centre of similitude lies on the circumcircle of one (and therefore of both) triangles.

C. Tweedie, Edin. M. S. Proc. 22 (1904), 22–26.

P. Pinkerton, Edin. M. S. Proc. 22 (1904), 27.

15. Inequalities for quadrilaterals

15.1 $s < p+q < 2s.$

F. A. Andreev, Matematika v škole 1953, No. 1, 58–69.

15.2 $s(p^t+q^t) > p^{t+1}+q^{t+1},$

where t is a positive number.

REMARK. This is a generalization of the problem proposed in Gaz. Mat. 12 (1961), 175.

15.3 For any convex quadrilateral there exists at least one side which is smaller than the greater among its diagonals.

15.4 $AC\cdot BD \leqslant AD\cdot BC+AB\cdot CD.$

PROOF. By the inversion with centre A and power $AB\cdot AC\cdot AD$ the points B, C, D are transformed into B_1, C_1, D_1 so that $B_1C_1 = AD\cdot BC$ etc. Theorem follows from the triangle inequality for B_1, C_1, D_1.

Equality holds if and only if A, B, C and D are on a circle.

15.5 If A, B, C and D are arbitrary points, $BC = a$, $CA = b$, $AB = c$, $AD = p$, $BD = q$, $CD = r$, then

$$(-ap+bq+cr)(ap-bq+cr)(ap+bq-cr) \geqslant 0.$$

Equality only if the four points are on a circle.

COROLLARY 1. If $BC = CA = AB$, then

$$(-p+q-r)(p-q+r)(p+q-r) \geqslant 0 \quad \text{(Pompeiu's theorem)}.$$

COROLLARY 2. If the triangles ABC and PQR are given a point D exists with the property $AD:BD:CD = p:q:r$ if, and only if

$$(-ap+bq+cr)(ap-bq+cr)(ap+bq-cr) \geqslant 0.$$

The two triangles may be interchanged.

O. Bottema, Euclides 38(1963/64), 129-137.

15.6 $$\frac{\max(a, b, c, d, p, q)}{\min(a, b, c, d, p, q)} \geqslant \sqrt{2}.$$

REMARK. This theorem has been a problem at the competition for secondary school students in Hungary in 1961.

15.7 $l^2-16F \geqslant 0.$

Equality only for a square.

REMARK. This is the isoperimetric inequality for quadrilaterals. See, for example, N. D. Kazarinoff, Geometric Inequalities, New York 1961, p. 51.

15.8 $4F \leqslant (a+c)(b+d).$

Equality only for a rectangle.

PROOF. Since

$$4F = ad\cdot\sin\alpha + ab\cdot\sin\beta + bc\cdot\sin\gamma + cd\cdot\sin\delta,$$

and

$$\sin\alpha,\ \sin\beta,\ \sin\gamma,\ \sin\delta \leqslant 1,$$

we obtain

$$4F \leqslant ab+ad+bc+cd = (a+c)(b+d).$$

15.9 If x and y are the segments connecting the midpoints of the opposite sides of a quadrilateral and if z is the segment connecting the midpoints of the diagonals, then

$$x^2+y^2+z^2 \geqslant 2F.$$

Ju. I. Gerasimov, Matematika v škole 1967, No. 1, 83.

15.10 If P and Q denote the midpoints of the sides BC and AD, then

$$|AB-CD| < 2PQ \leqslant AB+CD. \qquad (1)$$

PROOF. Consider the triangle MPQ, M being the midpoint of AC. Then

$$AB+CD = 2(PM+QM) \geqslant 2PQ.$$

Similarly, we have

$$|AB-CD| = 2|PM-QM| < 2PQ.$$

Equality in (1) holds if and only if AB is parallel to CD.

J. I. Nassar, Problem E 1617, Amer. Math. Monthly 70 (1963), 758.

15.11 $\dfrac{a}{c}+\dfrac{c}{a} > \dfrac{p}{q}+\dfrac{q}{p}$

unless $c = a$.

C. V. Durell and A. Robson, Advanced Trigonometry, London 1948, p. 281.

15.12 If $AB+BD = AC+CD$, then $AB < AC$.

PROOF. Adding the inequalities

$$PA+PB > AB, \ PC+PD > CD,$$

because $AB+BD = AC+CD$, we have

$$AC+BD > AB+AB+BD-AC,$$

i.e.,

$$AC > AB.$$

15.13 If P be the intersection of diagonals and if the distances of P from the sides AB, BC, CD, DA are x, y, z and t respectively, then

$$2(x+y+z+t) < 3s.$$

PROOF. Let the bisectors of the angles between the diagonals AC and BD meet AB, BC, CD, DA in R, S, T, U.

By a corollary of the Erdős-Mordell theorem (see: 12.13)

$$2(PS + PT) < PB + PC + PD$$
$$2(PT + PU) < PC + PD + PA$$
$$2(PU + PR) < PD + PA + PB$$
$$2(PR + PS) < PA + PB + PC$$

or

$$4(PR+PS+PT+PU) < 3(PA+PB+PC+PD).$$

This inequality is stronger than the one proposed because

$$x+y+z+t \leqslant PR+PS+PT+PU$$

and

$$PA+PB+PC+PD < a+b+c+d.$$

H. Demir-L. Bankoff, Math. Mag. 39 (1966), 306 and 40 (1967), 166.

15.14 Let $ABCD$ be a convex quadrilateral and let P be the intersection of diagonals AC and BD. Let the lines through P parallel to AB, BC, CD, DA intersect AD and BC at A_1 and B_2, AB and CD at B_1 and C_2, BC and DA at C_1 and D_2, CD and AB at D_1 and A_2 respectively. Then

$$\frac{A_1B_2}{AB} \cdot \frac{B_1C_2}{BC} \cdot \frac{C_1D_2}{CD} \cdot \frac{D_1A_2}{DA} \leqslant 1.$$

S. Zetel', Matematika v škole 1949, No. 4, 62.

15.15 If a quadrilateral has a circumscribed circle with radius R and an inscribed circle with radius r, then

$$R \geqslant r\sqrt{2}.$$

Ju. I. Gerasimov and O. A. Kotii, Matematika v škole 1964, No. 1, 83.

L. Carlitz, Math. Mag. 38 (1965), 33–35.

15.16 $4F \leqslant a^2+b^2+c^2+d^2.$

Equality only for a square.

Z. A. Skopec and V. A. Žarov, Zadači i teoremi po geometrii, Moskva 1962, p. 82.

15.17 $4F \leqslant p^2+q^2.$

Equality holds if and only if the diagonals are orthogonal and equal.

Z. A. Skopec and V. A. Žarov, Zadači i teoremi po geometrii, Moskva 1962, p. 82.

15.18 Let four points A, B, C, D be given.

1°. If D is in the interior or on an edge of triangle ABC, then, for all points P

$$PA+PB+PC+PD \geqslant DA+DB+DC.$$

Equality holds only if P coincides with D.

2°. If no vertex is on or in the triangle formed by the other three points, then

$$PA+PB+PC+PD$$

is minimal for the point of intersection of the diagonals of the convex quadrilateral defined by the four points.

H. W. Guggenheimer, Plane Geometry and its Groups, San Francisco, Cambridge, London, Amsterdam 1967, p. 178.

15.19 Let A_1, B_1, C_1, D_1 be the midpoints of the sides BC, CD, DA, AB respectively. If F_1 is the area of the quadrilateral enclosed by the lines AA_1, BB_1, CC_1, DD_1, then

$$\tfrac{1}{6}F < F_1 \leqslant \tfrac{1}{5}F.$$

Equality only for a parallelogram.

T. Popoviciu, Problem 5897, Gaz. Mat. 49 (1943), 322.

15.20 Suppose that the quadrilateral $ABCD$ is convex and AD is not parallel to BC. If $PQ \parallel AD$ (P-intersection of the diagonals AC and BD; Q on AB. $QR \parallel BC$ (R on AC), $RS \parallel AD$ (S on CD), and if F_1 is the area of $PQRS$, then

$$F_1 < \tfrac{8}{27}F. \tag{1}$$

PROOF. The quadrilateral $PQRS$ is a parallelogram.

$$\text{area } AQC = \frac{AC}{PR} \cdot \text{area } PQR = \frac{AC}{2PR} \cdot F_1 = \text{area } ASC = F_2.$$

Hence

$$\text{area } ABC = \frac{AB}{AQ} \cdot F_2 = \frac{AC}{AR} \cdot F_2 = \frac{AC^2}{2PR \cdot AR} \cdot F_1$$

and

$$\text{area } ADC = \frac{CD}{CS} \cdot F_2 = \frac{AC}{RC} \cdot F_2 = \frac{AC^2}{2PR \cdot RC} \cdot F_1.$$

Adding together the above-mentioned relations, we get

$$F = \frac{AC^3}{2PR \cdot AR \cdot RC} \cdot F_1,$$

whence

$$\frac{F_1}{F} = 2PR \cdot AR \frac{RC}{AC^3} < \frac{2AR^2 \cdot RC}{AC^3};$$

or

$$\frac{F_1}{F} < 2x^2(1-x), \text{ where } x = \frac{AR}{AC}.$$

By applying the arithmetic-geometric mean inequality to x, x and $2-2x$, we get (1).

Y. Hattori, Math. Gaz. 47 (1963), 148–150.

E. Szekeres, Math. Gaz. 48 (1964), 439–440.

15.21 Let $ABCD$ be a convex quadrilateral. Consider a point A_1 on the straight line AB so that the point B lies between the points A and A_1 and the equality $AA_1 = BC + CD + DA$ is satisfied. If B_1, C_1, D_1 are determined on the lines BC, CD, DA

in the same manner as point A_1, then

$$\frac{5}{3} < \frac{\operatorname{per} A_1B_1C_1D_1}{\operatorname{per} ABCD} < 5.$$

D. M. Batinetu, Problem E 1725, Gaz. Mat. B 12 (1961), 501.

15.22 Let $A_1A_2A_3A_4$ be a square and let P be an arbitrary point in its plane. Then

$$PA_1+PA_2+PA_3+PA_4 \geqslant (1+\sqrt{2}) \max_{(i)} PA_i + \min_{(i)} PA_i,$$

equality taking place when P lies on the circumcircle of the square.

PROOF. Without loss of generality we may assume

$$PA_4 \geqslant (PA_3, PA_1) \geqslant PA_2.$$

Then, by Ptolemy's theorem (see: **15.4**)

$$(PA_1)(A_3A_4)+(PA_3)(A_1A_4) \geqslant (PA_4)(A_1A_3),$$

equality occurring if and only if A_1, P, A_3, A_4 are concyclic.

Since $A_3A_4 = A_1A_4 = A_1A_3/\sqrt{2}$, we infer that

$$PA_1+PA_3 \geqslant PA_4\sqrt{2}.$$

Hence

$$PA_1+PA_2+PA_3+PA_4 \geqslant PA_4\sqrt{2}+PA_4+PA_2,$$

equality holding when P lies on the circumcircle of the square.

I. S. Gál-L. Bankoff, Problem E 1308, Amer. Math. Monthly 65 (1958), 205 and 710.

15.23 Let $ABCD$ be a quadrilateral. It need not be convex and may even possess a double-point. Let

$$P_1 = (-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)$$

and

$$P_2 = (a+b+c+d)(-a+b+c-d)(a-b+c-d)(a+b-c-d).$$

If $P_1 > 0$, then

$$P_2 \leqslant 16F^2 \leqslant P_1. \tag{1}$$

Equality at the right side occurs if $ABCD$ is a convex cyclic quadrilateral (Steiner's theorem). Such a quadrilateral exists always.

If $P_2 \geqslant 0$ there exists a non-convex cyclic quadrilateral the sides of which are (in a certain order) a, b, c, d. In that case the equality sign at the left-hand side (1) holds for that quadrilateral.

If $P_2 < 0$ no such quadrilateral exists. The inequalities (1) can be replaced by

$$0 \leqslant 16F^2 \leqslant P_1$$

and equality at the left takes place for the quadrilateral (always existing in this case) for which the diagonals are parallel.

O. Bottema, Euclides 15 (1938), 1–13.

15.24 $0 \leqslant \cos^2 \frac{1}{2}(\alpha+\gamma) = \cos^2 \frac{1}{2}(\beta+\delta) \leqslant 1$ if $P_2 \geqslant 0$,

$$0 \leqslant \cos^2 \tfrac{1}{2}(\alpha+\gamma) = \cos^2 \tfrac{1}{2}(\beta+\delta) \leqslant \frac{P_1}{16abcd} \text{ if } P_2 \leqslant 0,$$

where P_1 and P_2 are defined in **15.23**.

O. Bottema, Euclides 15 (1938), 1–13.

15.25 $pq \geqslant |ac-bd|$ if $P_2 > 0$,

$$pq \geqslant |a^2-b^2+c^2-d^2| \text{ if } P_2 \leqslant 0,$$

where P_2 is defined in **15.23**.

Equality for the non-convex cyclic quadrilateral, resp. for that with parallel diagonals.

O. Bottema, Euclides 15 (1938), 1–13.

15.26 If θ is one of the angles between the diagonals, then

$$\sin^2 \theta = 1 \text{ if } a^2-b^2+c^2-d^2 = 0.$$

In the other cases

$$\frac{P_2}{4(ac-bd)^2} \leqslant \sin^2 \theta \leqslant \frac{P_1}{4(ac+bd)^2} \quad \text{if } P_2 \geqslant 0,$$

$$0 \leqslant \sin^2 \theta \leqslant \frac{P_1}{4(ac+bd)^2} \quad \text{if } P_2 < 0,$$

where P_1 and P_2 are defined in 15.2I.

O. Bottema, Euclides 15 (1938), 1-13.

16. Inequalities for polygons

16.1 For any pentagon $A_1A_2A_3A_4A_5$

$$\sum_{i=1}^{5} (A_iA_{i+2}+A_{i+1}A_{i+4})A_iA_{i+1} > \sum_{i=1}^{5} A_iA_{i+2}^2 \quad (A_{i+5} = A_i). \qquad (1)$$

PROOF. Since

$$A_iA_{i+1}+A_{i+1}A_{i+2} > A_iA_{i+2} \qquad (i = 1, \ldots, 5)$$

or

$$(A_iA_{i+1}+A_{i+1}A_{i+2})A_iA_{i+2} > A_iA_{i+2}^2 \quad (i = 1, \ldots, 5),$$

we have (1).

16.2 Let A', B', C', D', E' be the midpoints of sides of a convex pentagon $ABCDE$. Then

$$\text{area } A'B'C'D'E' \geqslant \tfrac{1}{2} \text{ area } ABCDE.$$

16.3 Let $ABCDE$ be a convex pentagon inscribed within a unit circle with AE as diameter, then

$$a^2+b^2+c^2+d^2+abc+bcd < 4,$$

where $a = AB$, $b = BC$, $c = CD$, $d = DE$.

H. Demir, Problem 1877, Amer. Math. Monthly 73 (1966), 410.

16.4 Consider a regular hexagon of area F, which contains non-intersecting circles of radii $r_1, \ldots, r_n$. Then

$$(r_1 + \ldots + r_n)^2 \leqslant \frac{nF}{\sqrt{12}}.$$

L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, Berlin 1953, p. 75.

16.5 Let $(n-1)s$ be the perimeter of an n-gon with sides a_i $(i = 1, \ldots, n)$ and let $a_i \leqslant s$ $(i = 1, \ldots, n)$. Then

$$\frac{n}{n-1} \leqslant \frac{a_1}{a_2+\ldots+a_n} + \frac{a_2}{a_3+\ldots+a_n+a_1} + \ldots$$

$$+ \frac{a_n}{a_1+\ldots+a_{n-1}} \leqslant \frac{n-1}{n-2}. \qquad (1)$$

PROOF. Denote the expression whose bounds we want by S. Since

$$a_i \leqslant s \text{ for } i = 1, \ldots, n,$$

we have

$$a_1+\ldots+a_{i-1}+a_{i+1}+\ldots+a_n = (n-1)s-a_i \geqslant (n-2)s,$$

so that

$$S \leqslant \frac{1}{(n-2)s}(a_1+\ldots+a_n) = \frac{(n-1)s}{(n-2)s},$$

which represents the right-hand side of (1).

Since we can write S in the form

$$S = \left(\frac{1}{a_2+\ldots+a_n} + \frac{1}{a_3+\ldots+a_n+a_1} + \ldots + \frac{1}{a_1+\ldots+a_{n-1}}\right)(n-1)s-n \qquad (2)$$

and since by the arithmetic-harmonic mean inequality we have

$$\frac{1}{a_2+\ldots+a_n} + \frac{1}{a_3+\ldots+a_n+a_1} + \ldots + \frac{1}{a_1+\ldots+a_{n-1}}$$

$$\geqslant n^2[(a_2+\ldots+a_n)+(a_3+\ldots+a_n+a_1)+\ldots+(a_1+\ldots+a_{n-1})]^{-1}$$

$$= \frac{n^2}{(n-1)^2 s},$$

the left-hand side of (1) follows from (2).

This proof is due to P. M. Vasić.

M. Petrović, Bulletin Mathématique de la Société Roumaine des Sciences 40 (1938), 205–208.

S. Pavlović, Uvodjenje mladih u naučni rad, I, Matematička biblioteka, sv. 18, Beograd 1961, 139–142.

16.6 For every convex n-gon, with area F,

$$\sum_{i=1}^{n} a_i^2 \geqslant 4F \operatorname{tg} \frac{\pi}{n}.$$

PROOF. We use the fact that of all n-gons with the same perimeter, the regular n-gon has greatest area. Thus

$$F \leqslant \frac{na^2}{4 \operatorname{tg} \pi/n}$$

where $na = \sum_{i=1}^{n} a_i$. The Cauchy-Schwarz inequality, however, gives

$$na^2 = n^{-1}(\sum_{i=1}^{n} a_i)^2 \leqslant n^{-1}(\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} 1) = \sum_{i=1}^{n} a_i^2,$$

whence the desired inequality follows.

E. Just-N. Schaumberger, Problem 1634, Amer. Math. Monthly 70 (1963), 1005 and 71 (1964), 796.

16.7 Let P be an internal point of a convex n-gon. If r_k is the radius of the incircle of the triangle A_kPA_{k+1} ($k = 1, \ldots, n$ and $A_{n+1} = A_1$), then

$$r_1 + \ldots + r_n \leqslant \sqrt{\frac{nF}{3\sqrt{3}}}.$$

This inequality is due to Chr. Karanikolov.

16.8 Let R_i be the distances from an internal point P of an n-gon to its vertices, and r_i be the distances from P to its sides. Then

$$R_1 \ldots R_n \geqslant \left(\sec \frac{\pi}{n}\right)^n r_1 \ldots r_n.$$

L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, Berlin 1953, p. 33.

REMARK. For $n = 3$ we have 12.25.

16.9 Let $A_1, \ldots, A_n$ be the vertices of a convex n-gon and P an internal point. Let $R_k = PA_k$ and let r_k be the distance from P to the side A_kA_{k+1}.

L. Fejes Tóth has conjectured the following inequality

$$\left(\cos\frac{\pi}{n}\right)\sum_{k=1}^{n} R_k \geqslant \sum_{k=1}^{n} r_k. \tag{1}$$

For $n = 3$, (1) reduces to the Erdös-Mordell's inequality (see: **12.13**).

Inequality (1) for $n = 4$ was proved by A. Florian.

Later, Lenhard proved (1) for every natural number $n \geqslant 3$ and deduced the following stronger inequality

$$\left(\cos\frac{\pi}{n}\right)\sum_{k=1}^{n} R_k \geqslant \sum_{k=1}^{n} w_k,$$

where w_k is the segment of the bisector of the angle $A_kPA_{k+1} = 2\varphi_i$ from P to its intersection with the side A_kA_{k+1}.

Lenhard proved also that the following inequality holds:

$$\sqrt{R_kR_{k+1}}\cos\varphi_i \geqslant w_k \geqslant r_k.$$

L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, Berlin 1953, p. 33.

A. Florian, Elem. Math. 13 (1958), 55–58.

H. C. Lenhard, Arch. Math. 12 (1961), 311–314.

16.10 Let $A_1, \ldots, A_n$ be the vertices of a regular polygon and let P be an point in its interior. Then at least one of the angles A_iPA_j satisfies the following inequalities

$$\pi\left(1 - \frac{1}{n}\right) \leqslant A_iPA_j \leqslant \pi.$$

P. Erdõs, Problem 4086, Amer. Math. Monthly 50 (1943), 391; 51(1944), 480 and 54 (1947), 117.

16.11 Let $A_1, \ldots, A_n$ be the vertices of a regular polygon, P any point, then

$$\sum_{k=1}^{n} A_kP \geqslant \left[\left(\cos\frac{\pi}{2n}\right)^{\frac{1+(-1)^n}{2}}\cdot\operatorname{cosec}\frac{\pi}{2n}\right]\max_k A_kP + \frac{1+(-1)^n}{2}\min_k A_kP.$$

This result is the best possible.

I. S. Gál, Bull. Ecole Polytech. Jassy 3 (1948), ?–106.

REMARK. This inequality is a generalization of 15.22.

16.12 Let $A_1, \ldots, A_n$ be the vertices of a convex n-gon and P an internal point. Then

$$\frac{L}{2} < \sum_{k=1}^{n} A_kP < (n-1)\frac{L}{2},$$

where L is perimeter of the n-gon.

A. Natucci, Periodico Mat. (4) 29 (1951), 98–1??.

16.13 Let P be an internal point of the convex polygon $A_1, \ldots, A_n$. Let R_k denote the distance from P to the vertex A_k and r_k its distance to the side A_kA_{k+1}. Let $\alpha_k = \sphericalangle A_{k-1}A_kA_{k+1}$. Then

$$\sum_{k=1}^{n} R_k \sin\frac{\alpha_k}{2} \geqslant \sum_{k=1}^{n} r_k. \tag{1}$$

Equality holds if and only if all sides of the polygon are tangent to a circle with centre P.

PROOF. Let $\beta_k = \sphericalangle PA_kA_{k+1}$. From the triangle PA_kP_k, where P_k is the orthogonal projection of P on the sides A_kA_{k+1}, we get

$$r_k = R_k \sin\beta_k \quad (k = 1, \ldots, n). \tag{2}$$

From the triangle $PA_{k+1}P_k$ we obtain

$$r_k = R_{k+1}\sin(\alpha_{k+1}-\beta_{k+1}) \quad (k = 1, \ldots, n-1). \tag{3}$$
$$r_n = R_1 \sin(\alpha_1-\beta_1).$$

Adding together (2) and (3), we get

$$2\sum_{k=1}^{n} r_k = \sum_{k=1}^{n} R_k[\sin(\alpha_k-\beta_k)+\sin\beta_k].$$

i.e.

$$2\sum_{k=1}^{n} r_k = 2\sum_{k=1}^{n} R_k \sin\frac{\alpha_k}{2}\cos\left(\frac{\alpha_k}{2}-\beta_k\right). \tag{4}$$

Using $\cos[(\alpha_k/2)-\beta_k] \leqslant 1$, (4) gives (1).

Equality holds if and only if $\cos[(\alpha_k/2)-\beta_k] = 1$, i.e., if and only if $\alpha_k = 2\beta_k$. This condition is satisfied if the sides of the polygon are tangent to a circle with centre P.

This result is due to R. R. Janić.

16.14 If F is the area of an n-gon of diameter 1, then

$$F \leqslant \frac{n}{2}\cos\frac{\pi}{n}\,\mathrm{tg}\,\frac{\pi}{2n}.$$

N. D. Kazarinoff, Analytic Inequalities, New York 1961, p. 84.

16.15 Let $A_1 \ldots A_n$ be a convex polygon and let

$$\alpha_k = \sphericalangle A_{k-1}A_kA_{k+1} \quad (k = 2, \ldots, n-1)$$

$$\varphi = \sphericalangle A_{n-1}A_nA_1, \quad \psi = \sphericalangle A_nA_1A_2$$

be its internal angles. If

$$A_1A_2 = A_2A_3 = \ldots = A_{n-1}A_n \neq A_nA_1$$

and

$$\alpha_2 \geqslant \alpha_3 \geqslant \ldots \geqslant \alpha_{n-1} \tag{1}$$

then

$$\varphi \geqq \psi. \tag{2}$$

Inequality holds in (2) if and only if at least one inequality occurs in (1).

S. Bilinski, Glasnik matematičko-fisički i astronomski, 16 (1961), 195–201.

16.16 Let a polygon contain a circle with radius r. Then

$$\min\frac{a_1+a_2+\ldots+a_n}{n} \geqslant \min\frac{n}{1/a_1+1/a_2+\ldots+1/a_n}.$$

Equality holds only for $n = 3$.

F. Leuenberger, Elem. Math. 15 (1960), 77–79.

16.17 Let $a_1, \dots, a_n$ be the distances of any point on the circle, circumscribed with radius R about a regular n-gon, to its vertices. Then

$$R\sqrt{2n} \leqslant a_1+\dots+a_n \leqslant nR\sqrt{2}. \tag{1}$$

Proof. Since

$$a_1^2+\dots+a_n^2 = 2R^2n,$$

$$\frac{(a_1+\dots+a_n)^2}{n} \leqslant a_1^2+\dots+a_n^2 \leqslant (a_1+\dots+a_n)^2,$$

inequality (1) is indeed true.

M. Petrović, Računanje sa brojnim razmacima, Beograd 1932, p. 82.

16.18 With the notation of **16.17**, we have

$$2R \operatorname{cotg}\frac{\pi}{2n} \leqslant a_1+\dots+a_n \leqslant 2R \operatorname{cosec}\frac{\pi}{2n}.$$

P. M. Vasić and Ž. Živanović, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181–No. 196 (1967), 67–68.

16.19 About a circle is circumscribed a regular polygon of n sides each of length a_n, and within this circle is inscribed a regular polygon of k sides each of length b_k. Then the following inequalities hold:

$$a_{n-1} > b_n \qquad (n = 3, 4),$$

$$a_{n-1} < b_n \qquad (n = 5, 6, \dots),$$

$$2a_{n-1} > b_n+b_{n+1} \qquad (n = 3, \dots, 8),$$

$$2a_{n-1} < b_n+b_{n+1} \qquad (n = 9, 10, \dots).$$

N. Obreškov, Problems 30 and 31, Fiziko-matematičeskoe spisanie (Sofia) 3 (1960), 310–312.

16.20 If S_k is the sum of all the perpendiculars from the centre to the sides of a regular polygon of k sides which is inscribed in a circle of radius R, then

$$S_{k+1}-S_k > R.$$

16.21 Given a circle C of circumference U. Let p_n be the perimeter of the regular n-gon inscribed in C and P_n the perimeter of the circumscribed regular n-gon ($n > 3$). Then

1°. The sequence (p_n) is monotonously increasing, and the sequence (P_n) is monotonously decreasing;

2°. $\frac{2}{3}p_n+\frac{1}{3}P_n > U$.

A proof of this inequality, due to D. D. Adamović, can be found in: D. S. Mitrinović, Elementary Inequalities, Groningen 1964, pp. 115–117.

16.22 Given a circle C with radius R. Let o_n be the perimeter of the regular n-gon inscribed in C and O_n the perimeter of the circumscribed regular n-gon. Then

$$o_n+O_n > 4\pi R. \tag{1}$$

Proof. We shall first prove that

$$\sin x+\operatorname{tg} x > 2x \quad \text{for } 0 < x < \frac{\pi}{2}. \tag{2}$$

Consider the function $f(x) = \sin x+\operatorname{tg} x-2x$. Its derivative is

$$f'(x) = \cos x+\frac{1}{\cos^2 x}-2=\frac{(1-\cos x)(1+\cos x(1-\cos x))}{\cos^2 x}.$$

Since $f(0) = 0$ and $f'(x) > 0$ for $0 < x < \pi/2$, one gets $f(x) > 0$, i.e., (2).

Since $o_n = 2Rn \sin \pi/n$ and $O_n = 2Rn \operatorname{tg} \pi/n$, we have

$$o_n+O_n = 2Rn\left(\sin \frac{\pi}{n} + \operatorname{tg} \frac{\pi}{n}\right).$$

For $n \geq 3$, we get $0 < \pi/n < \pi/2$ and consequently

$$\sin \frac{\pi}{n} + \operatorname{tg} \frac{\pi}{n} > 2\frac{\pi}{n}.$$

Therefore (1).

Matematika v škole, 1965, No. 3, 76.

16.23 Let F, L, r, R be area, perimeter, radius of incircle, radius of circumcircle respectively of a convex n-gon. Then

$$nr^2 \operatorname{tg} \frac{\pi}{n} \leqslant F \leqslant \tfrac{1}{2} nR^2 \sin \frac{2\pi}{n},$$

$$2nr \operatorname{tg} \frac{\pi}{n} \leqslant L \leqslant 2nR \sin \frac{\pi}{n}.$$

Equalities hold if and only if the n-gon is equilateral.

J. Kürschak, Math. Ann. 30 (1887), 578–581.

16.24 The area of any n-gon, inscribed in an unit circle is at most equal to $\frac{1}{2} n \sin 2\pi/n$.

REMARK. This inequality was to be proved by high school students on competition in China in 1957.

16.25 Given a convex polygon inscribed in a circle, with vertices $A_1, \ldots, A_n$. Consider a second polygon, whose consecutive vertices are the midpoints of the arcs $A_1A_2, A_2A_3, \ldots, A_nA_1$. Then the area of the second polygon is greater or equal to the area of the first polygon, with equality if and only if the initial polygon is regular.

M. S. Klamkin, Math. Teacher 60 (1967), 323–328.

16.26 Let r, R, L be the radius of the incircle, radius of the circumcircle and the perimeter of a regular polygon, respectively. Then

$$2\pi \sqrt[3]{rR^2} < L < \tfrac{2}{3}\pi (r+2R).$$

E. Catalan, Question 65, Mathesis 2 (1882), 85–88 and 245–248.

16.27 If t_n denotes the area of a regular n-sided polygon inscribed in a circle of unit radius and if T_n denotes the area of a regular n-sided polygon circumscribed about this circle, then

$$\frac{3t_nT_n}{T_n+2t_n} < \pi < \frac{t_n+2T_n}{3}.$$

P. Szász, Mat. Lapok 5 (1954), 73–78.

16.28 Denote L and L' the perimeters of two polygons inscribed in the same circle. If the largest side of the second is less than the smallest side of the first, then

$$L < L'.$$

J. V. Uspensky, Amer. Math. Monthly 34 (1927), 247–250.

16.29 Let F_i be the area of the convex polygon inscribed in an unit circle and let F_c be the area of the circumscribed polygon whose points of contact with the circle coincide with the vertices of the inscribed polygon. Then

$$F_i + F_c \geqslant 6.$$

Equality holds only for squares.

REMARK. This inequality was conjectured by P. Szász and has been proved by: J. Aczél and L. Fuchs, Compositio Math. 8 (1950), 61–67.

17. Inequalities for a circle

17.1 Let P be a point inside the circle K of radius R. If arc AF = arc FB, with arc $AB < R\pi$ and $PA < PB$, then $\widehat{APF} > \widehat{FPB}$.

REMARK. This theorem was given to be proved by secondary school students at the competition in Hungary in 19[illegible].

17.2 Let A and B be two points outside a circle K and let the straight line segment AB not intersect this circle. If C and D are the points of contact of tangent lines drawn from A and B with opposite orientation to the circle K, then the following inequalities hold

$$|AC - BD| < AB < AC + BD. \qquad (1)$$

If the straight line AB intersects the circle K neither inequality in (1) holds.

PROOF. Let the straight line segment AB not intersect this circle. Consider the triangle ABM, M being the intersection of the tangent lines. We have

$$AM+BM > AB > |AM-BM|.$$

Since

$$AC > AM, \ BD > BM, \ MC = MD,$$

we get

$$AC+BD > AB > |AC-BD|.$$

If the straight line AB cuts the circle we shall have two cases:

1°. the points of intersection of the straight line AB and the circle lie between A and B;

2°. the points of intersection of the straight line AB and the circle do not lie between A and B.

If 1° holds, we have

$$AB > AE+FB > AC+BD,$$

where E and F are the intersection segment AB with OC and OD respectively.

If 2° holds, the segment AB lies in the angle CAC', where C' is the point of contact of the second tangent line drawn from point A. Let us draw another circle, concentric to given circle, through point B. Let this new circle cut AC and AC' in points E and E'. Then

$$AB < AE = AC-EC = AC-BD.$$

V. B. Lidskii, L. V. Ovsjannikov, A. N. Tulaĭkov, M. I. Šabunin, Problem 362, Zadači po elementarnoĭ matematike, Moskva 1962, p. 62.

17.3 A circle is inscribed in a right angle with vertex in A such that it touches the legs of the angle in points A and C. If a tangent line cuts the legs AB and AC in the points M and N $(AM < AB, \ AN < AC)$, then

$$\tfrac{1}{3}(AB+AC) < MB+NC < \tfrac{1}{2}(AB+AC).$$

Proof. If K is the point of contact of the tangent MN and circle, then we have $MB = MK$ and $NK = NC$, whence

$$MN = MB+NC. \qquad (1)$$

Since

$$MN < AM+AN,$$

we get

$$2MN < AM+MB+AN+NC = AB+BC.$$

On the other hand, $MN > AM$ and $MN > AN$, i.e.,

$$2MN > AM+AN,$$

and because of (1) we obtain

$$3MN > AM+MB+AN+NC = AB+AC.$$

17.4 Consider two circles of radii R with centres in points C_1 and C_2. Let A be a point on the first circle, and B_1 a point on the second.

If point B_2 is symmetrical to point B_1 in respect to the straight line C_1C_2 and if $C_1C_2 = R$, then

$$AB_1^2+AB_2^2 \geqslant 2R^2.$$

Proof. From triangles ABB_1 and ABB_2, where B is midpoint of the straight line B_1B_2, we have

$$AB_1^2 = AB^2+BB_1^2-2\,AB\,BB_1 \cos \widehat{ABB_1}, \qquad (1)$$

$$AB_2^2 = AB^2+BB_2^2-2\,AB\,BB_2 \cos \widehat{ABB_2}. \qquad (2)$$

Since $\widehat{ABB_1}+\widehat{ABB_2} = \pi$, adding together the equalities (1) and (2), we obtain

$$AB_1^2+AB_2^2 = 2(AB^2+BB_1^2). \qquad (3)$$

From (3) and inequality $AB \geqslant BC_2$, we get

$$AB_1^2+AB_2^2 \geqslant 2(BC_2^2+BB_1^2) = 2\cdot B_1C_2^2 = 2\cdot R^2.$$

17.5 If AC and BD are two mutually perpendicular chords of a circle with radius R, then

$$2R \leqslant AC+BD \leqslant 4R. \qquad (1)$$

Proof. Let E be the other end of the diameter which passes through the point A and let P denote the point of intersection

AC and BD. Then we get the equalities:

$$PA^2+PB^2 = AB^2, \quad PC^2+PD^2 = CD^2 = BE^2,$$

whence

$$PA^2+PB^2+PC^2+PD^2 = AB^2+BE^2 = 4R^2. \qquad (2)$$

On the basis of the inequalities

$$\frac{(x+y+z+t)^2}{4} \leqslant x^2+y^2+z^2+t^2 \leqslant (x+y+z+t)^2,$$

where x, y, z, t are non-negative real numbers, we see that the sum of squares on the left-hand side of identity (2) is between

$$\tfrac{1}{4}(PA+PB+PC+PD)^2 = \tfrac{1}{4}(AC+BD)^2$$

and

$$(PA+PB+PC+PD)^2 = (AC+BD)^2,$$

i.e.

$$\tfrac{1}{4}(AC+BD)^2 \leqslant 4R^2 \leqslant (AC+BD)^2.$$

Whence (1) follows.

M. Petrović, Računanje sa brojnim razmacima, Beograd 1932, pp. 84–85.

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AC and BD. Then we get the equalities:

$$PA^2+PB^2 = AB^2, \quad PC^2+PD^2 = CD^2 = BE^2,$$

whence

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On the basis of the inequalities

$$\frac{(x+y+z+t)^2}{4} \leqslant x^2+y^2+z^2+t^2 \leqslant (x+y+z+t)^2,$$

where x, y, z, t are non-negative real numbers, we see that the sum of squares on the left-hand side of identity (2) is between

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and

$$(PA+PB+PC+PD)^2 = (AC+BD)^2,$$

i.e.

$$\tfrac{1}{4}(AC+BD)^2 \leqslant 4R^2 \leqslant (AC+BD)^2.$$

Whence (1) follows.

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