MATHEMATICAL CONTEST

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METROPOLITAN NEW YORK SECTION

OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

PLEASE PRINT ON LINES BELOW

SCORE

NAME ____
FIRST NAME MIDDLE NAME LAST NAME

ADDRESS ____
STREET CITY STATE

SCHOOL ______________________ COUNTY ______________

ADDRESS ____

THURSDAY MORNING MAY 11, 1950

TIME OF CONTEST
1 HOUR 20 MINUTES

TOTAL CREDITS
150

INSTRUCTIONS

In each of the fifty exercises enclose the correct answer, including the letter that precedes it, in a box. The following is a sample problem. If the sum of two numbers is s and one of the numbers is n then the other number is:
(a) $n + s$ (b) $n - s$ [(c) $s - n$] (d) $s + n$ (e) none of the above

PART I. (2 credits each)

1. If 64 is divided into three parts proportional to 2, 4 and 6, the smallest part is:
   (a) 5 1/3 (b) 11 (c) 10 2/3 (d) 5 (e) none of these answers.

2. Let $R = gS - 4$. When $S = 8$, $R = 16$. When $S = 10$, R is equal to:
   (a) 11 (b) 14 (c) 20 (d) 21 (e) none of these answers.

3. The sum of the roots of the equation $4x^2 + 5 - 8x = 0$ is equal to:
   (a) 8 (b) −5 (c) −5/4 (d) −2 (e) none of these answers.

4. Reduced to lowest terms. $\dfrac{a^2 - b^2}{ab} - \dfrac{ab - b^2}{ab - a^2}$ is equal to:
   (a) $\dfrac{a}{b}$ (b) $\dfrac{a^2 - 2b^2}{ab}$ (c) a^2 (d) $a - 2b$ (e) none of these answers.

5. If five geometric means are inserted between 8 and 5832, the fifth term in the geometric series is:
   (a) 648 (b) 832 (c) 1168 (d) 1944 (e) none of these answers.

6. The values of y which will satisfy the equations

$$2x^2 + 6x + 5y + 1 = 0$$
$$2x + y + 3 = 0$$

may be found by solving:
   (a) $y^2 + 14y - 7 = 0$ (b) $y^2 + 8y + 1 = 0$ (c) $y^2 + 10y - 7 = 0$
   (d) $y^2 + y - 12 = 0$ (e) none of the above equations

7. If the digit 1 is placed after a two digit number whose tens' digit is t, and units' digit is u, the new number is:

(a) $10t + u + 1$ (b) $100t + 10u + 1$ (c) $1000t + 10u + 1$
(d) $t + u + 1$ (e) none of these answers.

8. If the radius of a circle is increased 100%, the area is increased:
(a) 100% (b) 200% (c) 300% (d) 400% (e) by none of these

9. The area of the largest triangle that can be inscribed in a semi-circle whose radius is r is:
(a) r^2 (b) r^3 (c) $2r^2$ (d) $2r^3$ (e) $\frac{1}{2}r^2$

10. After rationalizing the numerator of $\frac{\sqrt{3} - \sqrt{2}}{\sqrt{3}}$, the denominator in simplest form is:
(a) $\sqrt{3}(\sqrt{3} + \sqrt{2})$ (b) $\sqrt{3}\ (\sqrt{3} - \sqrt{2})$ (c) $3 - \sqrt{3}\sqrt{2}$
(d) $3 + \sqrt{6}$ (e) none of these answers.

11. If in the formula $C = \frac{en}{R + nr}$, n is increased while e, R and r are kept constant, then C:
(a) decreases (b) increases (c) remains constant
(d) increases and then decreases (e) decreases and then increases

12. As the number of sides of a polygon increases from 3 to n, the sum of the exterior angles formed by extending each side in succession:
(a) increases (b) decreases (c) remains constant
(d) cannot be predicted (e) becomes $(n - 3)$ straight angles.

13. The roots of $(x^2 - 3x + 2)(x)(x - 4) = 0$ are $x =$
(a) 4 (b) 0 and 4 (c) 1 and 2 (d) 0,1, 2 and 4
(e) 1, 2 and 4.

14. In the simultaneous equations $2x - 3y = 8$, $6y - 4x = 9$
(a) $x = 4,\ y = 0$ (b) $x = 0,\ y = 3/2$ (c) $x = 0,\ y = 0$
(d) there is no solution (e) there are an infinite number of solutions

15. The real factors of $x^2 + 4$ are:
(a) $(x^2 + 2)(x^2 + 2)$ (b) $(x^2 + 2)(x^2 - 2)$ (c) $x^2(x^2 + 4)$
(d) $(x^2 - 2x + 2)(x^2 + 2x + 2)$ (e) non-existent

PART II (3 credits each)

16. The number of terms in the expansion of $[(a + 3b)^2(a - 3b)^2]^2$ when simplified is:

(a) 4 (b) 5 (c) 6 (d) 7 (e) 8

17. The formula which expresses the relationship between x and y as shown in the accompanying table is:

(a) $y = 100 - 10x$ (b) $y = 100 - 5x^2$

(c) $y = 100 - 5x - 5x^2$

(d) $y = 20 - x - x^2$ (e) none of the above formulae.

x	0	1	2	3	4
y	100	90	70	40	0

18. Of the following (1) $a(x - y) = ax - ay$ (2) $a^{x-y} = a^x - a^y$

(3) $\log(x - y) = \log x - \log y$ (4) $\dfrac{\log x}{\log y} = \log x - \log y$

(5) $a(xy) = ax \cdot ay$

(a) only 1 and 4 are true (b) only 1 and 5 are true

(c) only 1 and 3 are true (d) only 1 and 2 are true

(e) only 1 is true

19. If m men can do a job in d days, then $m + r$ men can do the job in:

(a) $d + r$ days (b) $d - r$ days (c) $\dfrac{md}{m + r}$ days (d) $\dfrac{d}{m + r}$ days

(e) none of the above

20. When $x^{13} + 1$ is divided by $x - 1$, the remainder is:

(a) 1 (b) -1 (c) 0 (d) 2 (e) none of these answers.

21. The volume of a rectangular solid each of whose side, front, and bottom faces are 12 sq.in., 8 sq.in., and 6 sq.in. respectively is:

(a) 576 cu.in. (b) 24 cu.in. (c) 9 cu.in. (d) 104 cu.in.

(e) none of the above answers

22. Successive discounts of 10% and 20% are equivalent to a single discount of:

(a) 30% (b) 15% (c) 72% (d) 28% (e) none of these

23. A man buys a house for \$10,000 and rents it. He puts 12 1/2% of each month's rent aside for repairs and upkeep; pays \$325 a year taxes and realizes 5 1/2% on his investment. The monthly rent is:

(a) \$64.82 (b) \$83.33 (c) \$72.08 (d) \$45.83 (e) \$177.08

24. The equation $x + \sqrt{x - 2} = 4$ has:

(a) 2 real roots (b) 1 real and 1 imaginary root

(c) 2 imaginary roots (d) no roots

(e) 1 real root and 1 extraneous root.

25. The value of $\log_5 \frac{(125)(625)}{25}$ is equal to

(a) 725 (b) 6 (c) 3125 (d) 5 (e) none of these answers.

26. If $\log_{10} m = b - \log_{10} n$, then $m =$

(a) $\frac{b}{n}$ (b) bn (c) $10^b n$ (d) $b - 10^n$ (e) $\frac{10^b}{n}$

27. A car travels 120 miles from A to B at 30 miles per hour but returns the same distance at 40 miles per hour. The average speed for the round trip is closest to:

(a) 33 mph (b) 34 mph (c) 35 mph (d) 36 mph (e) 37 mph

28. Two boys A and B start at the same time to ride from Port Jervis to Poughkeepsie, 60 miles away. A travels 4 miles an hour slower than B. B reaches Poughkeepsie and at once turns back meeting A 12 miles from Poughkeepsie. The rate of A was:

(a) 4 mph (b) 8 mph (c) 12 mph (d) 16 mph (e) 20 mph

29. A manufacturer built a machine which will address 500 envelopes in 8 minutes. He wishes to build another machine so that when both are operating together they will address 500 envelopes in 2 minutes. The equations used to find how many minutes (x) it would require the second machine to address 500 envelopes alone is:

(a) $8 - x = 2$ (b) $\frac{1}{8} + \frac{1}{x} = \frac{1}{2}$ (c) $\frac{500}{8} + \frac{500}{x} = 250$ (d) $\frac{x}{2} + \frac{x}{8} = 1$

(e) none of these answers.

30. From a group of boys and girls, 15 girls leave. There are then left two boys for each girl. After this 45 boys leave. There are then 5 girls for each boy. The number of girls in the beginning was:

(a) 40 (b) 43 (c) 29 (d) 50 (e) none of these.

31. John ordered 4 pairs of black socks and some additional pairs of blue socks. The price of the black socks per pair was twice that of the blue. When the order was filled, it was found that the number of pairs of the two colors had been interchanged. This increased the bill by 50%. The ratio of the number of pairs of black socks to the number of pairs of blue socks in the original order was:

(a) 4:1 (b) 2:1 (c) 1:4 (d) 1:2 (e) 1:8

32. A 25 foot ladder is placed against a vertical wall of a building. The foot of the ladder is 7 feet from the base of the building. If the top of the ladder slips 4 feet, then the foot of the ladder will slide:

(a) 9 ft. (b) 15 ft. (c) 5 ft. (d) 8 ft. (e) 4 ft.

33. The number of circular pipes with an inside diameter of 1 inch which will carry the same amount of water as a pipe with an inside diameter of 6 inches is:

(a) 6π (b) 6 (c) 12 (d) 36 (e) 36π

34. When the circumference of a toy balloon is increased from 20 inches to 25 inches, the radius is increased by:

(a) 5 in. (b) 2 1/2 in. (c) $5/\pi$ in. (d) $5/2\pi$ in. (e) $\pi/5$ in.

35. In ΔABC, AC = 24", BC = 10", and AB = 26". The radius of the inscribed circle is:

(a) 26 in. (b) 4 in. (c) 13 in. (d) 8 in. (e) none of these answers

PART III. (4 credits each)

36. A merchant buys goods at 25% off the list price. He desires to mark the goods so that he can give a discount of 20% on the marked price and still clear a profit of 25% on the selling price. What per cent of the list price must he mark the goods?

(a) 125% (b) 100% (c) 120% (d) 80% (e) 75%

37. If $y = \log_a x$, and $a>1$, which of the following statements is incorrect?

(a) If $x = 1$, $y = 0$ (b) If $x = a$, $y = 1$

(c) If $x = -1$, y is imaginary (complex)

(d) If $0<x<1$, y is always less than 0 and decreases without limit as x approaches zero

(e) only some of the above statements are correct.

38. If the expression $\begin{vmatrix} a & c \\ d & b \end{vmatrix}$ has the value $ab - cd$ for all values of a, b, c and d, then the equation $\begin{vmatrix} 2x & 1 \\ x & x \end{vmatrix} = 3$

(a) is satisfied for only 1 value of x.

(b) is satisfied for 2 values of x.

(c) is satisfied for no values of x.

(d) is satisfied for an infinite number of values of x.

(e) is satisfied for none of these answers.

39. In the sequence 2, 1, 1/2, 1/4,......
    (1) the sum increases without limit.
    (2) the sum decreases without limit.
    (3) the difference between any term of the sequence and zero can be made less than any positive quantity no matter how small.
    (4) The difference between the sum and 4 can be made less than any positive quantity no matter how small.
    (5) the sum approaches a limit.

    Of the foregoing:

    (a) only (3) and (4) are correct statements

    (b) only (5) is a correct statement

    (c) only (2) and (4) are correct statements

    (d) only (2), (3) and (4) are correct statements

    (e) statements (a), (b), (c) and (d) are all incorrect.

40. The limit of $\frac{x^2 - 1}{x - 1}$ as x approaches 1 as a limit is:

    (a) $\frac{0}{0}$ (b) indeterminate (c) $x + 1$ (d) 2 (e) 1

41. The least value of the function $ax^2 + bx + c$ $(a>0)$ is:

    (a) $-\frac{b}{a}$ (b) $-\frac{b}{2a}$ (c) $b^2 - 4ac$ (d) $\frac{4ac - b^2}{4a}$ (e) none of the above.

42. If $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$, then x is equal to·

    (a) infinity (b) 2 (c) $\sqrt[4]{2}$ (d) $\sqrt{2}$ (e) none of these answers.

43. The sum to infinity of $\frac{1}{7} + \frac{2}{7^2} + \frac{1}{7^3} + \frac{2}{7^4} + \ldots$is:

    (a) $\frac{1}{5}$ (b) $\frac{1}{24}$ (c) $\frac{5}{48}$ (d) $\frac{1}{16}$ (e) none of these answers

44. The graph of $y = \log x$

    (a) cuts the y - axis (b) cuts all lines $\perp$ the x - axis

    (c) cuts the x - axis (d) cuts neither axis

    (e) cuts all circles whose center is at the origin

45. The number of diagonals that can be drawn in a polygon of 100 sides is:

    (a) 4850 (b) 4950 (c) 9900 (d) 98 (e) 8800

46. In triangle ABC, AB = 12, AC = 7, and BC = 10. If sides AB and AC are doubled while BC remains the same, then:

(a) the area is doubled (b) the altitude is doubled

(c) the area is four times the original area.

(d) the median is unchanged (e) the area of the triangle is 0.

47. A rectangle inscribed in a triangle has its base coinciding with the base (b) of the triangle. If the altitude of the triangle is h, and the altitude (x) of the rectangle is half the base of the rectangle, then:

(a) $x = \frac{1}{2}h$ (b) $x = \frac{bh}{h + b}$ (c) $x = \frac{bh}{2h + b}$ (d) $x = \sqrt{\frac{hb}{2}}$

(e) $x = \frac{1}{2}b$

48. A point is selected at random inside an equilateral triangle. From this point perpendiculars are dropped to each side. The sum of these perpendiculars is:

(a) least when the point is at the center of gravity of the Δ.

(b) greater than the altitude of the triangle.

(c) equal to the altitude of the triangle.

(d) one-half the sum of the sides of the triangle.

(e) the sum is greatest when the point is the center of gravity.

49. A triangle has a fixed base AB that is 2 inches long. The median from A to side BC is 1 1/2 inches long and can have any position emanating from A. The locus of the vertex C of the triangle is:

(a) a straight line AB, 1 1/2 in. from A.

(b) a circle with A as center and radius 2 in.

(c) a circle with A as center and radius 3 in.

(d) a circle with radius 3 in. and center 4 in. from B along BA.

(e) an ellipse with A as a focus.

50. A privateer discovers a merchantman 10 miles to leeward at 11:45 a.m. and there being a good breeze bears down upon her at 11 mph while the merchantman can only make 8 mph in her attempt to escape. After a two hour chase, the top sail of the privateer being carried away, she can make only 17 miles while the merchantman makes 15. The privateer will overtake the merchantman at:

(a) 3:45 p.m. (b) 3:30 p.m. (c) 5:00 p.m. (d) 2:45 p.m.

(e) 5:30 p.m.

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1 HOUR 20 MINUTES

TOTAL CREDITS
150

INSTRUCTIONS

In each of the fifty exercises enclose the correct answer, including the letter that precedes it, in a box. The following is a sample problem.

The sum of the exterior angles of a polygon depends on:

(a) the number of sides (b) the length of the sides (c) the sum of the interior angles (d) the equality of the interior angles (e) none of these

PART I. (2 credits each)

1. The per cent that M is greater than N, is:

(a) $\frac{100\ (M-N)}{M}$ (b) $\frac{100\ (M-N)}{N}$ (c) $\frac{M-N}{N}$ (d) $\frac{M-N}{M}$ (e) $\frac{100\ (M+N)}{N}$

2. A rectangular field is half as wide as it is long and is completely enclosed by x yards of fencing. The area in terms of x is:

(a) $\frac{x^2}{2}$ (b) $2x^2$ (c) $\frac{2x^2}{9}$ (d) $\frac{x^2}{18}$ (e) $\frac{x^2}{72}$

3. If the length of a diagonal of a square is $a+b$, then the area of the square is:

(a) $(a+b)^2$ (b) $\frac{1}{2}(a+b)^2$ (c) a^2+b^2 (d) $\frac{1}{2}(a^2+b^2)$ (e) none of these

4. A barn with a flat roof is rectangular in shape, 10 yd. wide, 13 yd. long and 5 yd. high. It is to be painted inside and outside, and on the ceiling, but not on the roof or floor. The total number of sq. yd. to be painted is:

(a) 360 (b) 460 (c) 490 (d) 590 (e) 720

5. Mr. A owns a home worth \$10,000. He sells it to Mr. B at a 10% profit. Mr. B sells the house back to Mr. A at a 10% loss. Then:

(a) A comes out even (b) A makes \$100 on the deal

(c) A makes \$1000 on the deal (d) A loses \$100 on the deal

(e) A loses \$1000 on the deal

6. The bottom, side, and front areas of a rectangular box are known. The product of these areas is equal to:

(a) The volume of the box (b) The square root of the volume

(c) Twice the volume (d) The square of the volume (e) The cube of the volume

7. An error of .02″ is made in the measurement of a line 10″ long, while an error of only .2″ is made in a measurement of a line 100″ long. In comparison with the relative error of the first measurement, the relative error of the second measurement is:

(a) greater by .18 (b) the same (c) less (d) 10 times as great
(e) correctly described by both (a) and (d)

8. The price of an article is cut 10%. To restore it to its former value, the new price must be increased by:

(a) 10% (b) 9% (c) $11\frac{1}{9}\%$ (d) 11% (e) none of these answers

9. An equilateral triangle is drawn with a side of length a. A new equilateral triangle is formed by joining the mid-points of the sides of the first one. Then a third equilateral triangle is formed by joining the mid-points of the sides of the second; and so on forever. The limit of the sum of the perimeters of all the triangles thus drawn is:

(a) Infinite (b) $5\frac{1}{4}a$ (c) $2a$ (d) $6a$ (e) $4\frac{1}{2}a$

10. Of the following statements, the one that is incorrect is:

(a) Doubling the base of a given rectangle doubles the area.
(b) Doubling the altitude of a triangle doubles the area.
(c) Doubling the radius of a given circle doubles the area.
(d) Doubling the divisor of a fraction and dividing its numerator by 2 changes the quotient.
(e) Doubling a given quantity may make it less than it originally was.

11. The limit of the sum of an infinite number of terms in a geometric progression is $\frac{a}{1-r}$ where a denotes the first term and $-1 < r < 1$ denotes the common ratio. The limit of the sum of their squares is:

(a) $\frac{a^2}{(1-r)^2}$ (b) $\frac{a^2}{1+r^2}$ (c) $\frac{a^2}{1-r^2}$ (d) $\frac{4a^2}{1+r^2}$ (e) none of these

12. At 2:15 o'clock, the hour and minute dials of a clock form an angle of:

(a) $30°$ (b) $5°$ (c) $22\frac{1}{2}°$ (d) $7\frac{1}{2}°$ (e) $28°$

13. A can do a piece of work in 9 days. B is 50% more efficient than A. The number of days it takes B to do the same piece of work is:

(a) $13\frac{1}{2}$ (b) $4\frac{1}{2}$ (c) 6 (d) 3 (e) none of these answers

14. In connection with proof in geometry, indicate which one of the following statements is incorrect:

(a) Some statements are accepted without being proved.
(b) In some instances there is more than one correct order in proving certain propositions.
(c) Every term used in a proof must have been defined previously.
(d) It is impossible to have a correct conclusion if in the beginning an incorrect proposition is introduced followed by correct reasoning.
(e) Indirect proof can be used whenever there are two or more contrary propositions.

15. The largest number by which the expression n^3-n is divisible for all possible positive values of n, is:

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

PART II (3 credits each)

16. If in applying the quadratic formula to a quadratic equation $f(x) = 0$, it happens that $c = \frac{b^2}{4a}$, then the graph of $y = f(x)$ will certainly:

(a) have a maximum (b) have a minimum (c) be tangent to the x-axis
(d) be tangent to the y-axis (e) lie in one quadrant only

17. Indicate in which one of the following equations y is neither directly nor inversely proportional to x:

(a) $x+y=0$ (b) $3xy=10$ (c) $x=5y$ (d) $3x+y=10$ (e) $\frac{x}{y}=\sqrt{3}$

18. $21x^2 + ax + 21$ is to be factored into prime binomial factors and without a numerical monomial factor. This can be done if the value ascribed to a is:

(a) any odd number (b) some odd number (c) any even number
(d) some even number (e) zero

19. A six place number is formed by repeating a three place number; for example, 256,256, or 678,678, etc. Any number of this form is always exactly divisible by:

(a) 7 only (b) 11 only (c) 13 only (d) 101 (e) 1001

20. When simplified and expressed with negative exponents, the expression $(x+y)^{-1}(x^{-1} + y^{-1})$ is equal to:

(a) $x^{-2} + 2x^{-1}y^{-1} + y^{-2}$ (b) $x^{-2} + 2^{-1}x^{-1}y^{-1} + y^{-2}$ (c) $x^{-1}y^{-1}$
(d) $x^{-2} + y^{-2}$ (e) $\frac{1}{x^{-1}y^{-1}}$

21. Given: $x > 0$, $y > 0$, $x > y$ and $z \neq 0$. The inequality which is not always correct is:

(a) $x+z > y+z$ (b) $x-z > y-z$ (c) $xz > yz$ (d) $\frac{x}{z^2} > \frac{y}{z^2}$ (e) $xz^2 > yz^2$

22. The values of a in the equation: $\log_{10}(a^2 - 15a) = 2$ is:

(a) $\frac{15 \pm \sqrt{233}}{2}$ (b) 20, -5 (c) $\frac{15 \pm \sqrt{305}}{2}$ (d) ± 20 (e) none of these

23. The radius of a cylindrical box is 8 inches and the height is 3 inches. The number of inches that may be added to either the radius or the height to give the same increase in volume is:

(a) 1 (b) $5\frac{1}{3}$ (c) any number (d) non-existent (e) none of these

24. $\dfrac{2^{n+4} - 2(2^n)}{2\ (2^{n+3})}$ when simplified is:

(a) $2^{n+1} - \frac{1}{8}$ (b) $-\ 2^{n+1}$ (c) $1 - 2^n$ (d) $\frac{7}{8}$ (e) $\frac{7}{4}$

25. The apothem of a square having its area numerically equal to its perimeter is compared with the apothem of an equilateral triangle having its area numerically equal to its perimeter. The first apothem will be:

(a) equal to the second (b) $\frac{4}{3}$ times the second (c) $\frac{2}{\sqrt{3}}$ times the second

(d) $\frac{\sqrt{2}}{\sqrt{3}}$ times the second (e) indeterminately related to the second

26. In the equation $\dfrac{x(x-1) - (m+1)}{(x-1)\ (m-1)} = \dfrac{x}{m}$ the roots are equal when

(a) $m = 1$ (b) $m = \frac{1}{2}$ (c) $m = 0$ (d) $m = -1$ (e) $m = -\frac{1}{2}$

27. Through a point inside a triangle, three lines are drawn from the vertices to the opposite sides forming six triangular sections. Then:

(a) The triangles are similar by opposite pairs.
(b) The triangles are congruent by opposite pairs.
(c) The triangles are equal in area by opposite pairs.
(d) Three similar quadrilaterals are formed.
(e) None of the above relations are true.

28. The pressure (P) of wind on a sail varies jointly as the area (A) of the sail and the square of the velocity (V) of the wind. The pressure on a square foot is 1 pound when the velocity is 16 miles per hour. The velocity of the wind when the pressure on a square yard is 36 pounds is:

(a) $10\frac{2}{3}$ mph (b) 96 mph (c) 32 mph (d) $1\frac{2}{3}$ mph (e) 16 mph

29. Of the following sets of data the only one that does not determine the shape of a triangle is:

(a) the ratio of two sides and the included angle
(b) the ratios of the three altitudes
(c) the ratios of the three medians
(d) the ratio of the altitude to the corresponding base
(e) two angles

30. If two poles 20″ and 80″ high are 100″ apart, then the height of the intersection of the lines joining the top of each pole to the foot of the opposite pole is:

(a) 50″ (b) 40″ (c) 16″ (d) 60″ (e) none of these

31. A total of 28 handshakes were exchanged at the conclusion of a party. Assuming that each participant was equally polite toward all the others, the number of people present was:

(a) 14 (b) 28 (c) 56 (d) 8 (e) 7

32. If ΔABC is inscribed in a semicircle whose diameter is AB, then $AC + BC$ must be:

(a) equal to AB (b) equal to $AB\sqrt{2}$ (c) $\geqslant AB\sqrt{2}$ (d) $\leqslant AB\sqrt{2}$ (e) $\overline{AB}^2$
(Hint: $a \geqslant b$ is read "a is equal to or greater than b.")

33. The roots of the equation $x^2 - 2x = 0$ can be obtained graphically by finding the abscissas of the points of intersection of each of the following pairs of equations except the pair:

(a) $y = x^2$, $y = 2x$ (b) $y = x^2 - 2x$, $y = 0$ (c) $y = x$, $y = x - 2$
(d) $y = x^2 - 2x + 1$, $y = 1$ (e) $y = x^2 - 1$, $y = 2x - 1$

34. The value of $10^{\log_{10}7}$ is:

(a) 7 (b) 1 (c) 10 (d) $\log_{10}7$ (e) $\log_7 10$

35. If $a^x = c^q = b$ and $c^y = a^z = d$, then:

(a) $xy = qz$ (b) $\frac{x}{y} = \frac{q}{z}$ (c) $x + y = q + z$ (d) $x - y = q - z$
(e) $x^y = q^z$

PART III. (4 credits each)

36. Which of the following methods of proving a geometric figure a locus is not correct?

(a) Every point on the locus satisfies the conditions and every point not on the locus does not satisfy the conditions.
(b) Every point not satisfying the conditions is not on the locus and every point on the locus does satisfy the conditions.
(c) Every point satisfying the conditions is on the locus and every point on the locus satisfies the conditions.
(d) Every point not on the locus does not satisfy the conditions and every point not satisfying the conditions is not on the locus.
(e) Every point satisfying the conditions is on the locus and every point not satisfying the conditions is not on the locus.

37. The number which when divided by 10 leaves a remainder of 9, when divided by 9 leaves a remainder of 8, by 8 leaves a remainder of 7, etc., down to where, when divided by 2, it leaves a remainder of 1, is:

(a) 59 (b) 419 (c) 1259 (d) 2519 (e) none of these answers

38. A rise of 600 feet is required to get a railroad line over a mountain. The grade can be kept down by lengthening the track and curving it around the mountain peak. The additional length of track required to reduce the grade from 3% to 2% is approximately:

(a) 10,000 feet (b) 20,000 feet (c) 30,000 feet (d) 12,000 feet

(e) none of these

39. A stone is dropped into a well and the report of the stone striking the bottom is heard 7.7 seconds after it is dropped. Assume that the stone falls $16t^2$ feet in t seconds and that the velocity of sound is 1,120 feet per second. The depth of the well is:

(a) 784 ft. (b) 342 ft. (c) 1568 ft. (d) 156.8 ft. (e) none of these

40. $\left[\frac{(x+1)^2(x^2-x+1)^2}{(x^3+1)^2}\right]^2 \cdot \left[\frac{(x-1)^2(x^2+x+1)^2}{(x^3-1)^2}\right]^2$ equals:

(a) $(x+1)^4$ (b) $(x^3+1)^4$ (c) 1 (d) $[(x^3+1)(x^3-1)]^2$ (e) $[(x^3-1)^2]^2$

41. The formula expressing the relationship between x and y in the table is:

x	2	3	4	5	6
y	0	2	6	12	20

(a) $y=2x-4$ (b) $y=x^2-3x+2$ (c) $y=x^3-3x^2+2x$ (d) $y=x^2-4x$ (e) $y=x^2-4$

42. If $x = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots\cdots}}}}$, then:

(a) $x=1$ (b) $0<x<1$ (c) $1<x<2$ (d) x is infinite (e) $x>2$ but finite

43. Of the following statements, the only one that is incorrect is:

(a) An inequality will remain true after each side is increased, decreased, multiplied or divided (zero excluded) by the same positive quantity.

(b) The arithmetic mean of two unequal positive quantities is greater than their geometric mean.

(c) If the sum of two positive quantities is given, their product is largest when they are equal.

(d) If a and b are positive and unequal, $\frac{a^2+b^2}{2}$ is greater than $\left(\frac{a+b}{2}\right)^2$.

(e) If the product of two positive quantities is given, their sum is greatest when they are equal.

44. If $\frac{xy}{x+y}=a$, $\frac{xz}{x+z}=b$ and $\frac{yz}{y+z}=c$, where a, b, and c are other than zero, then x equals:

(a) $\frac{abc}{ab+ac+bc}$ (b) $\frac{2abc}{ab+bc+ac}$ (c) $\frac{2abc}{ab+ac-bc}$ (d) $\frac{2abc}{ab+bc-ac}$ (e) $\frac{2abc}{ac+bc-ab}$

45. If you are given $\log 8 = .9031$ and $\log 9 = .9542$, then the only logarithm that cannot be found without the use of tables is:

(a) $\log 17$ (b) $\log \frac{5}{4}$ (c) $\log 15$ (d) $\log 600$ (e) $\log .4$

46. AB is a fixed diameter of a circle whose center is O. From C, any point on the circle, a chord CD is drawn perpendicular to AB. Then, as C moves over a semicircle, the bisector of angle OCD cuts the circle in a point that always:

(a) bisects the arc AB (b) trisects the arc AB (c) varies

(d) is as far from AB as from D (e) is equidistant from B and C.

47. If r and s are the roots of the equation $ax^2+bx+c=0$, the value of $\frac{1}{r^2}+\frac{1}{s^2}$ is:

(a) b^2-4ac (b) $\frac{b^2-4ac}{2a}$ (c) $\frac{b^2-4ac}{c^2}$ (d) $\frac{b^2-2ac}{c^2}$ (e) none of these answers

48. The area of a square inscribed in a semicircle is to the area of the square inscribed in the entire circle as

(a) 1:2 (b) 2:3 (c) 2:5 (d) 3:4 (e) 3:5

49. The medians of a right triangle which are drawn from the vertices of the acute angles are 5 and $\sqrt{40}$ respectively. The value of the hypotenuse is:

(a) 10 (b) $2\sqrt{40}$ (c) $\sqrt{13}$ (d) $2\sqrt{13}$ (e) none of these answers

50. Tom, Dick and Harry started out on a 100-mile journey. Tom and Harry went by automobile at the rate of 25 mph, while Dick walked at the rate of 5 mph. After a certain distance, Tom let Harry off, who walked on at 5 mph, while Tom went back for Dick and got him to the destination at the same time that Harry arrived. The number of hours required for the trip was:

(a) 5 (b) 6 (c) 7 (d) 8 (e) none of these answers

MATHEMATICAL CONTEST

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SCORE

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THURSDAY MORNING, MAY 1, 1952

TIME OF CONTEST 1 HOUR 20 MINUTES

TOTAL CREDITS 150

INSTRUCTIONS

In each of the fifty exercises enclose the correct answer, including the letter that precedes it, in a box. The following is a sample problem.

If the length of a median of a triangle is one-half the length of the side to which it is drawn, the triangle is:

(a) equilateral [(b) right] (c) isosceles (d) acute (e) obtuse

PART I. (2 credits each)

1. If the radius of a circle is a rational number, its area is given by a number which is:

   (a) rational (b) irrational (c) integral (d) a perfect square

   (e) none of these

2. Two high school classes take the same test. One class of 20 students made an average grade of 80%; the other class of 30 students made an average grade of 70%. The average grade for all students in both classes is:

   (a) 75% (b) 74% (c) 72% (d) 77% (e) none of these

3. The expression $a^3 - a^{-3}$ equals:

   (a) $(a - \frac{1}{a})(a^2 + 1 + \frac{1}{a^2})$ (b) $(\frac{1}{a} - a)(a^2 - 1 + \frac{1}{a^2})$

   (c) $(a - \frac{1}{a})(a^2 - 2 + \frac{1}{a^2})$ (d) $(\frac{1}{a} - a)(\frac{1}{a^2} + 1 + a^2)$ (e) none of these

4. The cost C of sending a parcel post package weighing P pounds, P an integer, is 10 cents for the first pound and 3 cents for each additional pound. The formula for the cost is:

   (a) $C = 10 + 3P$ (b) $C = 10P + 3$ (c) $C = 10 + 3(P-1)$

   (d) $C = 9 + 3P$ (e) $C = 10P - 7$

5. The points (6, 12) and (0, -6) are connected by a straight line. Another point on this line is:

   (a) (3, 3) (b) (2, 1) (c) (7, 16) (d) (-1, -4) (e) (-3, -8)

6. The difference of the roots of $x^2 - 7x - 9 = 0$ is:

   (a) +7 (b) $+\frac{7}{2}$ (c) +9 (d) $2\sqrt{85}$ (e) $\sqrt{85}$

7. When simplified, $(x^{-1} + y^{-1})^{-1}$ is equal to:

(a) $x + y$ (b) $\frac{xy}{x+y}$ (c) xy (d) $\frac{1}{xy}$ (e) $\frac{x+y}{xy}$

8. Two equal circles in the same plane cannot have the following number of common tangents.

(a) 1 (b) 2 (c) 3 (d) 4 (e) none of these

9. If $m = \frac{cab}{a-b}$, then b equals:

(a) $\frac{m(a-b)}{ca}$ (b) $\frac{cab - ma}{-m}$ (c) $\frac{1}{1+c}$ (d) $\frac{ma}{m+ca}$ (e) $\frac{m+ca}{ma}$

10. An automobile went up a hill at a speed of 10 miles an hour and down the same distance at a speed of 20 miles an hour. The average speed for the round trip was:

(a) $12\frac{1}{2}$ mph (b) $13\frac{1}{3}$ mph (c) $14\frac{1}{2}$ mph (d) 15 mph (e) none of these

11. If $y = f(x) = \frac{x+2}{x-1}$, then it is incorrect to say:

(a) $x = \frac{y+2}{y-1}$ (b) $f(0) = -2$ (c) $f(1) = 0$ (d) $f(-2) = 0$ (e) $f(y) = x$

12. The sum to infinity of the terms of an infinite geometric progression is 6. The sum of the first two terms is $4\frac{1}{2}$. The first term of the progression is:

(a) 3 or $1\frac{1}{2}$ (b) 1 (c) $2\frac{1}{2}$ (d) 6 (e) 9 or 3

13. The function $x^2 + px + q$ with p and q greater than zero has its minimum value when:

(a) $x = -p$ (b) $x = \frac{p}{2}$ (c) $x = -2p$ (d) $x = \frac{p^2}{4q}$ (e) $x = \frac{-p}{2}$

14. A house and store were sold for \$12,000 each. The house was sold at a loss of 20% of the cost, and the store at a gain of 20% of the cost. The entire transaction resulted in:

(a) no loss or gain (b) loss of \$1000 (c) gain of \$1000

(d) gain of \$2000 (e) none of these

15. The sides of a triangle are in the ratio 6 : 8 : 9. Then:

(a) the triangle is obtuse (b) the angles are in the ratio 6 : 8 : 9

(c) the triangle is acute (d) the angle opposite the largest side is double the angle opposite the smallest side

(e) none of these

PART II. (3 credits each)

16. If the base of a rectangle is increased by 10% and the area is unchanged, then the altitude is decreased by:

(a) 9% (b) 10% (c) 11% (d) $11\frac{1}{9}\%$ (e) $9\frac{1}{11}\%$

17. A merchant bought some goods at a discount of 20% off the list price. He wants to mark them at such a price that he can give a discount of 20% of the marked price and still make a profit of 20% of the selling price. The per cent of the list price at which he should mark them is:

(a) 20 (b) 100 (c) 125 (d) 80 (e) 120

18. Log p + log q = log $(p + q)$ only if:

(a) $p = q =$ zero (b) $p = \frac{q^2}{1-q}$ (c) $p = q = 1$ (d) $p = \frac{q}{q-1}$ (e) $p = \frac{q}{q+1}$

19. Angle B of triangle ABC is trisected by BD and BE which meet AC at D and E respectively. Then:

(a) $\frac{AD}{EC} = \frac{AE}{DC}$ (b) $\frac{AD}{EC} = \frac{AB}{BC}$ (c) $\frac{AD}{EC} = \frac{BD}{BE}$ (d) $\frac{AD}{EC} = \frac{(AB)(BD)}{(BE)(BC)}$

(e) $\frac{AD}{EC} = \frac{(AE)(BD)}{(DC)(BE)}$

20. If $\frac{x}{y} = \frac{3}{4}$, then the incorrect expression in the following is:

(a) $\frac{x+y}{y} = \frac{7}{4}$ (b) $\frac{y}{y-x} = \frac{4}{1}$ (c) $\frac{x+2y}{x} = \frac{11}{3}$ (d) $\frac{x}{2y} = \frac{3}{8}$ (e) $\frac{x-y}{y} = \frac{1}{4}$

21. The sides of a regular polygon of n sides, $n > 4$, are extended to form a star. The number of degrees at each point of the star is:

(a) $\frac{360}{n}$ (b) $\frac{(n-4)180}{n}$ (c) $\frac{(n-2)180}{n}$ (d) $180 - \frac{90}{n}$ (e) $\frac{180}{n}$

22. On hypotenuse AB of a right triangle ABC a second right triangle ABD is constructed with hypotenuse AB. If $BC = 1$, $AC = b$, and $AD = 2$, then BD equals:

(a) $\sqrt{b^2+1}$ (b) $\sqrt{b^2-3}$ (c) $\sqrt{b^2+1}+2$ (d) b^2+5 (e) $\sqrt{b^2+3}$

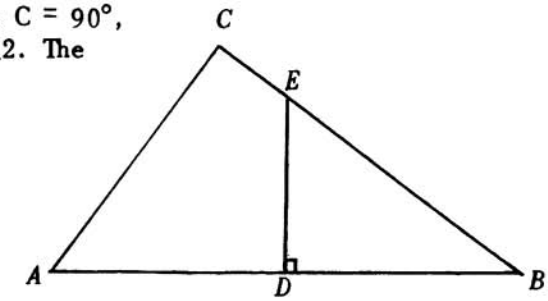
23. If $\frac{x^2-bx}{ax-c} = \frac{m-1}{m+1}$ has roots which are numerically equal but of opposite signs, the value of m must be:

(a) $\frac{a-b}{a+b}$ (b) $\frac{a+b}{a-b}$ (c) c (d) $\frac{1}{c}$ (e) 1

24. In the figure, it is given that angle C = 90°, $AD = DB$, $DE \perp AB$, $AB = 20$, and $AC = 12$. The area of quadrilateral $ADEC$ is:

(a) 75 (b) $58\frac{1}{2}$ (c) 48

(d) $37\frac{1}{2}$ (e) none of these



25. A powderman set a fuse for a blast to take place in 30 seconds. He ran away at a rate of 8 yards per second. Sound travels at the rate of 1080 feet per second. When the powderman heard the blast, he had run approximately:

(a) 200 yd. (b) 352 yd. (c) 300 yd. (d) 245 yd. (e) 512 yd.

26. If $(r + \frac{1}{r})^2 = 3$, then $r^3 + \frac{1}{r^3}$ equals:

(a) 1 (b) 2 (c) 0 (d) 3 (e) 6

27. The ratio of the perimeter of an equilateral triangle having an altitude equal to the radius of a circle, to the perimeter of an equilateral triangle inscribed in the circle is:

(a) 1 : 2 (b) 1 : 3 (c) $1 : \sqrt{3}$ (d) $\sqrt{3} : 2$ (e) 2 : 3

28. In the table shown, the formula relating x and y is:

(a) $y = 4x - 1$ (b) $y = x^3 - x^2 + x + 2$ (c) $y = x^2 + x + 1$

(d) $y = (x^2 + x + 1)(x - 1)$ (e) none of these

x	1	2	3	4	5
y	3	7	13	21	31

29. In a circle of radius 5 units, CD and AB are perpendicular diameters. A chord CH cutting AB at K is 8 units long. The diameter AB is divided into two segments whose dimensions are:

(a) 1.25, 8.75 (b) 2.75, 7.25 (c) 2, 8 (d) 4, 6 (e) none of these

30. When the sum of the first ten terms of an arithmetic progression is four times the sum of the first five terms, the ratio of the first term to the common difference is:

(a) 1 : 2 (b) 2 : 1 (c) 1 : 4 (d) 4 : 1 (e) 1 : 1

31. Given 12 points in a plane no three of which are collinear, the number of lines they determine is:

(a) 24 (b) 54 (c) 120 (d) 66 (e) none of these

32. K takes 30 minutes less time than M to travel a distance of 30 miles. K travels $\frac{1}{3}$ mile per hour faster than M. If x is K's rate of speed in miles per hour, then K's time for the distance is:

(a) $\frac{x+\frac{1}{3}}{30}$ (b) $\frac{x-\frac{1}{3}}{30}$ (c) $\frac{30}{x+\frac{1}{3}}$ (d) $\frac{30}{x}$ (e) $\frac{x}{30}$

33. A circle and a square have the same perimeter. Then:

(a) their areas are equal (b) the area of the circle is the greater

(c) the area of the square is the greater

(d) the area of the circle is π times the area of the square

(e) none of these

34. The price of an article was increased $p\%$. Later the new price was decreased $p\%$. If the last price was one dollar, the original price was:

(a) $\frac{1-p^2}{200}$ (b) $\frac{\sqrt{1-p^2}}{100}$ (c) one dollar (d) $1-\frac{p^2}{10,000-p^2}$

(e) $\frac{10,000}{10,000-p^2}$

35. When written with a rational denominator, the expression $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$ is equivalent to:

(a) $\frac{3+\sqrt{6}+\sqrt{15}}{6}$ (b) $\frac{\sqrt{6}-2+\sqrt{10}}{6}$ (c) $\frac{2+\sqrt{6}+\sqrt{10}}{10}$

(d) $\frac{2+\sqrt{6}-\sqrt{10}}{6}$ (e) none of these

PART III. (4 credits each)

36. To be continuous at $x = -1$, the value of $\frac{x^3+1}{x^2-1}$ is taken to be:

(a) -2 (b) 0 (c) $\frac{3}{2}$ (d) ∞ (e) $-\frac{3}{2}$

37. Two equal parallel chords are drawn 8 inches apart in a circle of radius 8 inches. The area of that part of the circle that lies between the chords is:

(a) $21\frac{1}{3}\pi - 32\sqrt{3}$ (b) $32\sqrt{3}+21\frac{1}{3}\pi$ (c) $32\sqrt{3}+42\frac{2}{3}\pi$

(d) $16\sqrt{3}+42\frac{2}{3}\pi$ (e) $42\frac{2}{3}\pi$

38. The area of a trapezoidal field is 1400 square yards. Its altitude is 50 yards. Find the two bases, if the number of yards in each base is an integer divisible by 8. The number of solutions to this problem is:

(a) none (b) one (c) two (d) three (e) more than three

39. If the perimeter of a rectangle is p and its diagonal is d, the difference between the length and width of the rectangle is:

(a) $\frac{\sqrt{8d^2 - p^2}}{2}$ (b) $\frac{\sqrt{8d^2 + p^2}}{2}$ (c) $\frac{\sqrt{6d^2 - p^2}}{2}$ (d) $\frac{\sqrt{6d^2 + p^2}}{2}$

(e) $\frac{\sqrt{8d^2 - p^2}}{4}$

40. In order to draw a graph of $f(x) = ax^2 + bx + c$, a table of values was constructed. These values of the function for a set of equally spaced increasing values of x were 3844, 3969, 4096, 4227, 4356, 4489, 4624, and 4761. The one which is incorrect is:

(a) 4096 (b) 4356 (c) 4489 (d) 4761 (e) none of these

41. Increasing the radius of a cylinder by 6 units increases the volume by y cubic units. Increasing the altitude of the cylinder by 6 units also increases the volume by y cubic units. If the original altitude is 2, then the original radius is:

(a) 2 (b) 4 (c) 6 (d) 6π (e) 8

42. Let D represent a repeating decimal. If P denotes the r figures of D which do not repeat themselves, and Q denotes the s figures which do repeat themselves, then the incorrect expression is:

(a) $D = .PQQQ\cdots$ (b) $10^r D = P.QQQ\cdots$ (c) $10^{r+s}D = PQ.QQQ\cdots$

(d) $10^r(10^s - 1)D = Q(P - 1)$ (e) $10^r \cdot 10^{2s}D = PQQ.QQQ\cdots$

43. The diameter of a circle is divided into n equal parts. On each part a semicircle is constructed. As n becomes very large, the sum of the lengths of the arcs of the semi-circles approaches:

(a) equal to the semi-circumference of the original circle

(b) equal to the diameter of the original circle

(c) greater than the diameter but less than the semi-circumference of the original circle

(d) infinite in length

(e) greater than the semi-circumference but finite

44. If an integer of two digits is k times the sum of its digits, the number formed by interchanging the digits is the sum of the digits multiplied by:

(a) $(9 - k)$ (b) $(10 - k)$ (c) $(11 - k)$ (d) $(k - 1)$ (e) $(k + 1)$

45. If a and b are two unequal positive numbers, then:

(a) $\frac{2ab}{a+b} > \sqrt{ab} > \frac{a+b}{2}$ (b) $\sqrt{ab} > \frac{2ab}{a+b} > \frac{a+b}{2}$ (c) $\frac{2ab}{a+b} > \frac{a+b}{2} > \sqrt{ab}$

(d) $\frac{a+b}{2} > \frac{2ab}{a+b} > \sqrt{ab}$ (e) $\frac{a+b}{2} > \sqrt{ab} > \frac{2ab}{a+b}$

46. The base of a new rectangle equals the sum of the diagonal and the greater side of a given rectangle, while the altitude of the new rectangle equals the difference of the diagonal and the greater side of the given rectangle. The area of the new rectangle is:

(a) greater than the area of the given rectangle

(b) equal to the area of the given rectangle

(c) equal to the area of a square with its side equal to the smaller side of the given rectangle

(d) equal to the area of a square with its side equal to the greater side of the given rectangle

(e) equal to the area of a rectangle whose dimensions are the diagonal and shorter side of the given rectangle

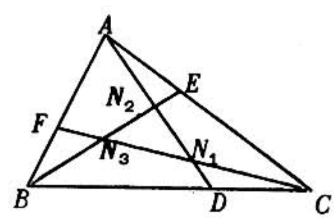
47. In the set of equations $z^x = y^{2x}$, $2^z = 2\cdot 4^x$, $x + y + z = 16$, the integral roots in the order x, y, z are:

(a) 3, 4, 9 (b) 9, -5, 12 (c) 12, -5, 9 (d) 4, 3, 9 (e) 4, 9, 3

48. Two cyclists, k miles apart, and starting at the same time, would be together in r hours if they traveled in the same direction, but would pass each other in t hours if they traveled in opposite directions. The ratio of the speed of the faster cyclist to that of the slower is:

(a) $\frac{r+t}{r-t}$ (b) $\frac{r}{r-t}$ (c) $\frac{r+t}{r}$ (d) $\frac{r}{t}$ (e) $\frac{r+k}{t-k}$

49. In the figure, CD, AE and BF are one-third of their respective sides. It follows that $AN_2 : N_2N_1 : N_1D = 3 : 3 : 1$, and similarly for lines BE and CF. Then the area of triangle $N_1N_2N_3$ is:



(a) $\frac{1}{10}\triangle ABC$ (b) $\frac{1}{9}\triangle ABC$

(c) $\frac{1}{7}\triangle ABC$ (d) $\frac{1}{6}\triangle ABC$

(e) none of these

50. A line initially 1 inch long, grows according to the following law, where the first term is the initial length.

$$1 + \frac{1}{4}\sqrt{2} + \frac{1}{4} + \frac{1}{16}\sqrt{2} + \frac{1}{16} + \frac{1}{64}\sqrt{2} + \frac{1}{64} + \ldots$$

If the growth process continued forever, the limit of the length of the line is:

(a) ∞ (b) $\frac{4}{3}$ (c) $\frac{8}{3}$ (d) $\frac{1}{3}(4+\sqrt{2})$ (e) $\frac{2}{3}(4+\sqrt{2})$

Fourth Annual

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THURSDAY MORNING, MAY 14, 1953

TIME OF CONTEST	TOTAL CREDITS
1 HOUR 20 MINUTES	150

INSTRUCTIONS

In each of the fifty exercises enclose the correct answer, including the letter that precedes it, in a box. The following is a sample problem.

The sum of any two fractions, $a/b + c/d$ is:

(a) $\frac{a+c}{b+d}$ (b) $\frac{a+c}{bd}$ **[(c) $\frac{ad+bc}{bd}$]** (d) $\frac{ab+cd}{bd}$ (e) $\frac{ad+bc}{b+d}$

PART I. (2 credits each)

1. A boy buys oranges at 3 for 10 cents. He will sell them for 5 for 20 cents. In order to make a profit of \$1.00, he must sell

   (a) 67 oranges (b) 150 oranges (c) 200 oranges

   (d) an infinite number of oranges (e) none of these

2. A refrigerator is offered for sale at \$250.00 less successive discounts of 20% and 15%. The sale price of the refrigerator is

   (a) 35% less than \$250.00 (b) 65% of \$250.00 (c) 77% of \$250.00

   (d) 68% of \$250.00 (e) none of these

3. The factors of the expression $x^2 + y^2$ are

   (a) $(x + y)(x - y)$ (b) $(x + y)^2$ (c) $(x^{2/3} + y^{2/3})(x^{4/3} + y^{4/3})$

   (d) $(x + iy)(x - iy)$ (e) none of these

4. The roots of $x(x^2 + 8x + 16)(4 - x) = 0$ are

   (a) 0 (b) 0, 4 (c) 0, 4, −4 (d) 0, 4, −4, −4 (e) none of these

5. If $\log_6 x = 2.5$, the value of x is

   (a) 90 (b) 36 (c) $36\sqrt{6}$ (d) 0.5 (e) none of these

6. Charles has $5q + 1$ quarters and Richard has $q + 5$ quarters. The difference in their money in dimes is

   (a) $10(q - 1)$ (b) $2/5(4q - 4)$ (c) $2/5(q - 1)$ (d) $5/2(q - 1)$

   (e) none of these

7. The fraction $\dfrac{\sqrt{a^2 + x^2} - \dfrac{x^2 - a^2}{\sqrt{a^2 + x^2}}}{a^2 + x^2}$ reduces to

(a) 0 (b) $\dfrac{2a^2}{a^2 + x^2}$ (c) $\dfrac{2x^2}{(a^2 + x^2)^{3/2}}$ (d) $\dfrac{2a^2}{(a^2 + x^2)^{3/2}}$ (e) $\dfrac{2x^2}{a^2 + x^2}$

8. The value of x at the intersection of $y = \dfrac{8}{x^2 + 4}$ and $x + y = 2$ is:

(a) $-2 + \sqrt{5}$ (b) $-2 - \sqrt{5}$ (c) 0 (d) 2 (e) none of these

9. The number of ounces of water needed to reduce 9 ounces of shaving lotion containing 50% alcohol to a lotion containing 30% alcohol is:

(a) 3 (b) 4 (c) 5 (d) 6 (e) 7

10. The number of revolutions of a wheel, with fixed center and with an outside diameter of 6 feet, required to cause a point on the rim to go one mile is:

(a) 880 (b) $\dfrac{440}{\pi}$ (c) $\dfrac{880}{\pi}$ (d) 440π (e) none of these

11. A running track is the ring formed by two concentric circles. It is 10 feet wide. The circumferences of the two circles differ by about:

(a) 10 feet (b) 30 feet (c) 60 feet (d) 100 feet (e) none of these

12. The diameters of two circles are 8 inches and 12 inches respectively. The ratio of the area of the smaller to the area of the larger circle is:

(a) 2/3 (b) 4/9 (c) 9/4 (d) 1/2 (e) none of these

13. A triangle and a trapezoid are equal in area. They also have the same altitude. If the base of the triangle is 18 inches, the median of the trapezoid is:

(a) 36 inches (b) 9 inches (c) 18 inches (d) not enough information

(e) none of these answers

14. Given the larger of two circles with center P and radius p and the smaller with center Q and radius q. Draw PQ. The following statement is false.

(a) $p - q$ can be equal to PQ (b) $p + q$ can be equal to PQ

(c) $p + q$ can be less than PQ (d) $p - q$ can be less than PQ

(e) none of these

15. A circular piece of metal of maximum size is cut out of a square piece and then a square piece of maximum size is cut out of the circular piece. The total amount of metal wasted is:

(a) 1/4 the area of the original square (b) 1/2 the area of the original square
(c) 1/2 the area of the circular piece (d) 1/4 the area of the circular piece
(e) none of these

PART II. (3 credits each)

16. Adams plans a profit of 10% on the selling price of an article and his expenses are 15% of sales. The rate of mark-up on an article that sells for $5.00 is:

(a) 20% (b) 25% (c) 30% (d) 33 1/3% (e) 35%

17. A man has part of $4500 invested at 4% and the rest at 6%. If his annual return on each investment is the same, the average rate of interest which he realizes on the $4500 is:

(a) 5% (b) 4.8% (c) 5.2% (d) 4.6% (e) none of these

18. One of the factors of $x^4 + 4$ is:

(a) $x^2 + 2$ (b) $x + 1$ (c) $x^2 - 2x + 2$ (d) $x^2 - 4$ (e) none of these

19. In the expression xy^2, the values of x and y are each decreased 25%, the value of the expression is:

(a) decreased 50% (b) decreased 75% (c) decreased 37/64 of its value
(d) decreased 27/64 of its value (e) none of these

20. If $y = x + 1/x$, then $x^4 + x^3 - 4x^2 + x + 1 = 0$ becomes:

(a) $x^2(y^2 + y - 2) = 0$ (b) $x^2(y^2 + y - 3) = 0$ (c) $x^2(y^2 + y - 4) = 0$
(d) $x^2(y^2 + y - 6) = 0$ (e) none of these

21. If $\log_{10} (x^2 - 3x + 6) = 1$, the value of x is:

(a) 10 or 2 (b) 4 or −2 (c) 3 or −1 (d) 4 or −1 (e) none of these

22. The logarithm of $27 \sqrt[4]{9} \sqrt[3]{9}$ to the base 3 is:

(a) 8 1/2 (b) 4 1/6 (c) 5 (d) 3 (e) none of these

23. The equation $\sqrt{x+10} - \dfrac{6}{\sqrt{x+10}} = 5$ has:

(a) an extraneous root between -5 and -1 (b) an extraneous root between -10 and -6

(c) a real root between 20 and 25 (d) two real roots (e) two extraneous roots

24. If a, b and c are positive integers less than 10, then $(10a + b)(10a + c)$ equals $100a(a + 1) + bc$ if:

(a) $b + c = 10$ (b) $b = c$ (c) $a + b = 10$ (d) $a = b$ (e) $a + b + c = 10$

25. In a geometric progression whose terms are positive, any term is equal to the sum of the next two following terms. Then the common ratio is:

(a) 1 (b) about $\dfrac{\sqrt{5}}{2}$ (c) $\dfrac{\sqrt{5}-1}{2}$ (d) $\dfrac{1-\sqrt{5}}{2}$ (e) $\dfrac{2}{\sqrt{5}}$

26. The base of a triangle is 15 inches. Two lines are drawn parallel to the base, terminating in the other two sides, and dividing the triangle into three equal areas. The length of the parallel close to the base is:

(a) $5\sqrt{6}$ inches (b) 10 inches (c) $4\sqrt{3}$ inches (d) 7.5 inches

(e) none of these

27. The radius of the first circle is 1 inch, that of the second ½ inch, that of the third ¼ inch and so on indefinitely. The sum of the areas of the circles is:

(a) $3\pi/4$ (b) 1.3π (c) 2π (d) $4\pi/3$ (e) none of these

28. In triangle ABC, sides a, b and c are opposite angles A, B and C respectively. AD bisects angle A and meets BC at D. Then if $x = CD$ and $y = BD$ the correct proportion is:

(a) $x/a = a/(b + c)$ (b) $x/b = a/(a + c)$ (c) $y/c = c/(b + c)$

(d) $y/c = a/(b + c)$ (e) $x/y = c/b$

29. The number of significant digits in the measurement of the side of a square whose computed area is 1.1025 square inches to the nearest ten-thousandth of a square inch is:

(a) 2 (b) 3 (c) 4 (d) 5 (e) 1

30. A house worth \$9000 is sold by Mr. A to Mr. B at a 10% loss. Mr. B sells the house back to Mr. A at a 10% gain. The result of the two transactions is:

(a) Mr. A breaks even (b) Mr. B gains \$900 (c) Mr. A loses \$900

(d) Mr. A loses \$810 (e) Mr. B gains \$1710

31. The rails on a railroad are 30 feet long. As the train passes over the point where the rails are joined, there is an audible click. The speed of the train in miles per hour is approximately the number of clicks heard in:

(a) 20 seconds (b) 2 minutes (c) 1 1/2 minutes (d) 5 minutes

(e) none of these

32. Each angle of a rectangle is trisected. The intersections of the pairs of trisectors adjacent to the same side always form:

(a) a square (b) a rectangle (c) a parallelogram with unequal sides

(d) a rhombus (e) a trapezium

33. The perimeter of an isosceles right triangle is $2p$. Its area is:

(a) $(2 + \sqrt{2})p$ (b) $(2 - \sqrt{2})p$ (c) $(3 - 2\sqrt{2})p^2$ (d) $(1 - 2\sqrt{2})p^2$

(e) $(3 + 2\sqrt{2})p^2$

34. If one side of a triangle is 12 inches and the opposite angle is 30 degrees, then the diameter of the circumscribed circle is:

(a) 18 inches (b) 30 inches (c) 24 inches (d) 20 inches

(e) none of these

35. If $f(x) = \frac{x(x-1)}{2}$, then $f(x+2)$ equals:

(a) $f(x) + f(2)$ (b) $(x+2)\,f(x)$ (c) $x(x+2)\,f(x)$ (d) $\frac{x\,f(x)}{x+2}$

(e) $\frac{(x+2)\,f(x+1)}{x}$

PART III. (4 credits each)

36. Determine m so that $4x^2 - 6x + m$ is divisible by $x - 3$. The obtained value, m, is an exact divisor of:

(a) 12 (b) 20 (c) 36 (d) 48 (e) 64

37. The base of an isosceles triangle is 6 inches and one of the equal sides is 12 inches. The radius of the circle through the vertices of the triangle is:

(a) $\frac{7\sqrt{15}}{5}$ (b) $4\sqrt{3}$ (c) $3\sqrt{5}$ (d) $6\sqrt{3}$ (e) none of these

38. If $f(a) = a - 2$ and $F(a, b) = b^2 + a$, then $F[3, f(4)]$ is:

(a) $a^2 - 4a + 7$ (b) 28 (c) 7 (d) 8 (e) 11

39. The product, $\log_a b \cdot \log_b a$ is equal to:

(a) 1 (b) a (c) b (d) ab (e) none of these

40. The contradictory of the statement, "all men are honest," is:

(a) no men are honest (b) all men are dishonest (c) some men are dishonest (d) no men are dishonest (e) some men are honest

41. A girls' camp is located 300 rods from a straight road. On this road, a boys' camp is located 500 rods from the girls' camp. It is desired to build a canteen on the road which shall be exactly the same distance from each camp. The distance of the canteen from each of the camps is:

(a) 400 rods (b) 250 rods (c) 87.5 rods (d) 200 rods (e) none of these answers

42. The centers of two circles are 41 inches apart. The smaller circle has a radius of 4 inches and the larger one has a radius of 5 inches. The length of the common internal tangent is:

(a) 41 inches (b) 39 inches (c) 39.8 inches (d) 40.1 inches (e) 40 inches

43. If the price of an article is increased by per cent p, then the decrease in per cent of sales must not exceed d in order to yield the same income. The value of d is:

(a) $1/(1 + p)$ (b) $1/(1 - p)$ (c) $p/(1 + p)$ (d) $p/(p - 1)$ (e) $(1 - p)/(1 + p)$

44. In solving a problem that reduces to a quadratic equation one student makes a mistake only in the constant term of the equation and obtains 8 and 2 for the roots. Another student makes a mistake only in the coefficient of the first degree term and finds -9 and -1 for the roots. The correct equation was:

(a) $x^2 - 10x + 9 = 0$ (b) $x^2 + 10x + 9$ (c) $x^2 - 10x + 16 = 0$

(d) $x^2 - 8x - 9 = 0$ (e) none of these

45. The lengths of two line segments are a units and b units respectively. Then the correct relation between them is:

(a) $\frac{(a+b)}{2} > \sqrt{ab}$ (b) $\frac{(a+b)}{2} < \sqrt{ab}$ (c) $\frac{(a+b)}{2} = \sqrt{ab}$

(d) $\frac{(a+b)}{2} \leqq \sqrt{ab}$ (e) $\frac{(a+b)}{2} \geqq \sqrt{ab}$

46. Instead of walking along two adjacent sides of a rectangular field, a boy took a short-cut along the diagonal of the field and saved a distance equal to 1/2 the longer side. The ratio of the shorter side of the rectangle to the longer side was:

(a) 1/2 (b) 2/3 (c) 1/4 (d) 3/4 (e) 2/5

47. If x is greater than zero, then the correct relationship is:

(a) $\log(1+x) = x/(1+x)$ (b) $\log(1+x) < x/(1+x)$

(c) $\log(1+x) > x$ (d) $\log(1+x) < x$ (e) none of these

48. If the larger base of an isosceles trapezoid equals a diagonal and the smaller base equals the altitude, then the ratio of the smaller base to the larger base is:

(a) 1/2 (b) 2/3 (c) 3/4 (d) 3/5 (e) 2/5

49. The coordinates of A, B and C are $(5,5)$, $(2,1)$ and $(0,k)$ respectively. The value of k that makes $AC + BC$ as small as possible is:

(a) 3 (b) 4 1/2 (c) 3 6/7 (d) 4 5/6 (e) 2 1/7

50. One of the sides of a triangle is divided into segments of 6 and 8 units by the point of tangency of the inscribed circle. If the radius of the circle is 4, then the length of the shortest side of the triangle is:

(a) 12 units (b) 13 units (c) 14 units (d) 15 units (e) 16 units

Committee on Mathematics Contests and Awards,
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Fifth Annual

MATHEMATICAL CONTEST

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INSTRUCTIONS

In each of the fifty exercises enclose the correct answer, including the letter that precedes it, in a box. The following is a sample problem.

The sum of any two fractions, $a/b + c/d$ is:

(a) $\frac{a+c}{b+d}$ (b) $\frac{a+c}{bd}$ **[(c) $\frac{ad+bc}{bd}$]** (d) $\frac{ab+cd}{bd}$ (e) $\frac{ad+bc}{b+d}$

PART I. (2 credits each)

1. The square of $5 - \sqrt{y^2 - 25}$ is:

(a) $y^2 - 5\sqrt{y^2-25}$ (b) $-y^2$ (c) y^2 (d) $(5-y)^2$ (e) $y^2 - 10\sqrt{y^2-25}$

2. The equation $\frac{2x^2}{x-1} - \frac{2x+7}{3} + \frac{4-6x}{x-1} + 1 = 0$ can be transformed by eliminating fractions to the equation $x^2 - 5x + 4 = 0$. The roots of the latter equation are 4 and 1. Then the roots of the first equation are:

(a) 4 and 1 (b) only 1 (c) only 4 (d) neither 4 nor 1

(e) 4 and some other root

3. If x varies as the cube of y, and y varies as the fifth root of z, then x varies as the nth power of z, where n is:

(a) 1/15 (b) 5/3 (c) 3/5 (d) 15 (e) 8

4. If the Highest Common Divisor of 6432 and 132 is diminished by 8, it will equal:

(a) -6 (b) 6 (c) -2 (d) 3 (e) 4

5. A regular hexagon is inscribed in a circle of radius 10 inches. Its area is:

(a) $150\sqrt{3}$ sq. in. (b) 150 sq. in. (c) $25\sqrt{3}$ sq. in. (d) 600 sq. in.

(e) $300\sqrt{3}$ sq. in.

6. The value of $\frac{1}{16}a^0 + \left(\frac{1}{16a}\right)^0 - (64^{-1/2}) - (-32)^{-4/5}$ is:

(a) $1\frac{13}{16}$ (b) $1\frac{3}{16}$ (c) 1 (d) $\frac{7}{8}$ (e) $\frac{1}{16}$

7. A housewife saved \$2.50 in buying a dress on sale. If she spent \$25 for the dress, she saved about:

(a) 8% (b) 9% (c) 10% (d) 11% (e) 12%

8. The base of a triangle is twice as long as a side of a square and their areas are the same. Then the ratio of the altitude of the triangle to the side of the square is:

(a) 1/4 (b) 1/2 (c) 1 (d) 2 (e) 4

9. A point P is outside a circle and is 13 inches from the center. A secant from P cuts the circle at Q and R so that the external segment of the secant PQ is 9 inches and QR is 7 inches. The radius of the circle is:

(a) 3" (b) 4" (c) 5" (d) 6" (e) 7"

10. The sum of the numerical coefficients in the expansion of the binomial $(a+b)^6$ is:

(a) 32 (b) 16 (c) 64 (d) 48 (e) 7

11. A merchant placed on display some dresses, each with a marked price. He then posted a sign "1/3 off on these dresses." The cost of the dresses was 3/4 of the price at which he actually sold them. Then the ratio of the cost to the marked price was:

(a) 1/2 (b) 1/3 (c) 1/4 (d) 2/3 (e) 3/4

12. The solution of the equations $\begin{matrix} 2x - 3y = 7 \\ 4x - 6y = 20 \end{matrix}$ is:

(a) $\begin{matrix} x = 18 \\ y = 12 \end{matrix}$ (b) $\begin{matrix} x = 0 \\ y = 0 \end{matrix}$ (c) There is no solution

(d) There is an unlimited number of solutions (e) $\begin{matrix} x = 8 \\ y = 5 \end{matrix}$

13. A quadrilateral is inscribed in a circle. If angles are inscribed in the four arcs cut off by the sides of the quadrilateral, their sum will be:

(a) 180° (b) 540° (c) 360° (d) 450° (e) 1080°

14. When simplified $\sqrt{1 + \left(\frac{x^4 - 1}{2x^2}\right)^2}$ equals:

(a) $\frac{x^4 + 2x^2 - 1}{2x^2}$ (b) $\frac{x^4 - 1}{2x^2}$ (c) $\frac{\sqrt{x^2 + 1}}{2}$ (d) $\frac{x^2}{\sqrt{2}}$ (e) $\frac{x^2}{2} + \frac{1}{2x^2}$

15. Log 125 equals:

(a) $100 \log 1.25$ (b) $5 \log 3$ (c) $3 \log 25$ (d) $3 - 3 \log 2$ (e) $(\log 25)(\log 5)$

PART II. (3 credits each)

16. If $f(x) = 5x^2 - 2x - 1$, then $f(x + h) - f(x)$ equals:
   (a) $5h^2 - 2h$ (b) $10xh - 4x + 2$ (c) $10xh - 2x - 2$
   (d) $h(10x + 5h - 2)$ (e) $3h$

17. The graph of the function $f(x) = 2x^3 - 7$ goes:
   (a) up to the right and down to the left
   (b) down to the right and up to the left
   (c) up to the right and up to the left
   (d) down to the right and down to the left
   (e) none of these ways.

18. Of the following sets, the one that includes all values of x which will satisfy $2x - 3 > 7 - x$ is:
   (a) $x > 4$ (b) $x < \frac{10}{3}$ (c) $x = \frac{10}{3}$ (d) $x > \frac{10}{3}$ (e) $x < 0$

19. If the three points of contact of a circle inscribed in a triangle are joined, the angles of the resulting triangle:
   (a) are always equal to 60°.
   (b) are always one obtuse angle and two unequal acute angles.
   (c) are always one obtuse angle and two equal acute angles.
   (d) are always acute angles.
   (e) are always unequal to each other.

20. The equation $x^3 + 6x^2 + 11x + 6 = 0$ has:
   (a) no negative real roots (b) no positive real roots (c) no real roots
   (d) 1 positive and 2 negative roots (e) 2 positive and 1 negative root

21. The roots of the equation $2\sqrt{x} + 2x^{-\frac{1}{2}} = 5$ can be found by solving:
   (a) $16x^2 - 92x + 1 = 0$ (b) $4x^2 - 25x + 4 = 0$ (c) $4x^2 - 17x + 4 = 0$
   (d) $2x^2 - 21x + 2 = 0$ (e) $4x^2 - 25x - 4 = 0$

22. The expression $\frac{2x^2 - x}{(x + 1)(x - 2)} - \frac{4 + x}{(x + 1)(x - 2)}$ cannot be evaluated for $x = -1$ or $x = 2$, since division by zero is not allowed. For other values of x:
   (a) The expression takes on many different values. (b) The expression has only the value 2. (c) The expression has only the value 1. (d) The expression always has a value between -1 and $+2$. (e) The expression has a value greater than 2 or less than -1.

23. If the margin made on an article costing C dollars and selling for S dollars is $M = \frac{1}{n}C$, then the margin is given by:

(a) $M = \frac{1}{n-1}S$ (b) $M = \frac{1}{n}S$ (c) $M = \frac{n}{n+1}S$ (d) $M = \frac{1}{n+1}S$

(e) $M = \frac{n}{n-1}S$

24. The values of k for which the equation $2x^2 - kx + x + 8 = 0$ will have real and equal roots are:

(a) 9 and −7 (b) only −7 (c) 9 and 7 (d) −9 and −7 (e) only 9

25. The two roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are 1 and:

(a) $\frac{b(c-a)}{a(b-c)}$ (b) $\frac{a(b-c)}{c(a-b)}$ (c) $\frac{a(b-c)}{b(c-a)}$ (d) $\frac{c(a-b)}{a(b-c)}$ (e) $\frac{c(a-b)}{b(c-a)}$

26. The straight line AB is divided at C so that $AC = 3CB$. Circles are described on AC and CB as diameters and a common tangent meets AB produced at D. Then BD equals:

(a) diameter of the smaller circle
(b) radius of the smaller circle
(c) radius of the larger circle
(d) $CB\sqrt{3}$
(e) the difference of the two radii.

27. A right circular cone has for its base a circle having the same radius as a given sphere. The volume of the cone is one-half that of the sphere. The ratio of the altitude of the cone to the radius of its base is:

(a) 1/1 (b) 1/2 (c) 2/3 (d) 2/1 (e) $\sqrt{5/4}$

28. If $m/n = 4/3$ and $r/t = 9/14$, the value of $\frac{3mr - nt}{4nt - 7mr}$ is:

(a) $-5\frac{1}{2}$ (b) $-\frac{11}{14}$ (c) $-1\frac{1}{4}$ (d) $\frac{11}{14}$ (e) $-\frac{2}{3}$

29. If the ratio of the legs of a right triangle is 1:2, then the ratio of the corresponding segments of the hypotenuse made by a perpendicular upon it from the vertex is:

(a) 1:4 (b) $1:\sqrt{2}$ (c) 1:2 (d) $1:\sqrt{5}$ (e) 1:5

30. A and B together can do a job in 2 days; B and C can do it in four days; and A and C in $2\frac{2}{5}$ days. The number of days required for A to do the job alone is:

(a) 1 (b) 3 (c) 6 (d) 12 (e) 2.8

31. In triangle ABC, $AB = AC$, angle $A = 40^\circ$. Point O is within the triangle with angle OBC = angle OCA. The number of degrees in angle BOC is:

(a) 110 (b) 35 (c) 140 (d) 55 (e) 70

32. The factors of $x^4 + 64$ are:

(a) $(x^2 + 8)^2$ (b) $(x^2 + 8)(x^2 - 8)$ (c) $(x^2 + 2x + 4)(x^2 - 8x + 16)$
(d) $(x^2 - 4x + 8)(x^2 - 4x - 8)$ (e) $(x^2 - 4x + 8)(x^2 + 4x + 8)$

33. A bank charges \$6 for a loan of \$120. The borrower receives \$114 and repays the loan in 12 installments of \$10 a month. The interest rate is approximately:

(a) 5% (b) 6% (c) 7% (d) 9% (e) 15%

34. The fraction, 1/3:

(a) equals 0.33333333 (b) is less than 0.33333333 by $\frac{1}{3\cdot 10^8}$

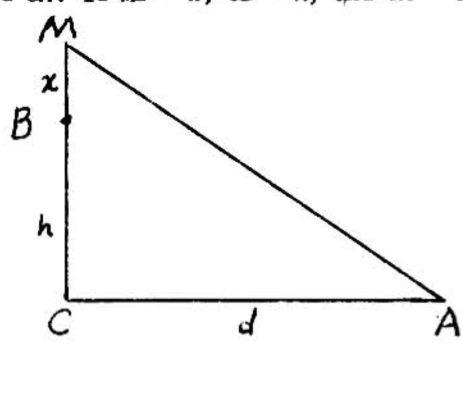
(c) is less than 0.33333333 by $\frac{1}{3\cdot 10^9}$

(d) is greater than 0.33333333 by $\frac{1}{3\cdot 10^8}$

(e) is greater than 0.33333333 by $\frac{1}{3\cdot 10^9}$

35. In the right triangle shown the sum of the distances BM and MA is equal to the sum of the distances BC and CA. If $MB = x$, $CB = h$, and $CA = d$, then x equals:

(a) $\frac{hd}{2h + d}$
(b) $d - h$
(c) $\frac{1}{2}d$
(d) $h + d - \sqrt{2d}$
(e) $\sqrt{h^2 + d^2} - h$

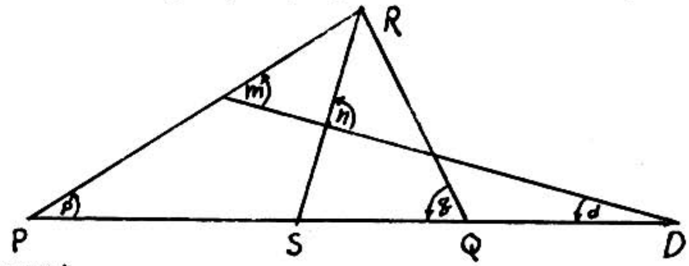


PART III. (4 credits each)

36. A boat has a speed of 15 m.p.h. in still water. In a stream that has a current of 5 m.p.h. it travels a certain distance downstream and returns. The ratio of the average speed for the round trip to the speed in still water is:

(a) 5/4 (b) 1/1 (c) 8/9 (d) 7/8 (e) 9/8

37. Given triangle PQR with RS bisecting angle R, PQ extended to D and angle n a right angle, then:



(a) $\angle m = \frac{1}{2}(\angle p - \angle q)$

(b) $\angle m = \frac{1}{2}(\angle p + \angle q)$

(c) $\angle d = \frac{1}{2}(\angle q + \angle p)$

(d) $\angle d = \frac{1}{2}\angle m$

(e) none of these is correct.

38. If log 2 = .3010 and log 3 = .4771, the value of x when $3^{x+3} = 135$ is approximately:

(a) 5 (b) 1.47 (c) 1.67 (d) 1.78 (e) 1.63

39. The locus of the mid-point of a line segment that is drawn from a given external point P to a given circle with center O and radius r, is:

(a) a straight line perpendicular to PO (b) a straight line parallel to PO (c) a circle with center P and radius r (d) a circle with center at the mid-point of PO and radius $2r$ (e) a circle with center at the mid-point PO and radius $\frac{1}{2}r$

40. If $(a + \frac{1}{a})^2 = 3$, then $a^3 + \frac{1}{a^3}$ equals:

(a) $\frac{10\sqrt{3}}{3}$ (b) $3\sqrt{3}$ (c) 0 (d) $7\sqrt{7}$ (e) $6\sqrt{3}$

41. The sum of all the roots of $4x^3 - 8x^2 - 63x - 9 = 0$ is:

(a) 8 (b) 2 (c) -8 (d) -2 (e) 0

42. Consider the graphs of (1) $y = x^2 - \frac{1}{2}x + 2$ and (2) $y = x^2 + \frac{1}{2}x + 2$ on the same set of axis. These parabolas are exactly the same shape. Then:

(a) the graphs coincide. (b) the graph of (1) is lower than the graph of (2). (c) the graph of (1) is to the left of the graph of (2). (d) The graph of (1) is to the right of the graph of (2). (e) The graph of (1) is higher than the graph of (2).

43. The hypotenuse of a right triangle is 10 inches and the radius of the inscribed circle is 1 inch. The perimeter of the triangle in inches is:

(a) 15 (b) 22 (c) 24 (d) 26 (e) 30

44. A man born in the first half of the nineteenth century was x years old in the year x^2. He was born in:

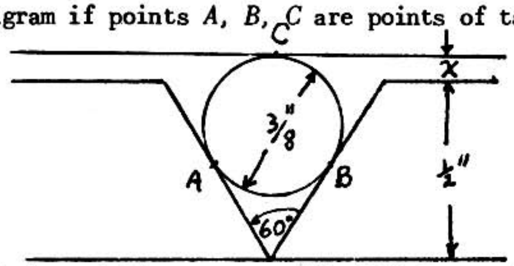
(a) 1849 (b) 1825 (c) 1812 (d) 1836 (e) 1806

45. In a rhombus, $ABCD$, line segments are drawn within the rhombus, parallel to diagonal BD, and terminated in the sides of the rhombus. A graph is drawn showing the length of a segment as a function of its distance from vertex A. The graph is:

(a) A straight line passing through the origin. (b) A straight line cutting across the upper right quadrant. (c) Two line segments forming an upright V. (d) Two line segments forming an inverted V, (Λ). (e) None of these.

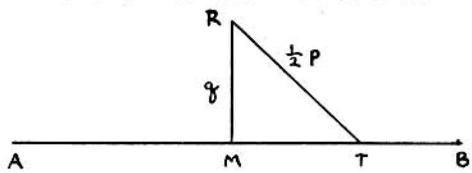
46. In the diagram if points A, B, C are points of tangency, then x equals:

(a) 3/16″
(b) 1/8″
(c) 1/32″
(d) 3/32″
(e) 1/16″



47. At the midpoint of line segment AB which is p units long, a perpendicular MR is erected with length q units. An arc is described from R with a radius equal to ½AB, meeting AB at T. Then AT and TB are the roots of:

(a) $x^2 + px + q^2 = 0$
(b) $x^2 - px + q^2 = 0$
(c) $x^2 + px - q^2 = 0$
(d) $x^2 - px - q^2 = 0$
(e) $x^2 - px + q = 0$



48. A train, an hour after starting, meets with an accident which detains it a half hour, after which it proceeds at 3/4 of its former rate and arrives 3½ hours late. Had the accident happened 90 miles farther along the line, it would have arrived only 3 hours late. The length of the trip in miles was:

(a) 400 (b) 465 (c) 600 (d) 640 (e) 550

49. The difference of the squares of two odd numbers is always divisible by 8. If $a > b$, and $2a + 1$ and $2b + 1$ are the odd numbers, to prove the given statement we put the difference of the squares in the form:

(a) $(2a + 1)^2 - (2b + 1)^2$ (b) $4a^2 - 4b^2 + 4a - 4b$ (c) $4[a(a+1) - b(b+1)]$
(d) $4(a - b)(a + b + 1)$ (e) $4(a^2 + a - b^2 - b)$

50. The times between 7 and 8 o'clock, correct to the nearest minute, when the hands of a clock will form an angle of 84 degrees are:

(a) 7:23 and 7:53 (b) 7:20 and 7:50 (c) 7:22 and 7:53
(d) 7:23 and 7:52 (e) 7:21 and 7:49

Committee on Mathematics Contests and Awards,
City College, New York 31, N.Y.

Sixth Annual

MATHEMATICAL CONTEST

Sponsored by

METROPOLITAN NEW YORK SECTION

of the

MATHEMATICAL ASSOCIATION OF AMERICA

PLEASE PRINT ON LINES BELOW

SCORE

NAME ____

FIRST NAME MIDDLE NAME LAST NAME

ADDRESS ____

STREET CITY STATE

SCHOOL ______________________ COUNTY ______________________

ADDRESS ____

THURSDAY MORNING, APRIL 21, 1955

TIME OF CONTEST 1 HOUR 20 MINUTES

INSTRUCTIONS

Do not guess at the answers as there will be a penalty for all incorrect answers. Scratch paper is permitted.

TOTAL CREDITS 150

In each of the fifty exercises enclose the correct answer, including the letter that precedes it, in a box. The following is a sample problem:

A mean proportional between 10 and 40 is:

(a) 15 $\boxed{\text{(b) } 20}$ (c) 25 (d) 30 (e) 50

PART I. (2 credits each)

1. Which one of the following is not equivalent to 0.000000375?

   (a) 3.75×10^{-7} (b) $3\frac{3}{4} \times 10^{-7}$ (c) 375×10^{-9} (d) $\frac{3}{8} \times 10^{-7}$ (e) $\frac{3}{80,000,000}$

2. The smaller angle between the hands of a clock at 12:25 p.m. is:

   (a) $132°30'$ (b) $137°30'$ (c) $150°$ (d) $137°32'$ (e) $137°$

3. If each number in a set of ten numbers is increased by 20, the arithmetic mean (average) of the original ten numbers:

   (a) remains the same (b) is increased by 20 (c) is increased by 200
   (d) is increased by 10 (e) is increased by 2

4. The equality $\frac{1}{x-1} = \frac{2}{x-2}$ is satisfied by:

   (a) no real values of x (b) either $x = 1$ or $x = 2$ (c) only $x = 1$
   (d) only $x = 2$ (e) only $x = 0$

5. y varies inversely as the square of x. When $y = 16$, $x = 1$. When $x = 8$, y equals

   (a) 2 (b) 128 (c) 64 (d) ¼ (e) 1024

6. A merchant buys a number of oranges at 3 for 10¢ and an equal number at 5 for 20¢. To "break even" he must sell all at:

   (a) 8 for 30¢ (b) 3 for 11¢ (c) 5 for 18¢ (d) 11 for 40¢ (e) 13 for 50¢

7. If a worker receives a 20 per cent cut in wages, he may regain his original pay exactly by obtaining a raise of:

   (a) 20 per cent (b) 25 per cent (c) 22½ per cent (d) \$20 (e) \$25

8. The graph of $x^2 - 4y^2 = 0$:

   (a) is a hyperbola intersecting only the x-axis (b) is a hyperbola intersecting only the y-axis (c) is a hyperbola intersecting neither axis (d) is a pair of straight lines (e) does not exist

9. A circle is inscribed in a triangle with sides 8, 15, and 17. The radius of the circle is:

(a) 6 (b) 2 (c) 5 (d) 3 (e) 7

10. How many hours does it take a train traveling at an average rate of 40 mph between stops to travel a miles if it makes n stops of m minutes each?

(a) $\frac{3a + 2mn}{120}$ (b) $3a + 2mn$ (c) $\frac{3a + 2mn}{12}$ (d) $\frac{a + mn}{40}$ (e) $\frac{a + 40mn}{40}$

11. The negation of the statement "No slow learners attend this school" is:

(a) All slow learners attend this school (b) All slow learners do not attend this school (c) Some slow learners attend this school (d) Some slow learners do not attend this school (e) No slow learners do not attend this school

12. The solution of $\sqrt{5x - 1} + \sqrt{x - 1} = 2$ is:

(a) $x = 2, x = 1$ (b) $x = 2/3$ (c) $x = 2$ (d) $x = 1$ (e) $x = 0$

13. The fraction $\frac{a^{-4} - b^{-4}}{a^{-2} - b^{-2}}$ is equal to:

(a) $a^{-6} - b^{-6}$ (b) $a^{-2} - b^{-2}$ (c) $a^{-2} + b^{-2}$ (d) $a^2 + b^2$ (e) $a^2 - b^2$

14. The length of rectangle R is 10 per cent more than the side of square S. The width of the rectangle is 10 per cent less than the side of the square. The ratio of the areas, R : S, is:

(a) 99 : 100 (b) 101 : 100 (c) 1 : 1 (d) 199 : 200 (e) 201 : 200

15. The ratio of the areas of two concentric circles is 1 : 3. If the radius of the smaller is r, then the difference between the radii is best approximated by:

(a) $0.41r$ (b) 0.73 (c) 0.75 (d) $0.73r$ (e) $0.75r$

PART II. (3 credits each)

16. The value of $\frac{3}{a + b}$ when $a = 4$ and $b = -4$ is:

(a) 3 (b) 3/8 (c) 0 (d) any finite number (e) meaningless

17. If $\log x - 5 \log 3 = -2$, then x equals:

(a) 1.25 (b) 0.81 (c) 2.43 (d) 0.8 (e) either 0.8 or 1.25

18. The discriminant of the equation $x^2 + 2x\sqrt{3} + 3 = 0$ is zero. Hence, its roots are:

(a) real and equal (b) rational and equal (c) rational and unequal (d) irrational and unequal (e) imaginary

19. Two numbers whose sum is 6 and the absolute value of whose difference is 8 are roots of the equation:

(a) $x^2 - 6x + 7 = 0$ (b) $x^2 - 6x - 7 = 0$ (c) $x^2 + 6x - 8 = 0$
(d) $x^2 - 6x + 8 = 0$ (e) $x^2 + 6x - 7 = 0$

20. The expression $\sqrt{25 - t^2} + 5$ equals zero for:

(a) no real or imaginary values of t (b) no real values of t only
(c) no imaginary values of t only (d) $t = 0$ (e) $t = \pm 5$

21. Represent the hypotenuse of a right triangle by c and the area by A. The altitude on the hypotenuse is:

(a) $\frac{A}{c}$ (b) $\frac{2A}{c}$ (c) $\frac{A}{2c}$ (d) $\frac{A^2}{c}$ (e) $\frac{A}{c^2}$

22. On a \$10,000 order a merchant has a choice between three successive discounts of 20%, 20%, and 10% and three successive discounts of 40%, 5%, and 5%. By choosing the better offer, he can save:

(a) nothing at all (b) \$400 (c) \$330 (d) \$345 (e) \$360

23. In checking the petty cash a clerk counts q quarters, d dimes, n nickels, and c cents. Later he discovers that x of the nickels were counted as quarters and x of the dimes were counted as cents. To correct the total obtained the clerk must:

(a) make no correction (b) subtract 11¢ (c) subtract $11x$¢ (d) add $11x$¢
(e) add x¢

24. The function $4x^2 - 12x - 1$:

(a) always increases as x increases (b) always decreases as x decreases to 1
(c) cannot equal 0 (d) has a maximum value when x is negative (e) has a minimum value of -10

25. One of the factors of $x^4 + 2x^2 + 9$ is:

(a) $x^2 + 3$ (b) $x + 1$ (c) $x^2 - 3$ (d) $x^2 - 2x - 3$ (e) none of these

26. Mr. A owns a house worth \$10,000. He sells it to Mr. B at 10% profit. Mr. B sells the house back to Mr. A at a 10% loss. Then:

(a) Mr. A comes out even (b) Mr. A makes \$100 (c) Mr. A makes \$1,000
(d) Mr. B loses \$100 (e) none of the above is correct

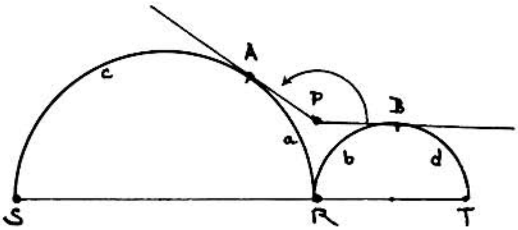
27. If r and s are the roots of $x^2 - px + q = 0$, then $r^2 + s^2$ equals:

(a) $p^2 + 2q$ (b) $p^2 - 2q$ (c) $p^2 + q^2$ (d) $p^2 - q^2$ (e) p^2

28. On the same set of axes are drawn the graph of $y = ax^2 + bx + c$ and the graph of the equation obtained by replacing x by $-x$ in the given equation. If $b \neq 0$ and $c \neq 0$ these two graphs intersect:

(a) in two points, one on the x-axis and one on the y-axis (b) in one point located on neither axis (c) only at the origin (d) in one point on the x-axis (e) in one point on the y-axis.

29. In the figure PA is tangent to semicircle SAR; PB is tangent to semicircle RBT; SRT is a straight line; the arcs are indicated in the figure. Angle APB is measured by:



(a) $\frac{1}{2}(a - b)$ (b) $\frac{1}{2}(a + b)$ (c) $(c - a) - (d - b)$ (d) $a - b$ (e) $a + b$

30. Each of the equations $3x^2 - 2 = 25$, $(2x - 1)^2 = (x - 1)^2$, $\sqrt{x^2 - 7} = \sqrt{x - 1}$ has:

(a) two integral roots (b) no root greater than 3 (c) no root zero (d) only one root (e) one negative root and one positive root

31. An equilateral triangle whose side is 2 is divided into a triangle and a trapezoid by a line drawn parallel to one of its sides. If the area of the trapezoid equals one-half of the area of the original triangle, the length of the median of the trapezoid is:

(a) $\sqrt{6}/2$ (b) $\sqrt{2}$ (c) $2 + \sqrt{2}$ (d) $\dfrac{2 + \sqrt{2}}{2}$ (e) $\dfrac{2\sqrt{3} - \sqrt{6}}{2}$

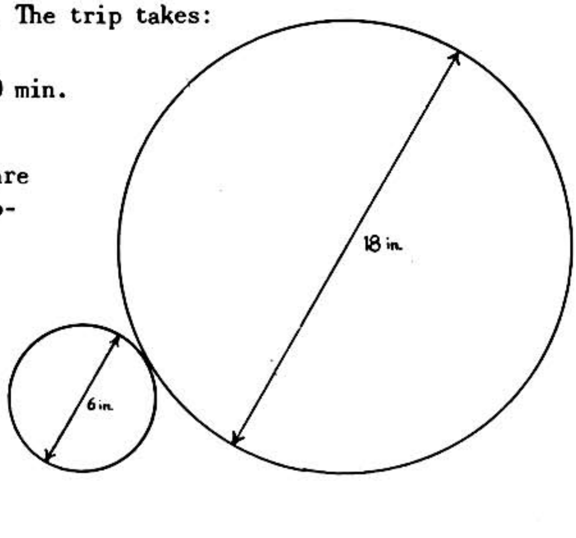
32. If the discriminant of $ax^2 + 2bx + c = 0$ is zero, then another true statement about a, b, and c is that:

(a) they form an arithmetic progression (b) they form a geometric progression (d) they are unequal (d) they are all negative numbers (e) only b is negative and a and c are positive.

33. Henry starts a trip when the hands of the clock are together between 8 a.m. and 9 a.m. He arrives at his destination between 2 p.m. and 3 p.m. when the hands of the clock are exactly 180° apart. The trip takes:

(a) 6 hr. (b) 6 hr. 43-7/11 min.
(c) 5 hr. 16-4/11 min. (d) 6 hr. 30 min.
(e) none of these.

34. A 6-inch and 18-inch diameter pole are placed as in the figure and bound together with wire. The length of the shortest wire that will go around them is:



(a) $12\sqrt{3} + 16\pi$ (b) $12\sqrt{3} + 7\pi$
(c) $12\sqrt{3} + 14\pi$ (d) $12 + 15\pi$
(e) 24π

35. Three boys agree to divide a bag of marbles in the following manner. The first boy takes one more than half the marbles. The second takes a third of the number remaining. The third boy finds that he is left with twice as many marbles as the second boy. The original number of marbles:

(a) is none of the following (b) cannot be determined from the given data
(c) is 20 or 26 (d) is 14 or 32 (e) is 8 or 38

PART III. (4 credits each)

36. A cylindrical oil tank, lying horizontally, has an interior length of 10 feet and an interior diameter of 6 feet. If the rectangular surface of the oil has an area of 40 square feet, the depth of the oil is:

(a) $\sqrt{5}$ (b) $2\sqrt{5}$ (c) $3-\sqrt{5}$ (d) $3+\sqrt{5}$ (e) either $3-\sqrt{5}$ or $3+\sqrt{5}$

37. A three-digit number has, from left to right, the digits h, t, and u with $h > u$. When the number with the digits reversed is subtracted from the original number, the units' digit in the difference is 4. The next two digits, from right to left, are:

(a) 5 and 9 (b) 9 and 5 (c) impossible to tell (d) 5 and 4 (e) 4 and 5

38. Four positive integers are given. Select any three of these integers, find their arithmetic average, and add this result to the fourth integer. Thus the numbers 29, 23, 21 and 17 are obtained. One of the original integers is:

(a) 19 (b) 21 (c) 23 (d) 29 (e) 17

39. If $y = x^2 + px + q$, then if the least possible value of y is zero q is equal to:

(a) 0 (b) $\frac{p^2}{4}$ (c) $\frac{p}{2}$ (d) $-\frac{p}{2}$ (e) $\frac{p^2}{4} - q$

40. The fractions $\frac{ax+b}{cx+d}$ and $\frac{b}{d}$ are unequal if:

(a) $a = c = 1$ and $x \neq 0$ (b) $a = b = 0$ (c) $a = c = 0$ (d) $x = 0$ (e) $ad = bc$

41. A train traveling from Aytown to Beetown meets with an accident after 1 hr. It is stopped for ½ hr., after which it proceeds at four-fifths of its usual rate, arriving at Beetown 2 hr. late. If the train had covered 80 miles more before the accident, it would have been just 1 hr. late. The usual rate of the train is:

(a) 20 mph (b) 30 mph (c) 40 mph (d) 50 mph (e) 60 mph

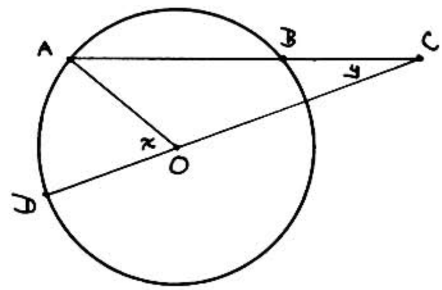
42. If a, b, and c are positive integers, the radicals $\sqrt{a + b/c}$ and $a\sqrt{b/c}$ are equal when and only when:

(a) $a = b = c = 1$ (b) $a = b$ and $c = a = 1$ (c) $c = \frac{b(a^2-1)}{a}$
(d) $a = b$ and c is any value (e) $a = b$ and $c = a - 1$

43. The pairs of values of x and y that are the common solutions of the equations $y = (x + 1)^2$ and $xy + y = 1$ are:

(a) 3 real pairs (b) 4 real pairs (c) 4 imaginary pairs (d) 2 real and 2 imaginary pairs (e) 1 real and 2 imaginary pairs

44. In circle O chord AB is produced so that BC equals a radius of the circle. CO is drawn and extended to D. AO is drawn. Which of the following expresses the relationship between x and y?



(a) $x = 3y$ (b) $x = 2y$ (c) $x = 60°$
(d) there is no special relationship between x and y (e) $x = 2y$ or $x = 3y$, depending upon the length of AB

45. Given a geometric sequence with the first term $\neq 0$ and $r \neq 0$ and an arithmetic sequence with the first term = 0. A third sequence 1, 1, 2, ... is formed by adding corresponding terms of the two given sequences. The sum of the first ten terms of the third sequence is:

(a) 978 (b) 557 (c) 467 (d) 1068 (e) not possible to determine from the information given.

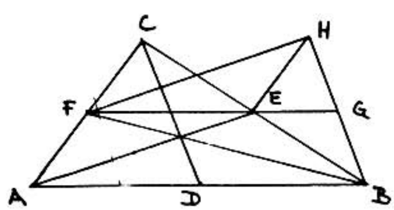
46. The graphs of $2x + 3y - 6 = 0$, $4x - 3y - 6 = 0$, $x = 2$, and $y = 2/3$ intersect in:

(a) 6 points (b) 1 point (c) 2 points (d) no points (e) an unlimited number of points

47. The expressions $a + bc$ and $(a + b)(a + c)$ are:

(a) always equal (b) never equal (c) equal whenever $a + b + c = 1$
(d) equal when $a + b + c = 0$ (e) equal only when $a = b = c = 0$

48. Given triangle ABC with medians AE, BF, CD; FH parallel and equal to AE; BH and HE are drawn; FE extended meets BH in G. Which one of the following statements is not necessarily correct?



(a) $AEHF$ is a parallelogram (b) $HE = HG$
(c) $BH = DC$ (d) $FG = \frac{3}{4}AB$ (e) FG is a median of triangle BFH

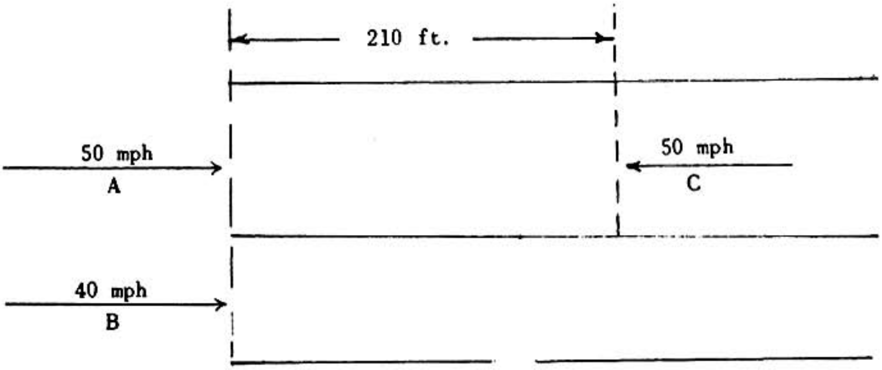
49. The graphs of $y = \dfrac{x^2 - 4}{x - 2}$ and $y = 2x$ intersect in:

(a) 1 point whose abscissa is 2 (b) 1 point whose abscissa is 0

(c) no points (d) two distinct points (e) two identical points

50. In order to pass B going 40 mph on a two-lane highway A, going 50 mph, must gain 30 feet. Meantime, C, 210 feet from A, is headed toward him at 50 mph. If B and C maintain their speeds, then, in order to pass safely, A must increase his speed by:

(a) 30 mph (b) 10 mph (c) 5 mph (d) 15 mph (e) 3 mph



This figure is not drawn to scale

Committee on Mathematics Contests and Awards,
City College, New York 31, N.Y.

SEVENTH ANNUAL
MATHEMATICAL CONTEST
Sponsored by
METROPOLITAN NEW YORK SECTION
of the
MATHEMATICAL ASSOCIATION OF AMERICA

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.

2. Do not mark answers in this booklet.

3. You will have an answer sheet on which you are to indicate the best answer to each question.

4. This is a multiple choice test. Each of the five answers to every question is preceeded by one of the capital letters, A, B, C, D or E. You are to decide which is the best answer and then write the capital letter that preceeds the best answer, in the box on the margin of your answer sheet directly under the number of the question. For example: In question No. 3 suppose that the best answer is preceeded by the letter C, then write the capital letter C in the box directly under No. 3. Fill in the answers as you find them.

5. If you are unable to solve a problem leave that box blank on the answer sheet. Do not guess at the answer as you will be penalized for all incorrect answers.

6. Use pencil. Scratch paper, ruler, compasses and eraser are permitted.

7. When your teacher gives the signal tear off the cover of this booklet along the dotted line and turn the cover over. Page 2 is your answer sheet.

8. Keep the questions covered with the answer sheet while you fill in your name and the name and address of your school.

9. When your teacher gives the signal begin working the problems. You will have 80 minutes for the test.

THURSDAY MORNING, MAY 3, 1956

PART I

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

PART II

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35

PART III

36	37	38	39	40	41	42	43	44	45	46	47	48	49	50

PRINT ____

FIRST NAME MIDDLE NAME LAST NAME

____

SCHOOL NUMBER STREET

____

CITY ZONE COUNTY STATE

Your teacher will fill in the following blanks.

	VALUE	POINTS CORRECT	POINTS ATTEMPTED
Part I	2 points each		
Part II	3 points each		
Part III	4 points each		
TOTAL			

$$\left(\underset{\text{POINTS CORRECT}}{\quad}\right) - \frac{1}{5}\left[\left(\underset{\text{POINTS ATTEMPTED}}{\quad}\right) - \left(\underset{\text{POINTS CORRECT}}{\quad}\right)\right] = \left(\underset{\text{SCORE}}{\quad}\right)$$

PART I. (2 credits each)

1. The value of $x + x(x^x)$ when $x = 2$ is:

   (A) 10 (B) 16 (C) 18 (D) 36 (E) 64

2. Mr. Jones sold two pipes at \$1.20 each. Based on the cost his profit on one was 20% and his loss on the other was 20%. On the sale of the pipes, he

   (A) broke even (B) lost 4¢ (C) gained 4¢ (D) lost 10¢ (E) gained 10¢

3. The distance light travels in one year is approximately 5,870,000,000,000 miles. The distance light travels in 100 years is:

   (A) 587×10^8 miles (B) 587×10^{10} miles (C) 587×10^{-10} miles
   (D) 587×10^{12} miles (E) 587×10^{-12} miles

4. A man has \$10,000 to invest. He invests \$4,000 at 5% and \$3,500 at 4%. In order to have a yearly income of \$500, he must invest the remainder at:

   (A) 6% (B) 6.1% (C) 6.2% (D) 6.3% (E) 6.4%

5. A nickel is placed on a table. The number of nickels which can be placed around it, each tangent to it and to two others is:

   (A) 4 (B) 5 (C) 6 (D) 8 (E) 12

6. In a group of cows and chickens, the number of legs was 14 more than twice the number of heads. The number of cows was:

   (A) 5 (B) 7 (C) 10 (D) 12 (E) 14

7. The roots of the equation $ax^2 + bx + c = 0$ will be reciprocal if:

   (A) $a = b$ (B) $a = bc$ (C) $c = a$ (D) $c = b$ (E) $c = ab$

8. If $8 \cdot 2^x = 5^{y+8}$, then, when $y = -8$, $x =$

   (A) -4 (B) -3 (C) 0 (D) 4 (E) 8

9. Simplify: $\left[\sqrt[3]{\sqrt[6]{a^9}}\right]^4 \left[\sqrt[6]{\sqrt[3]{a^9}}\right]^4$

   (A) a^{16} (B) a^{12} (C) a^8 (D) a^4 (E) a^2

10. A circle of radius 10 inches has its center at the vertex C of an equilateral triangle ABC and passes through the other two vertices. The side AC extended through C intersects the circle at D. The number of degrees of angle ADB is:

    (A) 15 (B) 30 (C) 60 (D) 90 (E) 120

11. The expression $1 - \frac{1}{1+\sqrt{3}} + \frac{1}{1-\sqrt{3}}$ equals

    (A) $1-\sqrt{3}$ (D) 1 (C) $-\sqrt{3}$ (D) $\sqrt{3}$ (E) $1+\sqrt{3}$

12. If $x^{-1} - 1$ is divided by $x - 1$ the quotient is:

(A) 1 (B) $\frac{1}{x-1}$ (C) $\frac{-1}{x-1}$ (D) $\frac{1}{x}$ (E) $-\frac{1}{x}$

13. The percent that x is less than y is

(A) $\frac{100(y-x)}{x}$ (B) $\frac{100(x-y)}{x}$ (C) $\frac{100(y-x)}{y}$ (D) $100(y-x)$

(E) $100(x-y)$

14. The points A, B, and C are on a circle O. The tangent line at A and the secant BC intersect at P. If BC = 20 and PA = $10\sqrt{3}$, then PB equals

(A) 5 (B) 10 (C) $10\sqrt{3}$ (D) 20 (E) 30

15. The roots of $\frac{15}{x^2-4} - \frac{2}{x-2} = 1$ are

(A) -5 and 3 (B) ± 2 (C) only 2 (D) -3 and 5 (E) only 3

PART II. (3 credits each)

16. The sum of three numbers is 98. The ratio of the first to the second is $\frac{2}{3}$, and the ratio of the second to the third is $\frac{5}{8}$. The second number is:

(A) 15 (B) 20 (C) 30 (D) 32 (E) 33

17. The fraction $\frac{5x-11}{2x^2+x-6}$ was obtained by adding the two fractions $\frac{A}{x+2}$ and $\frac{B}{2x-3}$. The values of A and B must be

(A) $A = 5x$, $B = -11$, (B) $A = -11$, $B = 5x$ (C) $A = -1$ $B = 3$ (D) $A = 3$ $B = -1$ (E) $A = 5$ $B = -11$

18. If $10^{2y} = 25$, then 10^{-y} equals

(A) $-\frac{1}{5}$ (B) $\frac{1}{625}$ (C) $\frac{1}{50}$ (D) $\frac{1}{25}$ (E) $\frac{1}{5}$

19. Two candles of the same height are lighted at the same time. The first is consumed in 4 hours and the second in 3 hours. Assuming that each candle burns at a constant rate, in how many hours after being lighted was the first candle twice the height of the second?

(A) $\frac{3}{4}$ hr. (B) $1\frac{1}{2}$ hr. (C) 2 hr. (D) $2\frac{2}{5}$ hr. (E) $2\frac{1}{2}$ hr.

20. If $(0.2)^x = 2$ and $\log 2 = 0.3010$, then the value of x to the nearest tenth is:

(A) -10 (B) -0.5 (C) -0.4 (D) -0.2, (E) 10

21. If each of two intersecting lines intersects a hyperbola and neither line is tangent to the hyperbola, then the possible number of points of intersection with the hyperbola is:

(A) 2 (B) 2 or 3 (C) 2 or 4 (D) 3 or 4 (E) 2, 3, or 4

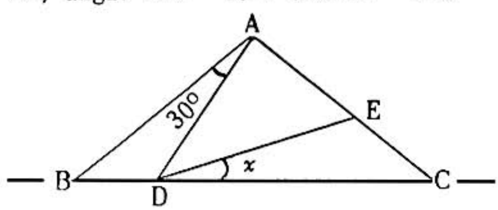
22. Jones covered a distance of 50 miles on his first trip. On a later trip he traveled 300 miles while going three times as fast. His new time compared with the old time was:

(A) three times as long (B) twice as long (C) the same
(D) half as long (E) a third as long

23. For the equation $x^2 - 2x\sqrt{2} + 2 = 0$ we have $b^2 - 4ac = 0$. A description of the roots is that they are

(A) equal and rational (B) unequal and rational (C) equal and irrational
(D) unequal and irrational (E) equal and imaginary

24. In the figure AB = AC, angle BAD = 30°, and AE = AD.



Then x equals:

(A) 7½° (B) 10° (C) 12½° (D) 15° (E) 20°

25. The sum of all numbers of the form 2k + 1, where k takes on integral values from 1 to n is:

(A) n^2 (B) $n(n+1)$ (C) $n(n+2)$ (D) $(n+1)^2$ (E) $(n+1)(n+2)$

26. Which one of the following combinations of given parts does not determine the indicated triangle?

(A) base angle and vertex angle; isosceles triangle
(B) vertex angle and the base; isosceles triangle
(C) the radius of the circumscribed circle; equilateral triangle
(D) one arm and the radius of the inscribed circle; right triangle
(E) two angles and a side opposite one of them; scalene triangle

27. If an angle of a triangle remains unchanged but each of its two including sides is doubled, then the area is multiplied by:

(A) 2 (B) 3 (C) 4 (D) 6 (E) more than 6

28. Mr. J left his entire estate to his wife, his daughter, his son, and the cook. His daughter and son got half the estate, sharing in the ratio of 4 to 3. His wife got twice as much as the son. If the cook received a bequest of \$500, then the entire estate was:

(A) \$3500 (B) \$5500 (C) \$6500 (D) \$7000 (E) \$7500

29. The points of intersection of $xy = 12$ and $x^2 + y^2 = 25$ are joined. The resulting figure is:

(A) a straight line (B) an equilateral triangle (C) a parallelogram
(D) a rectangle (E) a square

30. If the altitude of an equilateral triangle is $\sqrt{6}$, then the area is:

(A) $2\sqrt{2}$ (B) $2\sqrt{3}$ (C) $3\sqrt{3}$ (D) $6\sqrt{2}$ (E) 12

31. In our number system the base is ten. If the base were changed to four you would count as follows: 1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23, 30, ... The twentieth number would be:

(A) 20 (B) 38 (C) 44 (D) 104 (E) 110

32. George and Henry started a race from opposite ends of the pool. After a minute and a half, they passed each other in the center of the pool. If they lost no time in turning and maintained their respective speeds, how many minutes after starting did they pass each other the second time?

(A) 3 (B) $4\frac{1}{2}$ (C) 6 (D) $7\frac{1}{2}$ (E) 9

33. The number $\sqrt{2}$ is equal to:

(A) a rational fraction (B) a finite decimal (C) 1.41421 (D) an infinite repeating decimal (E) an infinite non-repeating decimal

34. If n is any whole number, $n^2(n^2 - 1)$ is always divisible by:

(A) 12 (B) 24 (C) Any multitude of 12 (D) $12 - n$ (E) 12 and 24

35. A rhombus is formed by two radii and two chords of a circle whose radius is 16 feet. The area of the rhombus in square feet is:

(A) 128 (B) $128\sqrt{3}$ (C) 256 (D) 512 (E) $512\sqrt{3}$

PART III. (4 credits each)

36. If the sum $1 + 2 + 3 + \cdots + K$ is a perfect square N^2 and if N is less than 100, then the possible values for K are:

(A) only 1 (B) 1 and 8 (C) only 8 (D) 8 and 49 (E) 1, 8, and 49

37. On a map whose scale is 400 miles to an inch and a half, a certain estate is represented by a rhombus having a $60°$ angle. The diagonal opposite $60°$ is 3/16 in. The area of the estate in square miles is:

(A) $\frac{2500}{\sqrt{3}}$ (B) $\frac{1250}{\sqrt{3}}$ (C) 1250 (D) $\frac{5625\sqrt{3}}{2}$ (E) $1250\sqrt{3}$

38. In a right triangle with sides a and b, and hypotenuse c, the altitude drawn on the hypotenuse is x. Then

(A) $a \cdot b = x^2$ (B) $\frac{1}{a} + \frac{1}{b} = \frac{1}{x}$ (C) $a^2 + b^2 = 2x^2$ (D) $\frac{1}{x^2} = \frac{1}{a^2} + \frac{1}{b^2}$ (E) $\frac{1}{x} = \frac{b}{a}$

39. The hypotenuse c and one arm a of a right triangle are consecutive integers. The square of the second arm is:

(A) ca (B) c/a (C) $c + a$ (D) $c - a$ (E) none of these

40. If $V = gt + V_0$ and $S = \frac{1}{2} gt^2 + V_0 t$ then t equals

(A) $\frac{2S}{V + V_0}$ (B) $\frac{2S}{V - V_0}$ (C) $\frac{2S}{V_0 - V}$ (D) $\frac{2S}{V}$ (E) $2S - V$

41. The equation $3y^2 + y + 4 = 2(6x^2 + y + 2)$ where $y = 2x$ is satisfied by:

(A) no value of x (B) all values of x (C) $x = 0$ only (D) all integral values of x only (E) all rational values of x only

42. The equation $\sqrt{x + 4} - \sqrt{x - 3} + 1 = 0$ has

(A) no root (B) one real root (C) one real root and one imaginary root (D) two imaginary roots (E) two real roots

43. The number of scalene triangles having all sides of integral lengths, and perimeter less than 13 is

(A) 1 (B) 2 (C) 3 (D) 4 (E) 18

44. If $x < a < 0$ means that x and a are numbers such that x is less than a and a is less than zero, then

(A) $x^2 < ax < 0$ (B) $x^2 > ax > a^2$ (C) $x^2 < a^2 < 0$ (D) $x^2 > ax$ but $ax < 0$ (E) $x^2 > a^2$ but $a^2 < 0$

45. A wheel with a rubber tire has an outside diameter of 25 in. When the radius has been decreased a quarter of an inch, the number of revolutions of the wheel in one mile will

(A) be increased about 2% (B) be increased about 1% (C) be increased about 20% (D) be increased by ½ % (E) remain the same

46. In the equation $\frac{1 + x}{1 - x} = \frac{N + 1}{N}$ where N is positive, x can have:

(A) any positive value less than 1 (B) any value less than 1 (C) the value zero only (D) any non-negative value (E) any value

47. An engineer said he could finish a highway section in 3 days with his present supply of a certain type of machine. However, with 3 more of these machines the job could be done in 2 days. If the machines all work at the same rate, how many days would it take to do the job with one machine?

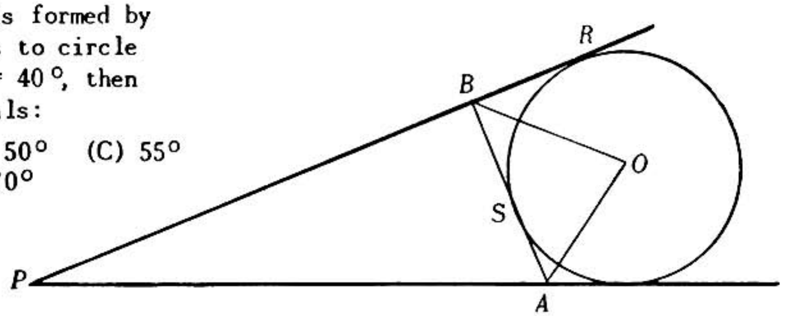
(A) 6 (B) 12 (C) 15 (D) 18 (E) 36

48. If p is a positive integer, then $\frac{3p + 25}{2p - 5}$ can be a positive integer, if and only if p is

(A) at least 3 (B) at least 3 and no more than 35 (C) no more than 35
(D) equal to 35 (E) equal to 3 or 35

49. Triangle PAB is formed by three tangents to circle O. Angle $APB = 40°$, then angle AOB equals:

(A) 45° (B) 50° (C) 55°
(D) 60° (E) 70°



50. In triangle ABC, $CA = CB$. On CB square $BCDE$ is constructed away from the triangle. If x is the number of degrees in angle DAB, then

(A) x depends upon triangle ABC (B) x is independent of the triangle
(C) x may equal angle CAD (D) x can never equal angle CAB (E) x is greater than 45° but less than 90°

Committee on Mathematics Contests and Awards
POLYTECHNIC INSTITUTE OF BROOKLYN
BROOKLYN 1, N.Y.

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EIGHTH ANNUAL MATHEMATICAL CONTEST

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the best answer to each question.
3. This is a multiple choice test. Each question is followed by five answers, marked A, B, C, D, E. For each question decide upon the best answer; then write the capital letter that precedes the best answer in the box on the margin of your answer sheet directly under the number of the question. For example: In question No. 3 suppose that the best answer is preceded by the letter C; you write the capital letter C in the box directly under No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. Avoid guessing since you are penalized for the incorrectly-guessed answers.
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. When your teacher gives the signal tear off the cover of this booklet along the perforated line and turn the cover over. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you fill in your name and the name of your school on it.
8. When your teacher gives the signal begin working the problems. You have 80 minutes for the test.

THURSDAY MORNING, MAY 9, 1957

16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35

PART II

PRINT ______________________________

FIRST NAME MIDDLE NAME LAST NAME

____

SCHOOL NUMBER STREET

____

CITY ZONE COUNTY STATE

Your teacher will fill in the following blanks.

	VALUE	POINTS CORRECT	POINTS ATTEMPTED
Part I	2 points each		
Part II	3 points each		
Part III	4 points each		
TOTAL			

$$\left(\underset{\text{POINTS CORRECT}}{\qquad}\right) - \frac{1}{5}\left[\left(\underset{\text{POINTS ATTEMPTED}}{\qquad}\right) - \left(\underset{\text{POINTS CORRECT}}{\qquad}\right)\right] = \left(\underset{\text{SCORE}}{\qquad}\right)$$

PART I

15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

PART III

36 37 38 39 40 41 42 43 44 45 46 47 48 49 50

PART I. (2 credits each)

1. The number of distinct lines representing the altitudes, medians, and interior angle bisectors of a triangle that is isosceles, but not equilateral, is:

   (A) 9 (B) 7 (C) 6 (D) 5 (E) 3

2. In the equation $2x^2 - hx + 2k = 0$, the sum of the roots is 4 and the product of the roots is -3. Then h and k have the values, respectively:

   (A) 8 and -6 (B) 4 and -3 (C) -3 and 4 (D) -3 and 8 (E) 8 and -3

3. The simplest form of $1 - \dfrac{1}{1 + \dfrac{a}{1 - a}}$ is:

   (A) a if $a \neq 0$ (B) 1 (C) a if $a \neq -1$ (D) $1 - a$ with no restriction on a
   (E) a if $a \neq 1$

4. The first step in finding the product $(3x + 2)(x - 5)$ by use of the distributive property in the form $a(b + c) = ab + ac$ is:

   (A) $3x^2 - 13x - 10$ (B) $3x(x - 5) + 2(x - 5)$ (C) $(3x + 2)x + (3x + 2)(-5)$
   (D) $3x^2 - 17x - 10$ (E) $3x^2 + 2x - 15x - 10$

5. Through the use of theorems of logarithms $\log \frac{a}{b} + \log \frac{b}{c} + \log \frac{c}{d} - \log \frac{ay}{dx}$ can be reduced to:

   (A) $\log \frac{y}{x}$ (B) $\log \frac{x}{y}$ (C) 1 (D) 0 (E) $\log \frac{a^2y}{d^2x}$

6. An open box is constructed by starting with a rectangular sheet of metal 10 in. by 14 in. and cutting a square of side x inches from each corner. The resulting projections are folded up and the seams welded. The volume of the resulting box is:

   (A) $140x - 48x^2 + 4x^3$ (B) $140x + 48x^2 + 4x^3$ (C) $140x + 24x^2 + x^3$
   (D) $140x - 24x^2 + x^3$ (E) none of these

7. The area of a circle inscribed in an equilateral triangle is 48π. The perimeter of this triangle is: (A) $72\sqrt{3}$ (B) $48\sqrt{3}$ (C) 36 (D) 24 (E) 72

8. The numbers x, y, z are proportional to 2, 3, 5. The sum of x, y, and z is 100. The number y is given by the equation $y = ax - 10$. Then a is:

   (A) 2 (B) 3/2 (C) 3 (D) 5/2 (E) 4

9. The value of $x - y^{x-y}$ when $x = 2$ and $y = -2$ is:

   (A) -18 (B) -14 (C) 14 (D) 18 (E) 256

10. The graph of $y = 2x^2 + 4x + 3$ has its:

    (A) lowest point at $(-1, 9)$ (B) lowest point at $(1, 1)$ (C) lowest point at $(-1, 1)$
    (D) highest point at $(-1, 9)$ (E) highest point at $(-1, 1)$

11. The angle formed by the hands of a clock at 2:15 is:

    (A) $30°$ (B) $27\frac{1}{2}°$ (C) $157\frac{1}{2}°$ (D) $172\frac{1}{2}°$ (E) none of these

12. Comparing the numbers 10^{-49} and $2 \cdot 10^{-50}$ we may say:

    (A) the first exceeds the second by $8 \cdot 10^{-1}$
    (B) the first exceeds the second by $2 \cdot 10^{-1}$
    (C) the first exceeds the second by $8 \cdot 10^{-50}$
    (D) the second is five times the first
    (E) the first exceeds the second by 5

13. A rational number between $\sqrt{2}$ and $\sqrt{3}$ is: (A) $\frac{\sqrt{2} + \sqrt{3}}{2}$ (B) $\frac{\sqrt{2} \cdot \sqrt{3}}{2}$

    (C) 1.5 (D) 1.8 (E) 1.4

14. If $y = \sqrt{x^2 - 2x + 1} + \sqrt{x^2 + 2x + 1}$, then y is:

    (A) $2x$ (B) $2(x + 1)$ (C) 0 (D) $|x - 1| + |x + 1|$ (E) none of these

15. The table below shows the distance s in feet a ball rolls down an inclined plane in t seconds

t	0	1	2	3	4	5
s	0	10	40	90	160	250

The distance s for $t = 2.5$ is: (A) 45 (B) 62.5 (C) 70 (D) 75 (E) 82.5

PART II. (3 credits each)

16. Goldfish are sold at 15¢ each. The rectangular coordinate graph showing the cost of 1 to 12 goldfish is:

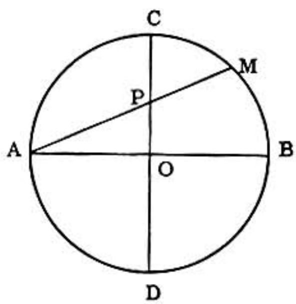
    (A) a straight line segment
    (B) a set of horizontal parallel line segments
    (C) a set of vertical parallel line segments
    (D) a finite set of distinct points
    (E) a straight line

17. A cube is made by soldering twelve 3-inch lengths of wire properly at the vertices of the cube. If a fly alights at one of the vertices and then walks along the edges, the greatest distance it could travel before coming to any vertex a second time, without retracing any distance, is:

    (A) 24 in. (B) 12 in. (C) 30 in. (D) 18 in. (E) 36 in.

18. Circle O has diameters AB and CD perpendicular to each other. AM is any chord intersecting CD at P. Then AP·AM is equal to:

    (A) AO·OB (B) AO·AB (C) CP·CD (D) CP·PD
    (E) CO·OP



19. The base of the decimal number system is ten, meaning, for example, that $123 = 1 \cdot 10^2 + 2 \cdot 10 + 3$. In the binary system, which has base two, the first five positive integers are 1, 10, 11, 100, 101. The numeral 10011 in the binary system would then be written in the decimal system as:

    (A) 19 (B) 40 (C) 10011 (D) 11 (E) 7

20. A man makes a trip by automobile at an average speed of 50 mph. He returns over the same route at an average speed of 45 mph. His average speed for the entire trip is:

(A) $47\frac{7}{19}$ (B) $47\frac{1}{4}$ (C) $47\frac{1}{2}$ (D) $47\frac{11}{19}$ (E) none of these

21. Start with the theorem "If two angles of a triangle are equal, the triangle is isosceles" and the following four statements
   1. If two angles of a triangle are not equal, the triangle is not isosceles
   2. The base angles of an isosceles triangle are equal
   3. If a triangle is not isosceles, then two of its angles are not equal
   4. A necessary condition that two angles of a triangle be equal is that the triangle be isosceles.

Which combination of statements contains only those which follow logically from the given theorem or which have the same verbal meaning as the given theorem?

(A) 1, 2, 3, 4 (B) 1, 2, 3 (C) 2, 3, 4 (D) 1, 2 (E) 3, 4

22. If $\sqrt{x-1} - \sqrt{x+1} + 1 = 0$, then 4x equals:

(A) 5 (B) $4\sqrt{-1}$ (C) 0 (D) $1\frac{1}{4}$ (E) no real value

23. The graph of $x^2 + y = 10$ and the graph of $x + y = 10$ meet in two points. The distance between these two points is:

(A) less than 1 (B) 1 (C) $\sqrt{2}$ (D) 2 (E) more than 2

24. If the square of a number of two digits is decreased by the square of the number formed by reversing the digits, then the result is not always divisible by:

(A) 9 (B) the product of the digits (C) the sum of the digits (D) the difference of the digits (E) 11

25. The vertices of triangle PQR have coordinates as follows: P(o,a), Q(b,o), R(c,d), where a, b, c and d are positive. The area of triangle PQR may be found from the expression:

(A) $\frac{ab + ac + bc + cd}{2}$ (B) $\frac{ac + bd - ab}{2}$ (C) $\frac{ab - ac - bd}{2}$
(D) $\frac{ac + bd + ab}{2}$ (E) $\frac{ac + bd - ab - cd}{2}$

26. From a point within a triangle, line segments are drawn to the vertices. A necessary and sufficient condition that the three triangles thus formed have equal areas is that the point be:

(A) the center of the inscribed circle
(B) the center of the circumscribed circle
(C) such that the three angles formed at the point each be 120°
(D) the intersection of the altitudes of the triangle
(E) the intersection of the medians of the triangle

27. The sum of the reciprocals of the roots of the equation $x^2 + px + q = 0$ is:

(A) $-p/q$ (B) q/p (C) p/q (D) $-q/p$ (E) pq

28. If a and b are positive and $a \neq 1$, $b \neq 1$, then the value of $b^{\log_b a}$ is:

(A) dependent upon b (B) dependent upon a (C) dependent upon a and b
(D) zero (E) one

29. The relation $x^2(x^2-1) \geqq 0$ is true for:

(A) $x \geqq 1$ only (B) $-1 \leqq x \leqq 1$ (C) $x = 0,\ x = 1,\ x = -1$

(D) $x = 0,\ x \leqq -1,\ x \geqq 1$ (E) $x \geqq 0$ only

Where $x \geqq a$ means that x can take on all values greater than a and the value equal to a, while $x \leqq a$ has a corresponding meaning with "less than."

30. The sum of the squares of the first n positive integers is given by the expression $\frac{n(n+c)(2n+k)}{6}$, if c and k are, respectively:

(A) 1 and 2 (B) 3 and 5 (C) 2 and 2 (D) 1 and 1 (E) 2 and 1

31. A regular octagon is to be formed by cutting equal isosceles right triangles from the corners of a square. If the square has sides of one unit, the leg of each of the triangles is:

(A) $\frac{2+\sqrt{2}}{3}$ (B) $\frac{2-\sqrt{2}}{2}$ (C) $\frac{1+\sqrt{2}}{2}$ (D) $\frac{1+\sqrt{2}}{3}$ (E) $\frac{2-\sqrt{2}}{3}$

32. The largest of the following integers which divides each of the members of the sequence $1^5-1,\ 2^5-2,\ 3^5-3,\ \ldots,\ n^5-n,\ \ldots$ is:

(A) 1 (B) 60 (C) 15 (D) 120 (E) 30

33. If $9^{x+2} = 240 + 9^x$, then the value of x is:

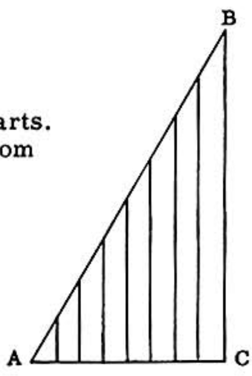
(A) 0.1 (B) 0.2 (C) 0.3 (D) 0.4 (E) 0.5

34. The points that satisfy the system $x + y = 1,\ x^2 + y^2 < 25$, where the symbol "$<$" means "less than," are:

(A) only two points
(B) an arc of a circle
(C) a straight line segment not including the end-points
(D) a straight line segment including the end-points
(E) a single point

35. Side AC of right triangle ABC is divided into 8 equal parts. Seven line segments parallel to BC are drawn to AB from the points of division. If BC = 10, then the sum of the lengths of the seven line segments:

(A) can not be found from the given information
(B) is 33 (C) is 34 (D) is 35 (E) is 45.

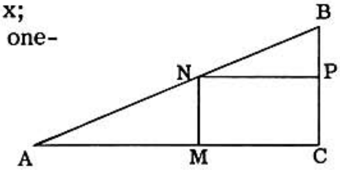


PART III. (4 credits each)

36. If x + y = 1, then the largest value of xy is:

(A) 1 (B) 0.5 (C) an irrational number about 0.4 (D) 0.25 (E) 0

37. In right triangle ABC, BC = 5, AC = 12, and AM = x; $MN \perp AC$, $NP \perp BC$; N is on AB. If y = MN + NP, one-half the perimeter of rectangle MCPN, then:



(A) $y = \frac{1}{2}(5 + 12)$ (B) $y = \frac{5x}{12} + \frac{12}{5}$

(C) $y = \frac{144 - 7x}{12}$ (D) $y = 12$ (E) $y = \frac{5x}{12} + 6$

38. From a two-digit number N we subtract the number with the digits reversed and find that the result is a positive perfect cube. Then:

(A) N cannot end in 5 (B) N can end in any digit other than 5
(C) N does not exist (D) there are exactly 7 values for N
(E) there are exactly 10 values for N

39. Two men set out at the same time to walk towards each other from M and N, 72 miles apart. The first man walks at the rate of 4 mph. The second man walks 2 miles the first hour, $2\frac{1}{2}$ miles the second hour, 3 miles the third hour, and so on in arithmetic progression. Then the men will meet:

(A) in 7 hours (B) in $8\frac{1}{4}$ hours (C) nearer M than N
(D) nearer N than M (E) midway between M and N

40. If the parabola $y = -x^2 + bx - 8$ has its vertex on the x-axis, then b must be:

(A) a positive integer (B) a positive or a negative rational number
(C) a positive rational number (D) a positive or a negative irrational number (E) a negative irrational number

41. Given the system of equations $\begin{matrix} ax + (a-1)y = 1 \\ (a+1)x - ay = 1 \end{matrix}$. For which one of the following values of a is there no solution for x and y?

(A) 1 (B) 0 (C) -1 (D) $\frac{\pm\sqrt{2}}{2}$ (E) $\pm\sqrt{2}$

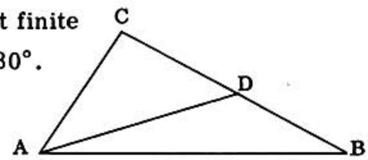
42. If $S = i^n + i^{-n}$, where $i = \sqrt{-1}$ and n is an integer, then the total number of possible distinct values for S is:

(A) 1 (B) 2 (C) 3 (D) 4 (E) more than 4

43. We define a lattice point as a point whose coordinates are integers, zero admitted. Then the number of lattice points on the boundary and inside the region bounded by the x-axis, the line x = 4, and the parabola $y = x^2$ is:

(A) 24 (B) 35 (C) 34 (D) 30 (E) not finite

44. In triangle ABC, AC = CD and $\angle CAB - \angle ABC = 30°$. Then $\angle BAD$ is:



(A) 30° (B) 20° (C) $22\frac{1}{2}°$ (D) 10°
(E) 15°

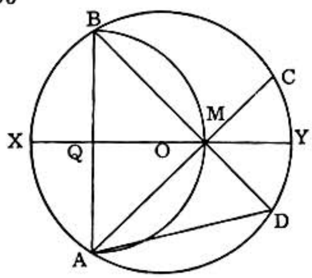
45. If two real numbers x and y satisfy the equation $\frac{x}{y} = x - y$, then:

(A) $x \geqq 4$ and $x \leqq 0$ where $x \geqq a$ means that x can take any value greater than a or equal to a
(B) y can equal 1 (C) both x and y must be irrational
(D) x and y cannot both be integers (E) both x and y must be rational

46. Two perpendicular chords intersect in a circle. The segments of one chord are 3 and 4; the segments of the other are 6 and 2. Then the diameter of the circle is:

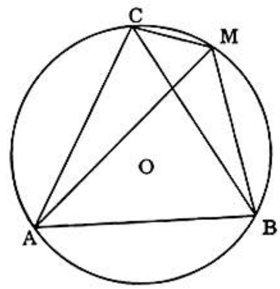
(A) $\sqrt{89}$ (B) $\sqrt{56}$ (C) $\sqrt{61}$ (D) $\sqrt{75}$ (E) $\sqrt{65}$

47. In circle O, the midpoint of radius OX is Q; at Q, $AB \perp XY$. The semi-circle on AB as diameter intersects XY in M. Line AM intersects circle O in C, and line BM intersects circle O in D. Line AD is drawn. Then, if the radius of circle O is r, AD is:



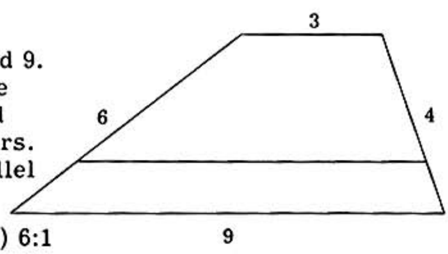
(A) $r\sqrt{2}$ (B) r (C) not a side of an inscribed regular polygon (D) $\frac{r\sqrt{3}}{2}$ (E) $r\sqrt{3}$

48. Let ABC be an equilateral triangle inscribed in circle O. M is a point on arc BC. Lines AM, BM, and CM are drawn. Then AM is:



(A) equal to BM + CM
(B) less than BM + CM
(C) greater than BM + CM
(D) equal, less than, or greater than BM + CM, depending upon the position of M
(E) none of these

49. The parallel sides of a trapezoid are 3 and 9. The non-parallel sides are 4 and 6. A line parallel to the bases divides the trapezoid into two trapezoids having equal perimeters. The ratio into which each of the non-parallel sides is divided is:



(A) 4:3 (B) 3:2 (C) 4:1 (D) 3:1 (E) 6:1

50. In circle O, G is a moving point on diameter AB. AA' is drawn perpendicular to AB and equal to AG. BB' is drawn perpendicular to AB, on the same side of diameter AB as AA', and equal to BG. Let O' be the midpoint of A'B'. Then, as G moves from A to B, point O':

(A) moves on a straight line parallel to AB
(B) remains stationary
(C) moves on a straight line perpendicular to AB
(D) moves in a small circle intersecting the given circle
(E) follows a path which is neither a circle nor a straight line

Committee on Mathematics Contests and Awards
POLYTECHNIC INSTITUTE OF BROOKLYN
BROOKLYN 1, N.Y.

Sponsored Jointly by

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and

THE SOCIETY OF ACTUARIES

NINTH ANNUAL MATHEMATICAL CONTEST 1958

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers, marked A, B, C, D, E. For each question decide upon the correct answer; then write the capital letter that precedes the correct answer in the box on the margin of your answer sheet directly under the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly under No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. Avoid guessing since you are penalized for incorrectly-guessed answers.
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. When your teacher gives the signal tear off the cover of this booklet along the perforated line and turn the cover over. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you fill in your name and the name of your school on it.
8. When your teacher gives the signal begin working the problems. You have 80 minutes for the test.

THURSDAY MORNING, MARCH 27, 1958

PART I

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

PART II

21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40

PART III

41	42	43	44	45	46	47	48	49	50

PRINT ____

first name middle name last name

____

school number street

____

city zone county state

Your teacher will fill in the following blanks:

	Value	Points Correct	Points Attempted
Part I	2 points each		
Part II	3 points each		
Part III	5 points each		
	TOTAL		

$$\left(\underset{\text{Points Correct}}{\qquad\qquad}\right) - \frac{1}{4}\left[\left(\underset{\text{Points Attempted}}{\qquad\qquad}\right) - \left(\underset{\text{Points Correct}}{\qquad\qquad}\right)\right] = \left(\underset{\text{SCORE}}{\qquad\qquad}\right)$$

PART I. (2 credits each)

1. The value of $[2 - 3(2 - 3)^{-1}]^{-1}$ is:

   (A) 5 (B) -5 (C) $\frac{1}{5}$ (D) $-\frac{1}{5}$ (E) $\frac{5}{3}$

2. If $\frac{1}{x} - \frac{1}{y} = \frac{1}{z}$, then z equals:

   (A) $y - x$ (B) $x - y$ (C) $\frac{y - x}{xy}$ (D) $\frac{xy}{y - x}$ (E) $\frac{xy}{x - y}$

3. Of the following expressions the one equal to $\frac{a^{-1}b^{-1}}{a^{-3} - b^{-3}}$ is:

   (A) $\frac{a^2b^2}{b^2 - a^2}$ (B) $\frac{a^2b^2}{b^3 - a^3}$ (C) $\frac{ab}{b^3 - a^3}$ (D) $\frac{a^3 - b^3}{ab}$ (E) $\frac{a^3b^3}{a - b}$

4. In the expression $\frac{x + 1}{x - 1}$ each x is replaced by $\frac{x + 1}{x - 1}$. The resulting expression, evaluated for $x = \frac{1}{2}$, equals:

   (A) 3 (B) -3 (C) 1 (D) -1 (E) none of these

5. The expression $2 + \sqrt{2} + \frac{1}{2 + \sqrt{2}} + \frac{1}{\sqrt{2} - 2}$ equals:

   (A) 2 (B) $2 - \sqrt{2}$ (C) $2 + \sqrt{2}$ (D) $2\sqrt{2}$ (E) $\sqrt{2}/2$

6. The arithmetic mean between $\frac{x + a}{x}$ and $\frac{x - a}{x}$, when $x \neq 0$, is (the symbol $\neq$ means "not equal to"):

   (A) 2, if $a \neq 0$ (B) 1 (C) 1, if $a = 0$ only (D) a/x (E) x

7. A straight line joins the points $(-1, 1)$ and $(3, 9)$. Its x-intercept is:

   (A) $-\frac{3}{2}$ (B) $-\frac{2}{3}$ (C) $\frac{2}{5}$ (D) 2 (E) 3

8. Which of these four numbers $\sqrt{\pi^2}$, $\sqrt[3]{.8}$, $\sqrt[4]{.00016}$, $\sqrt[3]{-1} \cdot \sqrt{(.09)^{-1}}$, is (are) rational;

   (A) none (B) all (C) the first and fourth (D) only the fourth (E) only the first

9. A value of x satisfying the equation $x^2 + b^2 = (a - x)^2$ is:

   (A) $\frac{b^2 + a^2}{2a}$ (B) $\frac{b^2 - a^2}{2a}$ (C) $\frac{a^2 - b^2}{2a}$ (D) $\frac{a - b}{2}$ (E) $\frac{a^2 - b^2}{2}$

. For what real values of k, other than $k = 0$, does the equation $x^2 + kx + k^2 = 0$ have real roots? (The symbol $x \geq a$ means that x can take on all values greater than a and the value a itself; $x \leq a$ has the corresponding meaning with "less than".)

   (A) $k < 0$ (B) $k > 0$ (C) $k \geq 1$ (D) all values of k
   (E) no values of k

. The number of roots satisfying the equation $\sqrt{5 - x} = x\sqrt{5 - x}$ is:

   (A) unlimited (B) 3 (C) 2 (D) 1 (E) 0

. If $P = \frac{s}{(1 + k)^n}$, then n equals:

   (A) $\frac{\log s/P}{\log(1 + k)}$ (B) $\log \frac{s}{P(1 + k)}$ (C) $\log \frac{s - P}{1 + k}$ (D) $\log \frac{s}{P} + \log(1 + k)$ (E) $\frac{\log s}{\log P(1 + k)}$

13. The sum of two numbers is 10; their product is 20. The sum of their reciprocals is:

(A) $\frac{1}{10}$ (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) 4

14. At a dance party a group of boys and girls exchange dances as follows: one boy dances with 5 girls, a second boy dances with 6 girls, and so on, the last boy dancing with all the girls. If b represents the number of boys and g, the number of girls, then:

(A) $b = g$ (B) $b = g/5$ (C) $b = g - 4$ (D) $b = g - 5$ (E) It is impossible to determine a relation between b and g without knowing the total number of boys and girls.

15. A quadrilateral is inscribed in a circle. If an angle is inscribed into each of the four segments outside the quadrilateral, the sum of these four angles, expressed in degrees, is:

(A) 1080 (B) 900 (C) 720 (D) 540 (E) 360

16. The area of a circle inscribed in a regular hexagon is 100π. The area of the hexagon is:

(A) 600 (B) 300 (C) $200\sqrt{2}$ (D) $200\sqrt{3}$ (E) $120\sqrt{5}$

17. If x is positive and $\log x \geq \log 2 + \frac{1}{2}\log x$, then:

(A) x has no minimum or maximum value (B) the maximum value of x is 1
(C) the minimum value of x is 1 (D) the maximum value of x is 4
(E) the minimum value of x is 4

18. The area of a circle is doubled when its radius r is increased by n. Then r equals:

(A) $n(\sqrt{2}+1)$ (B) $n(\sqrt{2}-1)$ (C) n (D) $n(2-\sqrt{2})$ (E) $\frac{n\pi}{\sqrt{2}+1}$

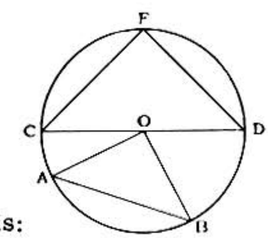
19. The sides of a right triangle are a and b and the hypotenuse is c. A perpendicular from the vertex divides c into segments r and s, adjacent respectively to a and b. If $a:b = 1:3$, then the ratio of r to s is:

(A) 1:3 (B) 1:9 (C) 1:10 (D) 3:10 (E) $1:\sqrt{10}$

20. If $4^x - 4^{x-1} = 24$, then $(2x)^x$ equals:

(A) $5\sqrt{5}$ (B) $\sqrt{5}$ (C) $25\sqrt{5}$ (D) 125 (E) 25

PART II. (3 credits each)



21. In the accompanying figure CE and DE are equal chords of a circle with center O. Arc AB is a quarter-circle. Then the ratio of the area of triangle CED to the area of triangle AOB is:

(A) $\sqrt{2}:1$ (B) $\sqrt{3}:1$ (C) 4:1 (D) 3:1 (E) 2:1

22. A particle is placed on the parabola $y = x^2 - x - 6$ at a point P whose ordinate is 6. It is allowed to roll along the parabola until it reaches the nearest point Q whose ordinate is -6. The horizontal distance traveled by the particle (the numerical value of the difference in the abscissas of P and Q) is:

(A) 5 (B) 4 (C) 3 (D) 2 (E) 1

23. If, in the expression $x^2 - 3$, x increases or decreases by a positive amount a, the expression changes by an amount:

(A) $\pm 2ax + a^2$ (B) $2ax \pm a^2$ (C) $\pm a^2 - 3$ (D) $(x + a)^2 - 3$ (E) $(x - a)^2 - 3$

24. A man travels m feet due north at 2 minutes per mile. He returns due south to his starting point at 2 miles per minute. The average rate in miles per hour for the entire trip is:

(A) 75 (B) 48 (C) 45 (D) 24 (E) impossible to determine without knowing the value of m

25. If $\log_k x \cdot \log_5 k = 3$, then x equals:

(A) k^5 (B) $5k^3$ (C) k^3 (D) 243 (E) 125

26. A set of n numbers has the sum s. Each number of the set is increased by 20, then multiplied by 5, and then decreased by 20. The sum of the numbers in the new set thus obtained is:

(A) $s + 20n$ (B) $5s + 80n$ (C) s (D) 5s (E) $5s + 4n$

27. The points $(2, -3)$, $(4, 3)$ and $(5, k/2)$ are on the same straight line. The value(s) of k is (are):

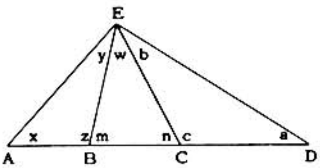
(A) 12 (B) -12 (C) ± 12 (D) 12 or 6 (E) 6 or $6\frac{2}{3}$

28. A 16-quart radiator is filled with water. Four quarts are removed and replaced with pure antifreeze liquid. Then four quarts of the mixture are removed and replaced with pure antifreeze. This is done a third and a fourth time. The fractional part of the final mixture that is water is:

(A) $\frac{1}{4}$ (B) $\frac{81}{256}$ (C) $\frac{27}{64}$ (D) $\frac{37}{64}$ (E) $\frac{175}{256}$

29. In a general triangle ADE (as shown in the diagram) lines EB and EC are drawn. Which of the following angle relations is true?

(A) $x + z = a + b$
(B) $y + z = a + b$
(C) $m + x = w + n$
(D) $x + z + n = w + c + m$
(E) $x + y + n = a + b + m$



30. If $xy = b$ and $\frac{1}{x^2} + \frac{1}{y^2} = a$, then $(x + y)^2$ equals:

(A) $(a + 2b)^2$ (B) $a^2 + b^2$ (C) $b(ab + 2)$ (D) $ab(b + 2)$ (E) $\frac{1}{a} + 2b$

31. The altitude drawn to the base of an isosceles triangle is 8, and the perimeter is 32. The area of the triangle is:

(A) 56 (B) 48 (C) 40 (D) 32 (E) 24

32. With $1000 a rancher is to buy steers at $25 each and cows at $26 each. If the number of steers s and the number of cows c are both positive integers, then:

(A) this problem has no solution
(B) there are two solutions with s exceeding c
(C) there are two solutions with c exceeding s
(D) there is one solution with s exceeding c
(E) there is one solution with c exceeding s

33. For one root of $ax^2 + bx + c = 0$ to be double the other, the coefficients a, b, c must be related as follows:

(A) $4b^2 = 9c$ (B) $2b^2 = 9ac$ (C) $2b^2 = 9a$ (D) $b^2 - 8ac = 0$ (E) $9b^2 = 2ac$

34. The numerator of a fraction is $6x + 1$, the denominator is $7 - 4x$, and x can have any value between -2 and 2, both included. The values of x for which the numerator is greater than the denominator are:

(A) $3/5 < x \le 2$ (B) $3/5 \le x \le 2$ (C) $0 < x \le 2$ (D) $0 \le x \le 2$ (E) $-2 \le x \le 2$

35. A triangle is formed by joining three points whose coordinates are integers. If the x-unit and the y-unit are each 1 inch, then the area of the triangle, in square inches,

(A) must be an integer
(B) may be irrational
(C) must be irrational
(D) must be rational
(E) will be an integer only if the triangle is equilateral.

36. The sides of a triangle are 30, 70, and 80 units. If an altitude is dropped upon the side 80, the larger segment cut off on this side is:

(A) 62 (B) 63 (C) 64 (D) 65 (E) 66

37. The first term of an arithmetic series of consecutive integers is $k^2 - 1$. The sum of $2k + 1$ terms of this series may be expressed as:

(A) $k^3 + (k+1)^3$ (B) $(k-1)^3 + k^3$ (C) $(k+1)^3$ (D) $(k+1)^2$ (E) $(2k+1)(k+1)^2$

38. Let r be the distance from the origin to a point P with coordinates x and y. Designate the ratio y/r by s and the ratio x/r by c. Then the values of $s^2 - c^2$ are limited to the numbers:

(A) less than -1 and greater than $+1$, both excluded
(B) less than -1 and greater than $+1$, both included
(C) between -1 and $+1$, both excluded
(D) between -1 and $+1$, both included
(E) -1 and $+1$ only

39. The symbol $|x|$ means x if x is not negative and $-x$ if x is not positive. We may then say concerning the solution of $|x|^2 + |x| - 6 = 0$ that:

(A) there is only one root
(B) the sum of the roots is $+1$
(C) the sum of the roots is 0
(D) the product of the roots is $+4$
(E) the product of the roots is -6

40. Given $a_0 = 1$, $a_1 = 3$, and the general relation $a_n^2 - a_{n-1}a_{n+1} = (-1)^n$ for $n \ge 1$. Then a_3 equals:

(A) $13/27$ (B) 33 (C) 21 (D) 10 (E) -17

PART III. (5 credits each)

41. The roots of $Ax^2 + Bx + C = 0$ are r and s. For the roots of $x^2 + px + q = 0$ to be r^2 and s^2, p must equal:

(A) $\frac{B^2 - 4AC}{A^2}$ (B) $\frac{B^2 - 2AC}{A^2}$ (C) $\frac{2AC - B^2}{A^2}$ (D) $B^2 - 2C$ (E) $2C - B^2$

42. In a circle with center O chord AB = chord AC. Chord AD cuts BC in E. If AC = 12 and AE = 8, then AD equals:

(A) 27 (B) 24 (C) 21 (D) 20 (E) 18

43. AB is the hypotenuse of a right triangle ABC. Median AD = 7 and median BE = 4. The length of AB is:

(A) 10 (B) $5\sqrt{3}$ (C) $5\sqrt{2}$ (D) $2\sqrt{13}$ (E) $2\sqrt{15}$

44. Given the true statements: (1) If a is greater than b, then c is greater than d (2) If c is less than d, then e is greater than f. A valid conclusion is:

(A) If a is less than b, then e is greater than f
(B) If e is greater than f, then a is less than b
(C) If e is less than f, then a is greater than b
(D) If a is greater than b, then e is less than f
(E) none of these

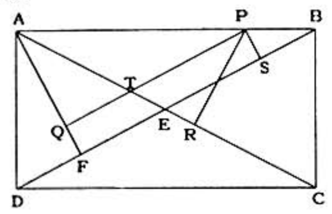
45. A check is written for x dollars and y cents, x and y both two-digit numbers. In error it is cashed for y dollars and x cents, the incorrect amount exceeding the correct amount by \$17.82. Then:

(A) x cannot exceed 70
(B) y can equal 2x
(C) the amount of the check cannot be a multiple of 5
(D) the incorrect amount can equal twice the correct amount
(E) the sum of the digits of the correct amount is divisible by 9

46. For values of x less than 1 but greater than -4, the expression $\frac{x^2 - 2x + 2}{2x - 2}$ has:

(A) no maximum or minimum value
(B) a minimum value of +1
(C) a maximum value of +1
(D) a minimum value of -1
(E) a maximum value of -1

47. ABCD is a rectangle (see the accompanying diagram) with P any point on AB. PS $\perp$ BD and PR $\perp$ AC. AF $\perp$ BD and PQ $\perp$ AF. Then PR + PS is equal to:



(A) PQ
(B) AE
(C) PT + AT
(D) AF
(E) EF

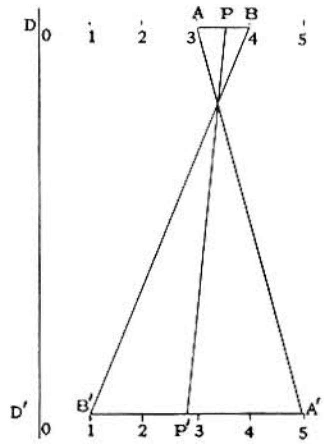
48. Diameter AB of a circle with center 0 is 10 units. C is a point 4 units from A, and on AB. D is a point 4 units from B, and on AB. P is any point on the circle. Then the broken-line path from C to P to D:

(A) has the same value for all positions of P
(B) exceeds 10 units for all positions of P
(C) cannot exceed 10 units
(D) is the least when CPD is a right triangle
(E) is the greatest when P is equidistant from C and D.

49. In the expansion of $(a + b)^n$ there are n + 1 dissimilar terms. The number of dissimilar terms in the expansion of $(a + b + c)^{10}$ is:

(A) 11 (B) 33 (C) 55 (D) 66 (E) 132

50. In this diagram a scheme is indicated for associating all the points of segment AB with those of segment A' B', and reciprocally. To describe this association scheme analytically let x be the distance from a point P on AB to D and let y be the distance from the associated point P' of A' B' to D'. Then for any pair of associated points, if x = a, x + y equals:



(A) 13a
(B) 17a − 51
(C) 17 − 3a
(D) $\frac{17 - 3a}{4}$
(E) 12a − 34

Annual High School Mathematics Contest
PAN AMERICAN COLLEGE
EDINBURG, TEXAS

Sponsored Jointly by

THE MATHEMATICAL ASSOCIATION OF AMERICA

and

THE SOCIETY OF ACTUARIES

TENTH ANNUAL MATHEMATICAL CONTEST 1959

INSTRUCTIONS

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2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers, marked A, B, C, D, E. For each question decide upon the correct answer; then write the capital letter that precedes the correct answer in the box on the margin of your answer sheet directly under the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly under No. 3. Fill in the answers as you find them.
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THURSDAY MORNING, MARCH 5, 1959

PART I

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

PART II

21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40

PART III

41	42	43	44	45	46	47	48	49	50

To be filled in by the student

PRINT ____
first name middle name last name

____
school number street

____
city zone county state

Not to be filled in by the student

	Value	Points Correct	Points Attempted
Part I	2 points each		
Part II	3 points each		
Part III	5 points each		
	TOTAL		

$$(\qquad)_{\text{Points Correct}} - \frac{1}{4}\left[(\qquad)_{\text{Points Attempted}} - (\qquad)_{\text{Points Correct}}\right] = (\qquad)_{\text{SCORE}}$$

PART I (2 credits each)

1. Each edge of a cube is increased by 50%. The percent of increase in the surface area of the cube is:

   (A) 50 (B) 125 (C) 150 (D) 300 (E) 750

2. Through a point P inside the triangle ABC a line is drawn parallel to the base AB, dividing the triangle into two equal areas. If the altitude to AB has a length of 1, then the distance from P to AB is:

   (A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $2-\sqrt{2}$ (D) $\frac{2-\sqrt{2}}{2}$ (E) $\frac{2+\sqrt{2}}{8}$

3. If the diagonals of a quadrilateral are perpendicular to each other, the figure would always be included under the general classification:

   (A) rhombus (B) rectangle (C) square (D) isosceles trapezoid (E) none of these

4. If 78 is divided into three parts which are proportional to 1, $\frac{1}{3}$, $\frac{1}{6}$, the middle part is:

   (A) $9\frac{1}{3}$ (B) 13 (C) $17\frac{1}{3}$ (D) $18\frac{1}{3}$ (E) 26

5. The value of $(256)^{.16} \cdot (256)^{.09}$ is:

   (A) 4 (B) 16 (C) 64 (D) 256.25 (E) -16

6. Given the true statement: If a quadrilateral is a square, then it is a rectangle. It follows that, of the converse and the inverse of this true statement,

   (A) only the converse is true (B) only the inverse is true (C) both are true (D) neither is true (E) the inverse is true, but the converse is sometimes true

7. The sides of a right triangle are a, $a+d$, and $a+2d$, with $\underline{a}$ and $\underline{d}$ both positive. The ratio of $\underline{a}$ to $\underline{d}$ is:

   (A) 1:3 (B) 1:4 (C) 2:1 (D) 3:1 (E) 3:4

8. The value of $x^2 - 6x + 13$ can never be less than:

   (A) 4 (B) 4.5 (C) 5 (D) 7 (E) 13

9. A farmer divides his herd of $\underline{n}$ cows among his four sons so that one son gets one-half the herd, a second son, one-fourth, a third son one-fifth, and the fourth son, 7 cows. Then $\underline{n}$ is:

   (A) 80 (B) 100 (C) 140 (D) 180 (E) 240

10. In triangle ABC, with AB = AC = 3.6, a point D is taken on AB at a distance 1.2 from A. Point D is joined to point E in the prolongation of AC so that triangle AED is equal in area to triangle ABC. Then AE equals:

    (A) 4.8 (B) 5.4 (C) 7.2 (D) 10.8 (E) 12.6

11. The logarithm of .0625 to the base 2 is:

    (A) .025 (B) .25 (C) 5 (D) -4 (E) -2

12. By adding the same constant to each of 20, 50, 100 a geometric progression results. The common ratio is:

(A) $5/3$ (B) $4/3$ (C) $3/2$ (D) $1/2$ (E) $1/3$

13. The arithmetic mean (average) of a set of 50 numbers is 38. If two numbers, namely, 45 and 55, are discarded, the mean of the remaining set of numbers is:

(A) 36.5 (B) 37 (C) 37.2 (D) 37.5 (E) 37.52

14. Given the set S whose elements are zero and the even integers, positive and negative. Of the five operations applied to any pair of elements: (1) addition (2) subtraction (3) multiplication (4) division (5) finding the arithmetic mean (average), those operations that yield only elements of S are:

(A) all (B) 1, 2, 3, 4 (C) 1, 2, 3, 5 (D) 1, 2, 3 (E) 1, 3, 5

15. In a right triangle the square of the hypotenuse is equal to twice the product of the legs. One of the acute angles of the triangle is:

(A) $15°$ (B) $30°$ (C) $45°$ (D) $60°$ (E) $75°$

16. The expression $\frac{x^2 - 3x + 2}{x^2 - 5x + 6} \div \frac{x^2 - 5x + 4}{x^2 - 7x + 12}$, when simplified, is:

(A) $\frac{(x-1)(x-6)}{(x-3)(x-4)}$ (B) $\frac{x+3}{x-3}$ (C) $\frac{x+1}{x-1}$ (D) 1 (E) 2

17. If $y = a + \frac{b}{x}$, where $\underline{a}$ and $\underline{b}$ are constants, and if $y = 1$ when $x = -1$, and $y = 5$ when $x = -5$, then $a + b$ equals:

(A) -1 (B) 0 (C) 1 (D) 10 (E) 11

18. Let $\underline{r}$ and $\underline{s}$ be positive numbers with $\underline{r}$ greater than $\underline{s}$. Let $R = r + 1$ and $S = s + 1$. The difference in percent by which r exceeds $\underline{s}$ and $\underline{R}$ exceeds $\underline{S}$ is:

(A) $\frac{100(r-s)}{r}$ (B) $\frac{100(r-s)}{s}$ (C) $\frac{100(r-s)}{r+1}$ (D) $\frac{100(r-s)}{s+1}$ (E) none of these

19. With the use of three different weights, namely, 1 lb., 3 lb., and 9 lb., how many objects of different weights can be weighed, if the objects to be weighed and the given weights may be placed in either pan of the scale?

(A) 15 (B) 13 (C) 11 (D) 9 (E) 7

20. It is given that $\underline{x}$ varies directly as $\underline{y}$ and inversely as the square of $\underline{z}$, and that $x = 10$ when $y = 4$ and $z = 14$. Then, when $y = 16$ and $z = 7$, $\underline{x}$ equals:

(A) 180 (B) 160 (C) 154 (D) 140 (E) 120

PART II (3 credits each)

21. If $\underline{p}$ is the perimeter of an equilateral triangle inscribed in a circle, the area of the circle is:

(A) $\frac{\pi p^2}{3}$ (B) $\frac{\pi p^2}{9}$ (C) $\frac{\pi p^2}{27}$ (D) $\frac{\pi p^2}{81}$ (E) $\frac{\pi p^2\sqrt{3}}{27}$

22. The line joining the midpoints of the diagonals of a trapezoid has length 3. If the longer base is 97, then the shorter base is:

(A) 94 (B) 92 (C) 91 (D) 90 (E) 89

23. The set of solutions for the equation $\log_{10}(a^2 - 15a) = 2$ consists of:

(A) two integers (B) one integer and one fraction (C) two irrational numbers (D) two non-real numbers (E) no numbers, that is, the set is empty.

24. A chemist has m ounces of salt water that is m% salt. How many ounces of salt must he add to make a solution that is 2m% salt?

(A) $\frac{m}{100+m}$ (B) $\frac{2m}{100-2m}$ (C) $\frac{m^2}{100-2m}$ (D) $\frac{m^2}{100+2m}$ (E) $\frac{2m}{100+m}$

25. The symbol $|a|$ means $+a$ if a is greater than or equal to zero, and $-a$ if a is less than or equal to zero; the symbol $<$ means "less than"; the symbol $>$ means "greater than."
The set of values x satisfying the inequality $|3-x| < 4$ consists of all x such that:

(A) $x^2 < 49$ (B) $x^2 > 1$ (C) $1 < x^2 < 49$ (D) $-1 < x < 7$ (E) $-7 < x < 1$

26. The base of an isosceles triangle is $\sqrt{2}$. The medians to the legs intersect each other at right angles. The area of the triangle is:

(A) 1.5 (B) 2 (C) 2.5 (D) 3.5 (E) 4

27. Which one of the following statements is not true for the equation $ix^2 - x + 2i = 0$ where $i = \sqrt{-1}$?

(A) The sum of the roots is 2 (B) The discriminant is 9 (C) The roots are imaginary (D) The roots can be found by using the quadratic formula (E) The roots can be found by factoring, using imaginary numbers.

28. In triangle ABC, AL bisects angle A and CM bisects angle C. Points L and M are on BC and AB, respectively. The sides of triangle ABC are a, b, and c. Then $\frac{AM}{MB} = k\frac{CL}{LB}$ where k is:

(A) 1 (B) $\frac{bc}{a^2}$ (C) $\frac{a^2}{bc}$ (D) $\frac{c}{b}$ (E) $\frac{c}{a}$

29. On an examination of n questions a student answers correctly 15 of the first 20. Of the remaining questions he answers one third correctly. All the questions have the same credit. If the student's mark is 50%, how many different values of n can there be?

(A) 4 (B) 3 (C) 2 (D) 1 (E) the problem cannot be solved

30. A can run around a circular track in 40 seconds. B, running in the opposite direction, meets A every 15 seconds. What is B's time to run around the track, expressed in seconds?

(A) $12\frac{1}{2}$ (B) 24 (C) 25 (D) $27\frac{1}{2}$ (E) 55

31. A square, with an area of 40, is inscribed in a semicircle. The area of a square that could be inscribed in the entire circle with the same radius, is:

(A) 80 (B) 100 (C) 120 (D) 160 (E) 200

32. The length $\underline{l}$ of a tangent, drawn from a point A to a circle, is $\frac{4}{3}$ of the radius $\underline{r}$. The (shortest) distance from A to the circle is:

(A) $\frac{1}{2}r$ (B) r (C) $\frac{1}{2}l$ (D) $\frac{2}{3}l$ (E) a value between $\underline{r}$ and $\underline{l}$.

33. A harmonic progression is a sequence of numbers such that their reciprocals are in arithmetic progression.

Let S_n represent the sum of the first $\underline{n}$ terms of the harmonic progression; for example, S_3 represents the sum of the first three terms. If the first three terms of a harmonic progression are 3, 4, 6, then:

(A) $S_4 = 20$ (B) $S_4 = 25$ (C) $S_5 = 49$ (D) $S_6 = 49$ (E) $S_2 = \frac{1}{2}S_4$

34. Let the roots of $x^2 - 3x + 1 = 0$ be $\underline{r}$ and $\underline{s}$. Then the expression $r^2 + s^2$ is:

(A) a positive integer (B) a positive fraction greater than 1 (C) a positive fraction less than 1 (D) an irrational number (E) an imaginary number

35. The symbol $\geq$ means "greater than or equal to"; the symbol $\leq$ means "less than or equal to".

In the equation $(x - m)^2 - (x - n)^2 = (m - n)^2$, $\underline{m}$ is a fixed positive number, and $\underline{n}$ is a fixed negative number. The set of values $\underline{x}$ satisfying the equation is:

(A) $x \geq 0$ (B) $x \leq n$ (C) $x = 0$ (D) the set of all real numbers (E) none of these

36. The base of a triangle is 80, and one of the base angles is 60°. The sum of the other two sides is 90. The shortest side of the triangle is:

(A) 45 (B) 40 (C) 36 (D) 17 (E) 12

37. When simplified the product $(1 - \frac{1}{3})(1 - \frac{1}{4})(1 - \frac{1}{5}) \dots (1 - \frac{1}{n})$ becomes:

(A) $\frac{1}{n}$ (B) $\frac{2}{n}$ (C) $\frac{2(n-1)}{n}$ (D) $\frac{2}{n(n+1)}$ (E) $\frac{3}{n(n+1)}$

38. If $4x + \sqrt{2x} = 1$, then x:

(A) is an integer (B) is fractional (C) is irrational (D) is imaginary (E) may have two different values

39. Let S be the sum of the first nine terms of the sequence, $x + a, x^2 + 2a, x^3 + 3a, \dots$. Then S equals:

(A) $\frac{50a + x + x^8}{x + 1}$ (B) $50a - \frac{x + x^{10}}{x - 1}$ (C) $\frac{x^9 - 1}{x + 1} + 45a$ (D) $\frac{x^{10} - x}{x - 1} + 45a$

(E) $\frac{x^{11} - x}{x - 1} + 45a$

40. In triangle ABC, BD is a median. CF intersects BD at E so that BE = ED. Point F is on AB. Then, if BF = 5, BA equals:

(A) 10 (B) 12 (C) 15 (D) 20 (E) none of these

PART III (5 credits each)

41. On the same side of a straight line three circles are drawn as follows: a circle with a radius of 4 inches is tangent to the line, the other two circles are equal, and each is tangent to the line and to the other two circles. The radius of the equal circles is:

(A) 24 (B) 20 (C) 18 (D) 16 (E) 12

42. Given three positive integers a, b, and c. Their greatest common divisor is D; their least common multiple is M. Then, which two of the following statements are true?

(1) the product MD cannot be less than abc
(2) the product MD cannot be greater than abc
(3) MD equals abc if and only if a, b, c are each prime
(4) MD equals abc if and only if a, b, c are relatively prime in pairs. (This means: no two have a common factor greater than 1)

(A) 1, 2 (B) 1, 3 (C) 1, 4 (D) 2, 3 (E) 2, 4

43. The sides of a triangle are 25, 39, and 40. The diameter of the circumscribed circle is:

(A) $^{133}/_3$ (B) $^{125}/_3$ (C) 42 (D) 41 (E) 40

44. The roots of $x^2 + bx + c = 0$ are both real and greater than 1. Let $s = b + c + 1$. Then s:

(A) may be less than zero (B) may be equal to zero (C) must be greater than zero (D) must be less than zero (E) must be between -1 and $+1$.

45. If $(\log_3 x)(\log_x 2x)(\log_{2x} y) = \log_x x^2$, then y equals:

(A) $^9/_2$ (B) 9 (C) 18 (D) 27 (E) 81

46. A student on vacation for d days observed that (1) it rained 7 times, morning or afternoon (2) when it rained in the afternoon, it was clear in the morning (3) there were five clear afternoons (4) there were six clear mornings. Then d equals:

(A) 7 (B) 9 (C) 10 (D) 11 (E) 12

47. Assume that the following three statements are true:
I. All freshmen are human. II. All students are human. III. Some students think
Given the following four statements:
(1) All freshmen are students. (2) Some humans think. (3) No freshmen think.
(4) Some humans who think are not students.
Those which are logical consequences of I, II, and III are:

(A) 2 (B) 4 (C) 2, 3 (D) 2, 4 (E) 1, 2

48. Given the polynomial $a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$, where n is a positive integer or zero, and a_0 is a positive integer. The remaining a's are integers or zero. Set $h = n + a_0 + |a_1| + |a_2| + \dots + |a_n|$. [See example 25 for the meaning of $|x|$.] The number of polynomials with $h = 3$ is:

(A) 3 (B) 5 (C) 6 (D) 7 (E) 9

49. For the infinite series $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} - \dots$ let S be the (limiting) sum. Then S equals:

(A) 0 (B) $\frac{2}{7}$ (C) $\frac{6}{7}$ (D) $\frac{9}{32}$ (E) $\frac{27}{32}$

50. A club with $\underline{x}$ members is organized into four committees in accordance with these two rules:

(1) Each member belongs to two and only two committees.
(2) Each pair of committees has one and only one member in common.
Then $\underline{x}$:

(A) cannot be determined (B) has a single value between 8 and 16 (C) has two values between 8 and 16 (D) has a single value between 4 and 8 (E) has two values between 4 and 8.

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ELEVENTH ANNUAL MATHEMATICS CONTEST 1960

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers, marked A, B, C, D, E. For each question decide upon the correct answer; then write the capital letter that precedes the correct answer in the box on the margin of your answer sheet directly under the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly under No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. Avoid guessing since you are penalized for incorrectly-guessed answers.
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. When your teacher gives the signal tear off the cover of this booklet along the perforated line and turn the cover over. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you fill in your name and the name of your school on it.
8. When your teacher gives the signal begin working the problems. You have 80 minutes for the test.

THURSDAY MORNING, MARCH 10, 1960

To be filled in by the student

PRINT ____

first name middle name last name

____

school number street

____

city zone county state

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

PART I

21	22	23	24	25	26	27	28	29	30

PART II

31	32	33	34	35	36	37	38	39	40

PART III

Not to be filled in by the student

	Value	Points Correct, C	Points Wrong, W (do not include omissions)
Part I	3 points each		
Part II	4 points each		
Part III	5 points each		
	TOTALS		
	SCORE =	$C - \frac{1}{4}W$ =	(Write Score Here)

ELEVENTH ANNUAL H. S. MATHEMATICS CONTEST — 1960

PART I (3 credits each)

1. If 2 is a solution (root) of $x^3 + hx + 10 = 0$, then $\underline{h}$ equals:

   (A) 10 (B) 9 (C) 2 (D) -2 (E) -9

2. It takes 5 seconds for a clock to strike 6 o'clock beginning at 6:00 o'clock precisely. If the strikings are uniformly spaced, how long, in seconds, does it take to strike 12 o'clock?

   (A) $9\frac{1}{5}$ (B) 10 (C) 11 (D) $14\frac{2}{5}$ (E) none of these

3. Applied to a bill for \$10,000 the difference between a discount of 40% and two successive discounts of 36% and 4%, expressed in dollars, is:

   (A) 0 (B) 144 (C) 256 (D) 400 (E) 416

4. Each of two angles of a triangle is 60° and the included side is 4 inches. The area of the triangle, in square inches, is:

   (A) $8\sqrt{3}$ (B) 8 (C) $4\sqrt{3}$ (D) 4 (E) $2\sqrt{3}$

5. The number of distinct points common to the graphs of $x^2 + y^2 = 9$ and $y^2 = 9$ is:

   (A) infinitely many (B) four (C) two (D) one (E) none

6. The circumference of a circle is 100 inches. The side of a square inscribed in this circle, expressed in inches, is:

   (A) $\frac{25\sqrt{2}}{\pi}$ (B) $\frac{50\sqrt{2}}{\pi}$ (C) $\frac{100}{\pi}$ (D) $\frac{100\sqrt{2}}{\pi}$ (E) $50\sqrt{2}$

7. Circle I passes through the center of, and is tangent to, circle II. The area of circle I is 4 square inches. Then the area of circle II, in square inches, is:

   (A) 8 (B) $8\sqrt{2}$ (C) $8\sqrt{\pi}$ (D) 16 (E) $16\sqrt{2}$

8. The number 2.5252525 . . . can be written as a fraction. When reduced to lowest terms the sum of the numerator and denominator of this fraction is:

   (A) 7 (B) 29 (C) 141 (D) 349 (E) none of these

9. The fraction $\frac{a^2 + b^2 - c^2 + 2ab}{a^2 + c^2 - b^2 + 2ac}$ is (with suitable restrictions on the values of a, b, and c):

(A) irreducible (B) reducible to -1 (C) reducible to a polynomial of three terms (D) reducible to $\frac{a - b + c}{a + b - c}$ (E) reducible to $\frac{a + b - c}{a - b + c}$

10. Given the following six statements:
(1) All women are good drivers (2) Some women are good drivers
(3) No men are good drivers (4) All men are bad drivers
(5) At least one man is a bad driver (6) All men are good drivers.
The statement that negates statement (6) is:

(A) (1) (B) (2) (C) (3) (D) (4) (E) (5)

11. For a given value of $\underline{k}$ the product of the roots of $x^2 - 3kx + 2k^2 - 1 = 0$ is 7. The roots may be characterized as:

(A) integral and positive (B) integral and negative (C) rational, but not integral (D) irrational (E) imaginary

12. The locus of the centers of all circles of given radius $\underline{a}$, in the same plane, passing through a fixed point, is:

(A) a point (B) a straight line (C) two straight lines
(D) a circle (E) two circles

13. The polygon(s) formed by $y = 3x + 2$, $y = -3x + 2$, and $y = -2$, is (are):

(A) an equilateral triangle (B) an isosceles triangle (C) a right triangle
(D) a triangle and a trapezoid (E) a quadrilateral

14. If $\underline{a}$ and $\underline{b}$ are real numbers, the equation $3x - 5 + a = bx + 1$ has a unique solution x [the symbol $a \neq 0$ means that $\underline{a}$ is different from zero]:

(A) for all $\underline{a}$ and $\underline{b}$ (B) if $a \neq 2b$ (C) if $a \neq 6$ (D) if $b \neq 0$
(E) if $b \neq 3$

15. Triangle I is equilateral with side $\underline{A}$, perimeter $\underline{P}$, area $\underline{K}$, and circumradius $\underline{R}$ (radius of the circumscribed circle). Triangle II is equilateral with side $\underline{a}$, perimeter $\underline{p}$, area $\underline{k}$, and circumradius $\underline{r}$. If $\underline{A}$ is different from $\underline{a}$, then:

(A) $P:p = R:r$ only sometimes (B) $P:p = R:r$ always (C) $P:p = K:k$ only sometimes (D) $R:r = K:k$ always (E) $R:r = K:k$ only sometimes

16. In the numeration system with base 5, counting is as follows: 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, ... The number whose description in the decimal system is 69, when described in the base 5 system, is a number with:

(A) two consecutive digits (B) two non-consecutive digits (C) three consecutive digits (D) three non-consecutive digits (E) four digits

17. The formula $N = 8 \cdot 10^8 \cdot x^{-\frac{3}{2}}$ gives, for a certain group, the number of individuals whose income exceeds $\underline{x}$ dollars. The lowest income, in dollars, of the wealthiest 800 individuals is at least:

(A) 10^4 (B) 10^6 (C) 10^8 (D) 10^{12} (E) 10^{16}

18. The pair of equations $3^{x+y} = 81$ and $81^{x-y} = 3$ has:

(A) no common solution (B) the solution $x = 2$, $y = 2$ (C) the solution $x = 2\frac{1}{2}$, $y = 1\frac{1}{2}$ (D) a common solution in positive and negative integers (E) none of these

19. Consider equation I: $x + y + z = 46$ where x, y, and z are positive integers, and equation II: $x + y + z + w = 46$, where x, y, z, and w are positive integers. Then

(A) I can be solved in consecutive integers (B) I can be solved in consecutive even integers (C) II can be solved in consecutive integers (D) II can be solved in oonsecutive even integers (E) II can be solved in consecutive odd integers.

20. The coefficient of x^7 in the expansion of $\left(\frac{x^2}{2} - \frac{2}{x}\right)^8$ is:

(A) 56 (B) -56 (C) 14 (D) -14 (E) 0

PART II (4 credits each)

21. The diagonal of square I is $a + b$. The perimeter of square II with <u>twice</u> the area of I is:

(A) $(a + b)^2$ (B) $\sqrt{2}(a + b)^2$ (C) $2(a + b)$ (D) $\sqrt{8}(a + b)$ (E) $4(a + b)$

22. The equality $(x + m)^2 - (x + n)^2 = (m - n)^2$, where <u>m and n are unequal</u> non-zero constants, is satisfied by $x = am + bn$ where:

(A) $a = 0$, $\underline{b}$ has a unique non-zero value (B) $a = 0$, $\underline{b}$ has two non-zero values
(C) $b = 0$, $\underline{a}$ has a unique non-zero value (D) $b = 0$, $\underline{a}$ has two non-zero values
(E) $\underline{a}$ and $\underline{b}$ each have a unique non-zero value.

23. The radius $\underline{R}$ of a cylindrical box is 8 inches, the height H, is 3 inches. The volume $V = \pi R^2 H$ is to be increased by the same fixed positive amount when $\underline{R}$ is increased by $\underline{x}$ inches as when $\underline{H}$ is increased by $\underline{x}$ inches. This condition is satisfied by:

(A) no real value of x (B) one integral value of x (C) one rational, but not integral, value of x (D) one irrational value of x (E) two real values of x

24. If $\log_{2x} 216 = x$, where x is real, then $\underline{x}$ is:

(A) A non-square, non-cube integer (B) a non-square, non-cube, non-integral rational number (C) an irrational number (D) a perfect square (E) a perfect cube

25. Let $\underline{m}$ and $\underline{n}$ be any two odd numbers, with $\underline{n}$ less than $\underline{m}$. The largest integer which divides all possible numbers of the form $m^2 - n^2$ is:

(A) 2 (B) 4 (C) 6 (D) 8 (E) 16

26. Find the set of x-values satisfying the inequality $\left|\frac{5-x}{3}\right| < 2$. [The symbol $|a|$ means +a if a is positive, −a if a is negative, 0 if a is zero. The notation $1 < a < 2$ means that $\underline{a}$ can have any value between 1 and 2, excluding 1 and 2.]

(A) $1 < x < 11$ (B) $-1 < x < 11$ (C) $x < 11$ (D) $x > 11$ (E) $|x| < 6$

27. Let S be the sum of the interior angles of a polygon P for which each interior angle is $7\frac{1}{2}$ times the exterior angle at the same vertex. Then

(A) $S = 2660°$ and P may be regular (B) $S = 2660°$ and P is not regular
(C) $S = 2700°$ and P is regular (D) $S = 2700°$ and P is not regular
(E) $S = 2700°$ and P may or may not be regular

28. The equation $x - \frac{7}{x-3} = 3 - \frac{7}{x-3}$ has:

(A) infinitely many integral roots (B) no root (C) one integral root
(D) two equal integral roots (E) two equal non-integral roots

29. Five times A's money added to B's money is more than \$51.00. Three times A's money minus B's money is \$21.00. If $\underline{a}$ represents A's money in dollars and $\underline{b}$ represents B's money in dollars, then:

(A) $a > 9$, $b > 6$ (B) $a > 9$, $b < 6$ (C) $a > 9$, $b = 6$
(D) $a > 9$, but we can put no bounds on $\underline{b}$ (E) $2a = 3b$

30. Given the line $3x + 5y = 15$ and a point on this line equidistant from the coordinate axes. Such a point exists in:

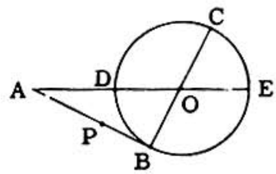
(A) none of the quadrants (B) quadrant I only (C) quadrants I, II only
(D) quadrants I, II, III only (E) each of the quadrants

PART III (5 credits each)

31. For $x^2 + 2x + 5$ to be a factor of $x^4 + px^2 + q$, the values of p and q must be, respectively:

(A) $-2, 5$ (B) $5, 25$ (C) $10, 20$ (D) $6, 25$ (E) $14, 25$

32. In this figure the center of the circle is O. $AB \perp BC$, ADOE is a straight line, $AP = AD$, and AB has a length twice the radius. Then:



(A) $\overline{AP}^2 = PB \cdot AB$ (B) $AP \cdot DO = PB \cdot AD$
(C) $\overline{AB}^2 = AD \cdot DE$ (D) $AB \cdot AD = OB \cdot AO$
(E) none of these

33. You are given a sequence of 58 terms, each of which is of the form $2 \cdot 3 \cdot 5 \cdots 61 + n$, where the first term is the product of all prime numbers (a prime number is one divisible only by 1 and itself) beginning with 2 and ending with 61, and the second term n is one of the natural numbers, taken in order, beginning with 2 and ending with 59, inclusive. Let N be the number of primes appearing in this sequence. Then N is:

(A) zero (B) sixteen (C) seventeen (D) fifty-seven (E) fifty-eight

34. Two swimmers, at opposite ends of a 90-foot pool, start to swim the length of the pool, one at the rate of 3 feet per second, the other at 2 feet per second. They swim back and forth for 12 minutes. Allowing no loss of time at the turns, find the number of times they pass each other.

(A) 24 (B) 21 (C) 20 (D) 19 (E) 18

35. From point P outside a circle, with a circumference of 10 units, a tangent is drawn. Also from P a secant is drawn dividing the circle into unequal arcs with lengths m and n. It is found that t, the length of the tangent, is the mean proportional between m and n. If m and t are integers, then t may have the following number of values:

(A) zero (B) one (C) two (D) three (E) infinitely many

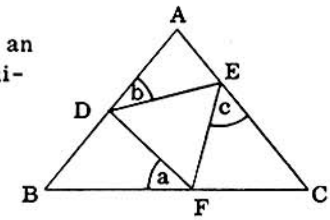
36. Let s_1, s_2, s_3 be the respective sums of $\underline{n}$, 2n, 3n terms of the same arithmetic progression with $\underline{a}$ as the first term and $\underline{d}$ as the common difference. Let $R = s_3 - s_2 - s_1$. Then $\underline{R}$ is dependent on:

(A) $\underline{a}$ and $\underline{d}$ (B) $\underline{d}$ and $\underline{n}$ (C) $\underline{a}$ and $\underline{n}$ (D) $\underline{a}$, $\underline{d}$, and $\underline{n}$
(E) neither $\underline{a}$ nor $\underline{d}$ nor $\underline{n}$

37. The base of a triangle is of length $\underline{b}$, and the altitude is of length $\underline{h}$. A rectangle of height $\underline{x}$ is inscribed in the triangle with the base of the rectangle in the base of the triangle. The area of the rectangle is:

(A) $\frac{bx}{h}(h-x)$ (B) $\frac{hx}{b}(b-x)$ (C) $\frac{bx}{h}(h-2x)$ (D) $x(b-x)$ (E) $x(h-x)$

38. In this diagram AB and AC are the equal sides of an isosceles triangle ABC, in which is inscribed equilateral triangle DEF. Designate angle BFD by $\underline{a}$, angle ADE by $\underline{b}$, and angle FEC by $\underline{c}$. Then



(A) $b = \frac{a+c}{2}$ (B) $b = \frac{a-c}{2}$ (C) $a = \frac{b-c}{2}$ (D) $a = \frac{b+c}{2}$ (E) none of these

39. To satisfy the equation $\frac{a+b}{a} = \frac{b}{a+b}$, $\underline{a}$ and $\underline{b}$ must be:

(A) both rational (B) both real but not rational (C) both not real
(D) one real, one not real (E) one real, one not real or both not real

40. Given right triangle ABC with legs 3 and 4. Find the length of the <u>angle trisector</u> to the hypotenuse nearer the shorter leg.

(A) $\frac{32\sqrt{3} - 24}{13}$ (B) $\frac{12\sqrt{3} - 9}{13}$ (C) $6\sqrt{3} - 8$ (D) $\frac{5\sqrt{10}}{6}$ (E) $\frac{25}{12}$

COMMITTEE ON THE NATIONAL H. S. CONTEST

National Office:
Pan American College
Edinburg, Texas

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THE SOCIETY OF ACTUARIES

TWELFTH ANNUAL MATHEMATICS CONTEST 1961

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers, marked A, B, C, D, E. For each question decide upon the correct answer; then write the capital letter that precedes the correct answer in the box on the margin of your answer sheet directly under the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly under No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. Avoid guessing since you are penalized for incorrectly-guessed answers.
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. When your teacher gives the signal tear off the cover of this booklet along the perforated line and turn the cover over. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you fill in your name and the name of your school on it.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

THURSDAY MORNING, MARCH 9, 1961

To be filled in by the student

PRINT ____

first name   middle name   last name

______________ ______________________________

school   number   street

______________ ______________________________

city   zone   county   state

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

PART I

21	22	23	24	25	26	27	28	29	30

PART II

31	32	33	34	35	36	37	38	39	40

PART III

Not to be filled in by the student

	Value	Points Correct, C	Points Wrong, W (do not include omissions)
Part I	3 points each		
Part II	4 points each		
Part III	5 points each		
	TOTALS		
	SCORE =	$C - \frac{1}{4}W$ =	(Write Score Here)

PART I (3 credits each)

1. When simplified $(-\frac{1}{125})^{-2/3}$ becomes:

(A) $\frac{1}{25}$ (B) $-\frac{1}{25}$ (C) 25 (D) -25 (E) $25\sqrt{-1}$

2. An automobile travels $\frac{a}{6}$ *feet* in r *seconds*. If this rate is maintained for 3 minutes, how many *yards* does it travel in the 3 *minutes*?

(A) $\frac{a}{1080r}$ (B) $\frac{30r}{a}$ (C) $\frac{30a}{r}$ (D) $\frac{10r}{a}$ (E) $\frac{10a}{r}$

3. If the graphs of $2y + x + 3 = 0$ and $3y + ax + 2 = 0$ are to meet at right angles, the value of a is:

(A) $\pm\frac{2}{3}$ (B) $-\frac{2}{3}$ (C) $-\frac{3}{2}$ (D) 6 (E) -6

4. Let the set consisting of the squares of the positive integers be called u; thus u is the set 1, 4, 9, ... If a certain operation on one or more members of the set always yields a member of the set, we say that the set is closed under that operation. Then u is closed under:

(A) addition (B) multiplication (C) division
(D) extraction of a positive integral root (E) none of these

5. Let $S = (x-1)^4 + 4(x-1)^3 + 6(x-1)^2 + 4(x-1) + 1$. Then S equals:

(A) $(x-2)^4$ (B) $(x-1)^4$ (C) x^4 (D) $(x+1)^4$ (E) $x^4 + 1$

6. When simplified $\log 8 \div \log \frac{1}{8}$ becomes:

(A) $6 \log 2$ (B) $\log 2$ (C) 1 (D) 0 (E) -1

7. The third term in the expansion of $\left(\frac{a}{\sqrt{x}} - \frac{\sqrt{x}}{a^2}\right)^6$ is, when simplified:

(A) $\frac{15}{x}$ (B) $-\frac{15}{x}$ (C) $-\frac{6x^2}{a^9}$ (D) $\frac{20}{a^3}$ (E) $-\frac{20}{a^3}$

8. Let the two base angles of a triangle be A and B, with B larger than A. The altitude to the base divides the vertex angle C into two parts, C_1 and C_2, with C_2 adjacent to side a. Then:

(A) $C_1 + C_2 = A + B$ (B) $C_1 - C_2 = B - A$ (C) $C_1 - C_2 = A - B$
(D) $C_1 + C_2 = B - A$ (E) $C_1 - C_2 = A + B$

9. Let r be the result of doubling both the base and the exponent of a^b, $b \neq 0$. If r equals the product of a^b by x^b, then x equals:

(A) a (B) 2a (C) 4a (D) 2 (E) 4

10. Each side of triangle ABC is 12 units. D is the foot of the perpendicular dropped from A on BC, and E is the midpoint of AD. The length of BE, in the same unit, is:

(A) $\sqrt{18}$ (B) $\sqrt{28}$ (C) 6 (D) $\sqrt{63}$ (E) $\sqrt{98}$

11. Tangents AB and AC are drawn to a circle from exterior point A. Tangent PQR intersects AB in P and AC in R, and touches the circle at Q. If AB = 20, then the perimeter of triangle APR is:

(A) 42 (B) 40.5 (C) 40 (D) $39\frac{7}{8}$ (E) not determined by the given information.

12. The first three terms of a geometric progression are $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt[6]{2}$. The fourth term is:

(A) 1 (B) $\sqrt[7]{2}$ (C) $\sqrt[8]{2}$ (D) $\sqrt[9]{2}$ (E) $\sqrt[10]{2}$

13. The symbol $|a|$ means a if a is a positive number or zero, and $-a$ if a is a negative number. For all real values of t the expression $\sqrt{t^4 + t^2}$ is equal to:

(A) t^3 (B) $t^2 + t$ (C) $|t^2 + t|$ (D) $t\sqrt{t^2 + 1}$ (E) $|t|\sqrt{1 + t^2}$

14. A rhombus is given with one diagonal twice the length of the other diagonal. Express the side of the rhombus in terms of K, where K is the area of the rhombus in square inches.

(A) $\sqrt{K}$ (B) $\frac{1}{2}\sqrt{2K}$ (C) $\frac{1}{3}\sqrt{3K}$ (D) $\frac{1}{4}\sqrt{4K}$ (E) none of these is correct

15. If x men working x hours a day for each of x days produce x articles, then the number of articles (not necessarily an integer) produced by y men working y hours a day for each of y days is:

(A) $\frac{x^3}{y^2}$ (B) $\frac{y^3}{x^2}$ (C) $\frac{x^2}{y^3}$ (D) $\frac{y^2}{x^3}$ (E) y

16. An altitude h of a triangle is increased by a length m. How much must be taken from the corresponding base b so that the area of the new triangle is one-half that of the original triangle?

(A) $\frac{bm}{h + m}$ (B) $\frac{bh}{2(h + m)}$ (C) $\frac{b(2m + h)}{m + h}$ (D) $\frac{b(m + h)}{2m + h}$ (E) $\frac{b(2m + h)}{2(h + m)}$

17. In the base ten number system the number 526 means $5 \cdot 10^2 + 2 \cdot 10 + 6$. In the Land of Mathesis, however, numbers are written in the base r. Jones purchases an automobile there for 440 monetary units (abbreviated m.u.). He gives the salesman a 1000 m.u. bill, and receives, in change, 340 m.u. The base r is:

(A) 2 (B) 5 (C) 7 (D) 8 (E) 12

18. The yearly changes in the population census of a town for four consecutive years are, respectively, 25% increase, 25% increase, 25% decrease, 25% decrease. The net change over the four years, to the nearest percent, is:

(A) -12 (B) -1 (C) 0 (D) 1 (E) 12

19. Consider the graphs of $y = 2 \log x$ and $y = \log 2x$. We may say that.

(A) They do not intersect (B) They intersect in one point only (C) They intersect in two points only (D) They intersect in a finite number of points but more than two (E) They coincide

20. The set of points satisfying the pair of inequalities $y > 2x$ and $y > 4 - x$ is contained entirely in quadrants:

(A) I and II (B) II and III (C) I and III (D) III and IV (E) I and IV

PART II (4 credits each)

21. Medians AD and CE of triangle ABC intersect in M. The midpoint of AE is N. Let the area of triangle MNE = k times the area of triangle ABC. Then k equals:

(A) $\frac{1}{6}$ (B) $\frac{1}{8}$ (C) $\frac{1}{9}$ (D) $\frac{1}{12}$ (E) $\frac{1}{16}$

22. If $3x^3 - 9x^2 + kx - 12$ is divisible by $x - 3$, then it is also divisible by:

(A) $3x^2 - x + 4$ (B) $3x^2 - 4$ (C) $3x^2 + 4$ (D) $3x - 4$ (E) $3x + 4$

23. Points P and Q are both in the line AB and on the same side of the midpoint of line AB. P divides AB in the ratio 2 : 3 and Q divides AB in the ratio 3 : 4. If PQ = 2, then the length of AB is:

(A) 60 (B) 70 (C) 75 (D) 80 (E) 85

24. Thirty-one books are arranged from left to right in order of increasing prices. The price of each book differs by $2 from that of each adjacent book. For the price of the book at the extreme right a customer can buy the middle book and an adjacent one. Then:

(A) The adjacent book referred to is at the left of the middle book
(B) The middle book sells for $36 (C) The cheapest book sells for $4
(D) The most expensive book sells for $64 (E) None of these is correct

25. Triangle ABC is isosceles with base AC. Points P and Q are respectively in CB and AB and such that AC = AP = PQ = QB. The number of degrees in angle B is:

(A) $25\frac{5}{7}$ (B) $26\frac{1}{3}$ (C) 30 (D) 40 (E) not determined by the information given.

26. For a given arithmetic series the sum of the first 50 terms is 200, and the sum of the next 50 terms is 2700. The first term of the series is:

(A) -1221 (B) -21.5 (C) -20.5 (D) 3 (E) 3.5

27. Given two equiangular polygons P_1 and P_2 with different numbers of sides; each angle of P_1 is x degrees and each angle of P_2 is kx degrees, where k is an integer greater than 1. The number of possibilities for the pair (x, k) is:

(A) infinite (B) finite, but more than two (C) two (D) one (E) zero

28. If 2137^{753} is multiplied out, the units' digit in the final product is:

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

29. Let the roots of $ax^2 + bx + c = 0$ be r and s. The equation with roots $ar + b$ and $as + b$ is:

(A) $x^2 - bx - ac = 0$ (B) $x^2 - bx + ac = 0$ (C) $x^2 + 3bx + ca + 2b^2 = 0$
(D) $x^2 + 3bx - ca + 2b^2 = 0$ (E) $x^2 + bx(2 - a) + a^2c + b^2(a + 1) = 0$

30. If $\log_{10} 2 = a$ and $\log_{10} 3 = b$, then $\log_5 12$ equals:

(A) $\frac{a + b}{1 + a}$ (B) $\frac{2a + b}{1 + a}$ (C) $\frac{a + 2b}{1 + a}$ (D) $\frac{2a + b}{1 - a}$ (E) $\frac{a + 2b}{1 - a}$

PART III (5 credits each)

31. In triangle ABC the ratio AC : CB = 3 : 4. The bisector of the exterior angle at C intersects BA extended at P (A is between P and B). The ratio PA : AB is:

(A) 1 : 3 (B) 3 : 4 (C) 4 : 3 (D) 3 : 1 (E) 7 : 1

32. A regular polygon of n sides is inscribed in a circle of radius R. The area of the polygon is $3R^2$. Then n equals:

(A) 8 (B) 10 (C) 12 (D) 15 (E) 18

33. The number of solutions of $2^{2x} - 3^{2y} = 55$, in which x and y are integers, is:

(A) zero (B) one (C) two (D) three (E) more than three, but finite

34. Let S be the set of values assumed by the function $\frac{2x + 3}{x + 2}$ when x is any member of the interval $x \geqq 0$. If there exists a number M such that no number of the set S is greater than M, then M is an upper bound of S. If there exists a number m such that no number of the set S is less than m, then m is a lower bound of S. We may then say:

(A) m is in S, M is not in S (B) M is in S, m is not in S (C) both m and M are in S (D) neither m nor M is in S (E) M does not exist either in or outside S.

35. The number 695 is to be written with a factorial base of numeration, that is, $695 = a_1 + a_2 \cdot 2! + a_3 \cdot 3! + \cdots + a_n \cdot n!$ where $a_1, a_2, \cdots, a_n$ are integers such that $0 \leqq a_k \leqq k$, and $n!$ means $n(n-1)(n-2) \cdots 2 \cdot 1$. Find a_4.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

36. In triangle ABC the median from A is given perpendicular to the median from B. If BC = 7 and AC = 6, then the length of AB is:

(A) 4 (B) $\sqrt{17}$ (C) 4.25 (D) $2\sqrt{5}$ (E) 4.5

37. In racing over a given distance d, at uniform speed, A can beat B by 20 yards, B can beat C by 10 yards, and A can beat C by 28 yards. Then d, in yards, equals:

(A) not determined by the given information (B) 58 (C) 100 (D) 116 (E) 120

38. Triangle ABC is inscribed in a semicircle of radius r. Base AB coincides with diameter AB. Point C does not coincide with either A or B. Let $s = AC + BC$. Then, for all permissible positions of C:

(A) $s^2 \leqq 8r^2$ (B) $s^2 = 8r^2$ (C) $s^2 \geqq 8r^2$ (D) $s^2 \leqq 4r^2$ (E) $s^2 = 4r^2$

39. Any five points are taken inside or on a square of side 1. Let a be the *smallest* possible number with the property that it is always possible to select one pair of points from these five such that the distance between them is equal to or less than a. Then a is:

(A) $\sqrt{3}/3$ (B) $\sqrt{2}/2$ (C) $2\sqrt{2}/3$ (D) 1 (E) $\sqrt{2}$

40. Find the minimum value of $\sqrt{x^2 + y^2}$ if $5x + 12y = 60$.

(A) $\frac{60}{13}$ (B) $\frac{13}{5}$ (C) $\frac{13}{12}$ (D) 1 (E) 0

COMMITTEE ON THE NATIONAL H. S. CONTEST

National Office: Pan American College, Edinburg, Texas
New York Office: Polytechnic Institute of Brooklyn, Brooklyn 1, N. Y.

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THE SOCIETY OF ACTUARIES

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THIRTEENTH ANNUAL MATHEMATICS CONTEST 1962

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers, marked A, B, C, D, E. For each question decide upon the correct answer; then write the capital letter that precedes the correct answer in the box on the margin of your answer sheet directly under the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly under No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid guessing since you are penalized for incorrectly-guessed answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. When your teacher gives the signal tear off the cover of this booklet along the perforated line and turn the cover over. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you fill in your name and the name of your school on it.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

THURSDAY MORNING, MARCH 8, 1962

To be filled in by the student

PRINT ____

first name   middle name   last name

____

school   number   street

____

city   zone   county   state

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

PART I

21	22	23	24	25	26	27	28	29	30

PART II

31	32	33	34	35	36	37	38	39	40

PART III

Not to be filled in by the student

	Value	Points Correct, C	Points Wrong, W (do not include omissions)
Part I	3 points each	3 times =	3 times =
Part II	4 points each	4 times =	4 times =
Part III	5 points each	5 times =	5 times =
	TOTALS		
	SCORE =	$C - \frac{1}{4}W$ =	Write Score Here (2 dec. places)

PART I (3 credits each)

1. The expression $\dfrac{1^{4y-1}}{5^{-1}+3^{-1}}$ is equal to:

(A) $\dfrac{4y-1}{8}$ (B) 8 (C) $\frac{15}{2}$ (D) $\frac{15}{8}$ (E) $\frac{1}{8}$

2. The expression $\sqrt{\dfrac{4}{3}}-\sqrt{\dfrac{3}{4}}$ is equal to:

(A) $\sqrt{3}/6$ (B) $-\sqrt{3}/6$ (C) $\sqrt{-3}/6$ (D) $5\sqrt{3}/6$ (E) 1

3. The first three terms of an arithmetic progression are $x - 1$, $x + 1$, $2x + 3$, in the order shown. The value of x is:

(A) -2 (B) 0 (C) 2 (D) 4 (E) undetermined

4. If $8^x = 32$, then x equals:

(A) 4 (B) $\frac{5}{3}$ (C) $\frac{3}{2}$ (D) $\frac{3}{5}$ (E) $\frac{1}{4}$

5. If the radius of a circle is increased by 1 unit, the ratio of the new circumference to the new diameter is:

(A) $\pi + 2$ (B) $\dfrac{2\pi + 1}{2}$ (C) π (D) $\dfrac{2\pi - 1}{2}$ (E) $\pi - 2$

6. A square and an equilateral triangle have equal perimeters. The area of the triangle is $9\sqrt{3}$ square inches. Expressed in inches the diagonal of the square is:

(A) 9/2 (B) $2\sqrt{5}$ (C) $4\sqrt{2}$ (D) $9\sqrt{2}/2$ (E) none of these

7. Let the bisectors of the exterior angles at B and C of triangle ABC meet at D. Then, if all measurements are in degrees, angle BDC equals:

(A) $\frac{1}{2}(90 - A)$ (B) $90 - A$ (C) $\frac{1}{2}(180 - A)$ (D) $180 - A$ (E) $180 - 2A$

8. Given the set of $\underline{n}$ numbers, $n > 1$, of which one is $1 - \dfrac{1}{n}$ and all the others are 1. The arithmetic mean of the $\underline{n}$ numbers is:

(A) 1 (B) $n - \dfrac{1}{n}$ (C) $n - \dfrac{1}{n^2}$ (D) $1 - \dfrac{1}{n^2}$ (E) $1 - \dfrac{1}{n} - \dfrac{1}{n^2}$

9. When $x^9 - x$ is factored as completely as possible into polynomials and monomials with integral coefficients, the number of factors is:

(A) mcre than 5 (B) 5 (C) 4 (D) 3 (E) 2

10. A man drives 150 miles to the seashore in 3 hours and 20 minutes. He returns from the shore to the starting point in 4 hours and 10 minutes. Let $\underline{r}$ be the average rate for the entire trip. Then the average rate for the trip going exceeds $\underline{r}$, in miles per hour, by:

(A) 5 (B) $4\frac{1}{2}$ (C) 4 (D) 2 (E) 1

11. The difference between the larger root and the smaller root of $x^2 - px + (p^2 - 1)/4 = 0$ is:

(A) 0 (B) 1 (C) 2 (D) p (E) p + 1

12. When $\left(1 - \frac{1}{a}\right)^6$ is expanded the sum of the last three coefficients is:

(A) 22 (B) 11 (C) 10 (D) − 10 (E) − 11

13. R varies directly as $\underline{S}$ and inversely as $\underline{T}$. When $R = \frac{4}{3}$ and $T = \frac{9}{14}$, $S = \frac{3}{7}$. Find $\underline{S}$ when $R = \sqrt{48}$ and $T = \sqrt{75}$.

(A) 28 (B) 30 (C) 40 (D) 42 (E) 60

14. Let $\underline{s}$ be the limiting sum of the geometric series $4 - \frac{8}{3} + \frac{16}{9} - \ldots$, as the number of terms increases without bound. Then $\underline{s}$ equals:

(A) a number between 0 and 1 (B) 2.4 (C) 2.5 (D) 3.6 (E) 12

15. Given triangle ABC with base AB fixed in length and position. As the vertex C moves on a straight line, the intersection point of the three medians moves on:

(A) a circle (B) a parabola (C) an ellipse (D) a straight line
(E) a curve here not listed

16. Given rectangle R_1 with one side 2 inches and area 12 square inches. Rectangle R_2 with diagonal 15 inches is similar to R_1. Expressed in square inches the area of R_2 is:

(A) 9/2 (B) 36 (C) 135/2 (D) $9\sqrt{10}$ (E) $27\sqrt{10}/4$

17. If $a = \log_8 225$ and $b = \log_2 15$, then $\underline{a}$, in terms of $\underline{b}$, is:

(A) b/2 (B) 2b/3 (C) b (D) 3b/2 (E) 2b

18. A regular dodecagon (12 sides) is inscribed in a circle with radius $\underline{r}$ inches. The area of the dodecagon, in square inches, is:

(A) $3r^2$ (B) $2r^2$ (C) $3r^2\sqrt{3}/4$ (D) $r^2\sqrt{3}$ (E) $3r^2\sqrt{3}$

19. If the parabola $y = ax^2 + bx + c$ passes through the points $(-1, 12)$, $(0,5)$, and $(2, -3)$, the value of $a + b + c$ is:

(A) -4 (B) -2 (C) 0 (D) 1 (E) 2

20. The angles of a pentagon are in arithmetic progression. One of the angles, in degrees, must be:

(A) 108 (B) 90 (C) 72 (D) 54 (E) 36

PART II (4 credits each)

21. It is given that one root of $2x^2 + rx + s = 0$, with $\underline{r}$ and $\underline{s}$ real numbers, is $3 + 2i$ $(i = \sqrt{-1})$. The value of $\underline{s}$ is:

(A) undetermined (B) 5 (C) 6 (D) -13 (E) 26

22. The number 121_b, written in the integral base $\underline{b}$, is the square of an integer, for

(A) $b = 10$, only (B) $b = 10$ and $b = 5$, only (C) $2 \leq b \leq 10$ (D) $b > 2$
(E) no value of b

23. In triangle ABC, CD is the altitude to AB and AE is the altitude to BC. If the lengths of AB, CD, and AE are known, the length of DB is:

(A) not determined by the information given
(B) determined only if A is an acute angle
(C) determined only if B is an acute angle
(D) determined only if ABC is an acute triangle.
(E) none of these is correct

24. Three machines P, Q, and R, working together, can do a job in $\underline{x}$ hours. When working alone P needs an additional 6 hours to do the job; Q, one additional hour; and $\underline{R}$, x additional hours. The value of $\underline{x}$ is:

(A) $\frac{2}{3}$ (B) $\frac{11}{12}$ (C) $\frac{3}{2}$ (D) 2 (E) 3

25. Given square ABCD with side 8 feet. A circle is drawn through vertices A and D and tangent to side BC. The radius of the circle, in feet, is:

(A) 4 (B) $4\sqrt{2}$ (C) 5 (D) $5\sqrt{2}$ (E) 6

26. For any real value of x the maximum value of $8x - 3x^2$ is:

(A) 0 (B) $\frac{8}{3}$ (C) 4 (D) 5 (E) $\frac{16}{3}$

27. Let a Ⓛ b represent the operation on two numbers, $\underline{a}$ and $\underline{b}$, which selects the larger of the two numbers, with a Ⓛ a = a. Let a Ⓢ b represent the operation which selects the smaller of the two numbers, with a Ⓢ a = a.
Which of the following three rules is (are) correct?
(1) a Ⓛ b = b Ⓛ a (2) a Ⓛ (b Ⓛ c) = (a Ⓛ b) Ⓛ c (3) a Ⓢ (b Ⓛ c) = (a Ⓢ b) Ⓛ (a Ⓢ c)

(A) (1) only (B) (2) only (C) (1) and (2) only (D) (1) and (3) only
(E) all three

28. The set of x-values satisfying the equation $x^{\log_{10} x} = \frac{x^3}{100}$ consists of:

(A) $\frac{1}{10}$, only (B) 10, only (C) 100, only (D) 10 or 100, only
(E) more than two real numbers.

29. Which of the following sets of x-values satisfy the inequality $2x^2 + x < 6$?

(A) $-2 < x < \frac{3}{2}$ (B) $x > \frac{3}{2}$ or $x < -2$ (C) $x < \frac{3}{2}$ (D) $\frac{3}{2} < x < 2$
(E) $x < -2$

30. Form I

Consider the statements: (1) $p \wedge q$ (2) $p \wedge \sim q$ (3) $\sim p \wedge q$ (4) $\sim p \wedge \sim q$, where p and q are statements each of which may be true or false.
How many of these imply the truth of $\sim (p \wedge q)$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Form II

Consider the statements: (1) p and q are both true (2) p is true and q is false (3) p is false and q is true (4) p is false and q is false.
How many of these imply the negation of the statement "p and q are both true"?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

PART III (5 credits each)

31. The ratio of the interior angles of two regular polygons with sides of unit length is 3 : 2. How many such pairs are there?

(A) 1 (B) 2 (C) 3 (D) 4 (E) infinitely many

32. If $x_{k+1} = x_k + \frac{1}{2}$ for $k = 1, 2, \ldots, n-1$ and $x_1 = 1$, find $x_1 + x_2 + \ldots + x_n$.

(A) $\frac{n+1}{2}$ (B) $\frac{n+3}{2}$ (C) $\frac{n^2-1}{2}$ (D) $\frac{n^2+n}{4}$ (E) $\frac{n^2+3n}{4}$

33. The set of x-values satisfying the inequality $2 \leq |x - 1| \leq 5$ is:

(A) $-4 \leq x \leq -1$ or $3 \leq x \leq 6$ (B) $3 \leq x \leq 6$ or $-6 \leq x \leq -3$ (C) $x \leq -1$ or $x \geq 3$ (D) $-1 \leq x \leq 3$

(E) $-4 \leq x \leq 6$

34. For what real values of K does $x = K^2(x - 1)(x - 2)$ have real roots?

(A) none (B) $-2 < K < 1$ (C) $-2\sqrt{2} < K < 2\sqrt{2}$ (D) $K > 1$ or $K < -2$
(E) all

35. A man on his way to dinner shortly after 6:00 p.m. observes that the hands of his watch form an angle of 110°. Returning before 7:00 p.m. he notices that again the hands of his watch form an angle of 110°. The number of minutes that he has been away is:

(A) $36\frac{2}{3}$ (B) 40 (C) 42 (D) 42.4 (E) 45

36. If both x and y are integers, how many pairs of solutions are there of the equation $(x - 8)(x - 10) = 2^y$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) more than 3

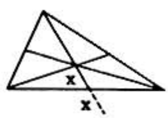
37. ABCD is a square with side of unit length. Points E and F are taken respectively on sides AB and AD so that AE = AF and the quadrilateral CDFE has maximum area. In square units this maximum area is:

(A) $\frac{1}{2}$ (B) $\frac{9}{16}$ (C) $\frac{19}{32}$ (D) $\frac{5}{8}$ (E) $\frac{2}{3}$

38. The population of Nosuch Junction at one time was a perfect square. Later, with an increase of 100, the population was one more than a perfect square. Now, with an additional increase of 100, the population is again a perfect square.
The original population is a multiple of:

(A) 3 (B) 7 (C) 9 (D) 11 (E) 17

39. Two medians of a triangle with unequal sides are 3 inches and 6 inches. Its area is $3\sqrt{15}$ square inches. The length of the third median, in inches, is:



This diagram contains a hint for the solution.

(A) 4 (B) $3\sqrt{3}$ (C) $3\sqrt{6}$ (D) $6\sqrt{3}$ (E) $6\sqrt{6}$

40. The limiting sum of the infinite series, $\frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \ldots$ whose n th term is $\frac{n}{10^n}$ is:

(A) $\frac{1}{9}$ (B) $\frac{10}{81}$ (C) $\frac{1}{8}$ (D) $\frac{17}{72}$ (E) larger than any finite quantity

COMMITTEE ON HIGH SCHOOL CONTESTS

National Office: Pan American College, Edinburg, Texas
New York Office: Polytechnic Institute of Brooklyn, Brooklyn 1, N.Y.

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THE SOCIETY OF ACTUARIES

M A
S A^2

FOURTEENTH ANNUAL MATHEMATICS CONTEST 1963

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers, marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box on the margin of your answer sheet directly under the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly under No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off the cover of this booklet along the dotted line and turn the cover over. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you fill in your name and the name of your school on it.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

*This instruction may be modified if machine-scoring is used.

THURSDAY MORNING, MARCH 14, 1963

To be filled in by the student

PRINT ____

first name middle name last name

____

school number street

____

city zone county state

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

PART I

21	22	23	24	25	26	27	28	29	30

PART II

31	32	33	34	35	36	37	38	39	40

PART III

Not to be filled in by the student

	Value	Points Correct, C	Points Wrong, W (do not include omissions)
Part I	3 points each	3 times =	3 times =
Part II	4 points each	4 times =	4 times =
Part III	5 points each	5 times =	5 times =
	TOTALS		
	SCORE =	$C - \frac{1}{4}W$ =	Write Score Here (2 dec. places)

PART I (3 credits each)

1. Which one of the following points is not on the graph of $y = \frac{x}{x+1}$?

(A) (0,0) (B) $\left(-\frac{1}{2}, -1\right)$ (C) $\left(\frac{1}{2}, \frac{1}{3}\right)$ (D) (−1,1) (E) (−2,2)

2. Let $n = x - y^{x-y}$. Find n when x = 2 and y = −2.

(A) −14 (B) 0 (C) 1 (D) 18 (E) 256

3. If the reciprocal of x + 1 is x − 1, then x equals:

(A) 0 (B) 1 (C) −1 (D) + 1 or −1 (E) none of these

4. For what value(s) of k does the pair of equations $y = x^2$ and $y = 3x + k$ have two identical solutions?

(A) $\frac{4}{9}$ (B) $-\frac{4}{9}$ (C) $\frac{9}{4}$ (D) $-\frac{9}{4}$ (E) $\frac{9}{4}$ or $-\frac{9}{4}$

5. If x and $\log_{10} x$ are real numbers and $\log_{10} x < 0$, then:

(A) $x < 0$ (B) $-1 < x < 1$ (C) $0 < x \leq 1$ (D) $-1 < x < 0$ (E) $0 < x < 1$

6. Triangle ABD is right-angled at B. On AD there is a point C for which AC = CD and AB = BC. The magnitude of angle DAB, in degrees, is:

(A) $67\frac{1}{2}$ (B) 60 (C) 45 (D) 30 (E) $22\frac{1}{2}$

7. Given the four equations: (1) $3y - 2x = 12$, (2) $-2x - 3y = 10$, (3) $3y + 2x = 12$, (4) $2y + 3x = 10$. The pair representing perpendicular lines is:

(A) (1) and (4) (B) (1) and (3) (C) (1) and (2) (D) (2) and (4) (E) (2) and (3)

8. The smallest positive integer x for which $1260x = N^3$, where N is an integer, is:

(A) 1050 (B) 1260 (C) 1260^2 (D) 7350 (E) 44,100

9. In the expansion of $\left(a - \frac{1}{\sqrt{a}}\right)^7$ the coefficient of $a^{-1/2}$ is:

(A) −7 (B) 7 (C) −21 (D) 21 (E) 35

10. Point P is taken interior to a square of side-length $\underline{a}$ and such that it is equally distant from two consecutive vertices and from the side opposite these vertices. If d represents the common distance, then d equals:

(A) $\frac{3a}{5}$ (B) $\frac{5a}{8}$ (C) $\frac{3a}{8}$ (D) $\frac{a\sqrt{2}}{2}$ (E) $\frac{a}{2}$

11. The arithmetic mean of a set of 50 numbers is 38. If two numbers of the set, namely 45 and 55, are discarded, the arithmetic mean of the remaining set of numbers is:

(A) 38.5 (B) 37.5 (C) 37 (D) 36.5 (E) 36

12. Three vertices of parallelogram PQRS are $P(-3,-2)$, $Q(1,-5)$, $R(9,1)$ with P and R diagonally opposite. The sum of the coordinates of vertex S is:

(A) 13 (B) 12 (C) 11 (D) 10 (E) 9

13. If $2^a + 2^b = 3^c + 3^d$, the number of integers a, b, c, d which can possibly be negative, is, at most:

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

14. Given the equations $x^2 + kx + 6 = 0$ and $x^2 - kx + 6 = 0$. If, when the roots of the equations are suitably listed, each root of the second equation is 5 more than the corresponding root of the first equation, then k equals:

(A) 5 (B) -5 (C) 7 (D) -7 (E) none of these

15. A circle is inscribed in an equilateral triangle, and a square is inscribed in the circle. The ratio of the area of the triangle to the area of the square is:

(A) $\sqrt{3}:1$ (B) $\sqrt{3}:\sqrt{2}$ (C) $3\sqrt{3}:2$ (D) $3:\sqrt{2}$ (E) $3:2\sqrt{2}$

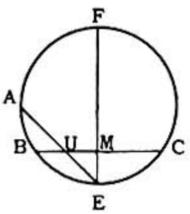
16. Three numbers a, b, c, none zero, form an arithmetic progression. Increasing a by 1 <u>or</u> increasing c by 2 results in a geometric progression. Then b equals:

(A) 16 (B) 14 (C) 12 (D) 10 (E) 8

17. The expression $\dfrac{\dfrac{a}{a+y}+\dfrac{y}{a-y}}{\dfrac{y}{a+y}-\dfrac{a}{a-y}}$, a real, $a \neq 0$, has the value -1 for:

(A) all but two real values of y
(B) only two real values of y
(C) all real values of y
(D) only one real value of y
(E) no real values of y

18. Chord EF is the perpendicular bisector of chord BC, intersecting it in M. Between B and M point U is taken, and EU extended meets the circle in A. Then, for any selection of U, as described, triangle EUM is similar to triangle:



(A) EFA (B) EFC (C) ABM (D) ABU (E) FMC

19. In counting n colored balls, some red and some black, it was found that 49 of the first 50 counted were red. Thereafter, 7 out of every 8 counted were red. If, in all, 90% or more of the balls counted were red, the maximum value of n is:

(A) 225 (B) 210 (C) 200 (D) 180 (E) 175

20. Two men at points R and S, 76 miles apart, set out at the same time to walk towards each other. The man at R walks uniformly at the rate of $4\frac{1}{2}$ miles per hour; the man at S walks at the constant rate of $3\frac{1}{4}$ miles per hour for the first hour, at $3\frac{3}{4}$ miles per hour for the second hour, and so on, in arithmetic progression. The men will meet x miles nearer R than S where x is:

(A) 10 (B) 8 (C) 6 (D) 4 (E) 2

PART II (4 credits each)

21. The expression $x^2 - y^2 - z^2 + 2yz + x + y - z$ has:

(A) no linear factor with integer coefficients and integer exponents
(B) the factor $-x + y + z$
(C) the factor $x - y - z + 1$
(D) the factor $x + y - z + 1$
(E) the factor $x - y + z + 1$

22. Acute-angled triangle ABC is inscribed in a circle with center at O, and $\widehat{AB} = 120^\circ$ and $\widehat{BC} = 72^\circ$. A point E is taken in minor arc AC such that OE is perpendicular to AC. Then the ratio of the magnitudes of angles OBE and BAC is:

(A) $\frac{5}{18}$ (B) $\frac{2}{9}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) $\frac{4}{9}$

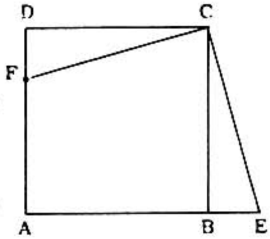
23. A gives B as many cents as B has and C as many cents as C has. Similarly, B then gives A and C as many cents as each then has. C, similarly, then gives A and B as many cents as each then has. If each finally has 16 cents, with how many cents does A start?

(A) 24 (B) 26 (C) 28 (D) 30 (E) 32

24. Consider equations of the form $x^2 + bx + c = 0$. How many such equations have real roots and have coefficients b and c selected from the set of integers $\{1, 2, 3, 4, 5, 6\}$?

(A) 20 (B) 19 (C) 18 (D) 17 (E) 16

25. Point F is taken in side AD of square ABCD. At C a perpendicular is drawn to CF, meeting AB extended at E. The area of ABCD is 256 square inches and the area of triangle CEF is 200 square inches. Then the number of inches in $\overline{BE}$ is:



(A) 12 (B) 14 (C) 15 (D) 16 (E) 20

26. Form I

Consider the statements (1) $p \wedge \sim q \wedge r$ (2) $\sim p \wedge \sim q \wedge r$ (3) $p \wedge \sim q \wedge \sim r$ (4) $\sim p \wedge q \wedge r$, where p, q, and r are propositions. How many of these imply the truth of $(p \rightarrow q) \rightarrow r$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Form II

Consider the statements (1) p and r are true and q is false (2) r is true and p and q are false (3) p is true and q and r are false (4) q and r are true and p is false. How many of these imply the truth of the statement "r is implied by the statement that p implies q"?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

27. Six straight lines are drawn in a plane with no two parallel and no three concurrent. The number of regions into which they divide the plane is:

(A) 16 (B) 20 (C) 22 (D) 24 (E) 26

28. Given the equation $3x^2 - 4x + k = 0$ with real roots. The value of k for which the product of the roots of the equation is a maximum, is:

(A) $\frac{16}{9}$ (B) $\frac{16}{3}$ (C) $\frac{4}{9}$ (D) $\frac{4}{3}$ (E) $-\frac{4}{3}$

29. A particle projected vertically upward reaches, at the end of t seconds, an elevation of s feet where $s = 160\,t - 16t^2$. The highest elevation is:

(A) 800 (B) 640 (C) 400 (D) 320 (E) 160

30. Let $F = \log \frac{1+x}{1-x}$. Replace each x in F by $\frac{3x + x^3}{1 + 3x^2}$, and simplify. The simplified expression is equal to:

(A) $-F$ (B) F (C) $3F$ (D) F^3 (E) $F^3 - F$

PART III (5 credits each)

31. The number of solutions in positive integers of $2x + 3y = 763$ is:

(A) 255 (B) 254 (C) 128 (D) 127 (E) 0

32. The dimensions of a rectangle R are a and b, $a < b$. It is required to obtain a rectangle with dimensions x and y, $x < a$, $y < a$, so that its perimeter is one-third that of R, and its area is one-third that of R. The number of such (different) rectangles is:

(A) 0 (B) 1 (C) 2 (D) 4 (E) infinitely many

33. Given the line $y = \frac{3}{4}x + 6$ and a line L parallel to the given line and 4 units from it. A possible equation for L is:

(A) $y = \frac{3}{4}x + 1$ (B) $y = \frac{3}{4}x$ (C) $y = \frac{3}{4}x - \frac{2}{3}$ (D) $y = \frac{3}{4}x - 1$ (E) $y = \frac{3}{4}x + 2$

34. In triangle ABC, side $a = \sqrt{3}$, side $b = \sqrt{3}$, and side $c > 3$. Let x be the largest number such that the magnitude, in degrees, of the angle opposite side c exceeds x. Then x equals:

(A) 150 (B) 120 (C) 105 (D) 90 (E) 60

35. The sides of a triangle are integers, and its area is also an integer. One side is 21 and the perimeter is 48. The shortest side is:

(A) 8 (B) 10 (C) 12 (D) 14 (E) 16

36. A person starting with 64 cents and making 6 bets, wins three times and loses three times, the wins and losses occurring in random order. The chance for a win is equal to the chance for a loss. If each wager is for half the money remaining at the time of the bet, then the final result is:

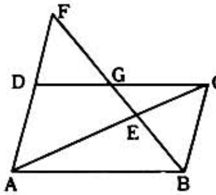
(A) a loss of 27¢ (B) a gain of 27¢ (C) a loss of 37¢
(D) neither a gain nor a loss (E) a gain or a loss depending upon the order in which the wins and losses occur.

37. Given seven points $P_1, P_2, \ldots, P_7$ on a straight line, in the order stated (not necessarily evenly spaced). Let P be an arbitrarily selected point on the line and let s be the sum of the undirected lengths $PP_1, PP_2, \ldots PP_7$. Then s is smallest if and only if the point P is:

(A) midway between P_1 and P_7 (B) midway between P_2 and P_6
(C) midway between P_3 and P_5 (D) at P_4
(E) at P_1

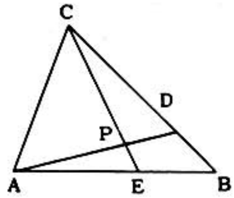
38. Point F is taken on the extension of side AD of parallelogram ABCD. BF intersects diagonal AC at E and side DC at G. If $\overline{EF} = 32$ and $\overline{GF} = 24$, then $\overline{BE}$ equals:

(A) 4 (B) 8 (C) 10 (D) 12 (E) 16



39. In triangle ABC lines CE and AD are drawn so that $\frac{\overline{CD}}{\overline{DB}} = \frac{3}{1}$ and $\frac{\overline{AE}}{\overline{EB}} = \frac{3}{2}$. Let $r = \frac{\overline{CP}}{\overline{PE}}$ where P is the intersection point of CE and AD. Then r equals:

(A) 3 (B) $\frac{3}{2}$ (C) 4 (D) 5 (E) $\frac{5}{2}$



40. If x is a number satisfying the equation $\sqrt[3]{x+9} - \sqrt[3]{x-9} = 3$, then x^2 is between:

(A) 55 and 65 (B) 65 and 75 (C) 75 and 85 (D) 85 and 95 (E) 95 and 105

COMMITTEE ON HIGH SCHOOL CONTESTS

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Sponsored Jointly by

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and

THE SOCIETY OF ACTUARIES

M A
S A^2

15

FIFTEENTH

ANNUAL

MATHEMATICS

CONTEST

1964

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers, marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box on the margin of your answer sheet directly under the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly under No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
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THURSDAY MORNING, MARCH 5, 1964

To be filled in by the student

PRINT ____

first name   middle name   last name

____

school   number   street

____

city   zone   county   state

PART I

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

PART II

21	22	23	24	25	26	27	28	29	30

PART III

31	32	33	34	35	36	37	38	39	40

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
	TOTALS	C =	W =
	SCORE = $C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

PART I (3 credits each)

1. What is the value of $[\log_{10}(5 \log_{10} 100)]^2$?

   (A) $\log_{10}50$ (B) 25 (C) 10 (D) 2 (E) 1

2. The graph of $x^2 - 4y^2 = 0$ is:

   (A) a parabola (B) an ellipse (C) a pair of straight lines
   (D) a point (E) none of these

3. When a positive integer x is divided by a positive integer y, the quotient is u and the remainder is v, u and v integers. What is the remainder when $x + 2uy$ is divided by y?

   (A) 0 (B) 2u (C) 3u (D) v (E) 2v

4. The expression $\frac{P+Q}{P-Q} - \frac{P-Q}{P+Q}$, where $P = x + y$ and $Q = x - y$, is equivalent to:

   (A) $\frac{x^2 - y^2}{xy}$ (B) $\frac{x^2 - y^2}{2xy}$ (C) 1 (D) $\frac{x^4 + y^2}{xy}$ (E) $\frac{x^2 + y^2}{2\,xy}$

5. If y varies directly as x and if $y = 8$ when $x = 4$, the value of y when $x = -8$ is:

   (A) -16 (B) -4 (C) -2 (D) $4k, k = \pm 1, \pm 2, \ldots$
   (E) $16k, k = \pm 1, \pm 2, \ldots$

6. If $x, 2x + 2, 3x + 3, \ldots$ are in geometric progression, the fourth term is:

   (A) -27 (B) $-13\frac{1}{2}$ (C) 12 (D) $13\frac{1}{2}$ (E) 27

7. Let n be the number of real values of p for which the roots of $x^2 - px + p = 0$ are equal. Then n equals:

   (A) 0 (B) 1 (C) 2 (D) a finite number greater than 2
   (E) an infinitely large number

8. The smaller root of the equation $(x - \frac{3}{4})(x - \frac{3}{4}) + (x - \frac{3}{4})(x - \frac{1}{2}) = 0$ is:

   (A) $-\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{5}{8}$ (D) $\frac{3}{4}$ (E) 1

9. A jobber buys an article at "\$24 less $12\frac{1}{2}\%$". He then wishes to sell the article at a gain of $33\frac{1}{3}\%$ of his cost after allowing a 20% discount on his marked price. At what price, in dollars, should the article be marked?

   (A) 25.20 (B) 30.00 (C) 33.60 (D) 40.00 (E) none of these

10. Given a square with side of length s. On a diagonal as base a triangle with three unequal sides is constructed so that its area equals that of the square. The length of the altitude drawn to the base is:

(A) $s\sqrt{2}$ (B) $s/\sqrt{2}$ (C) 2 s (D) $2\sqrt{s}$ (E) $2/\sqrt{s}$

11. Given $2^x = 8^{y+1}$ and $9^y = 3^{x-9}$; the value of x + y is:

(A) 18 (B) 21 (C) 24 (D) 27 (E) 30

12. Which of the following is the negation of the statement: For all x of a certain set, $x^2 > 0$?

(A) For all x, $x^2 < 0$ (B) For all x, $x^2 \leqq 0$ (C) For no x, $x^2 > 0$
(D) For some x, $x^2 > 0$ (E) For some x, $x^2 \leqq 0$

13. A circle is inscribed in a triangle with sides of lengths 8, 13, and 17. Let the segments of the side of length 8, made by a point of tangency, be r and s, with $r < s$. Then the ratio r : s is:

(A) 1 : 3 (B) 2 : 5 (C) 1 : 2 (D) 2 : 3 (E) 3 : 4

14. A farmer bought 749 sheep. He sold 700 of them for the price paid for the 749 sheep. The remaining 49 sheep were sold at the same price per head as the other 700. Based on the cost, the percent gain on the entire transaction is:

(A) 6.5 (B) 6.75 (C) 7.0 (D) 7.5 (E) 8.0

15. A line through the point (− a, o) cuts from the second quadrant a triangular region with area T. The equation of the line is:

(A) $2Tx + a^2y + 2aT = 0$ (B) $2Tx - a^2y + 2aT = 0$ (C) $2Tx + a^2y - 2aT = 0$
(D) $2Tx - a^2y - 2aT = 0$ (E) none of these

16. Let the expression $x^2 + 3x + 2$ have a remainder of zero when divided by 6, and let S be the set of integers $\{0, 1, 2, \ldots, 25\}$. The number of members of S satisfying the given condition is:

(A) 25 (B) 22 (C) 21 (D) 18 (E) 17

17. Given the distinct points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_1 + x_2, y_1 + y_2)$. Line segments are drawn connecting these points to each other and to the origin 0. Of the three possibilities: (1) parallelogram (2) straight line (3) trapezoid, figure OPRQ, depending upon the location of the points P, Q, and R, can be:

(A) (1) only (B) (2) only (C) (3) only (D) (1) or (2) only (E) all three

18. Let n be the number of pairs of values of b and c such that $3x + by + c = 0$ and $cx - 2y + 12 = 0$ have the same graph. Then n is:

(A) 0 (B) 1 (C) 2 (D) finite but more than 2
(E) greater than any finite number

19. If $2x - 3y - z = 0$ and $x + 3y - 14z = 0$, $z \neq 0$, the numerical value of

$\dfrac{x^2 + 3xy}{y^2 + z^2}$ is:

(A) 7 (B) 2 (C) 0 (D) $-20/17$ (E) -2

20. The sum of the numerical coefficients of all the terms in the expansion of $(x - 2y)^{18}$ is:

(A) 0 (B) 1 (C) 19 (D) -1 (E) -19

PART II (4 credits each)

21. If $\log_{b^2} x + \log_{x^2} b = 1$, $b > 0$, $b \neq 1$, $x \neq 1$, then x equals:

(A) $1/b^2$ (B) $1/b$ (C) b^2 (D) b (E) $\sqrt{b}$

22. Given parallelogram ABCD with E the midpoint of diagonal BD. Point E is connected to a point F in DA so that $\overline{DF} = \frac{1}{3}\overline{DA}$. What is the ratio of the area of triangle DFE to the area of quadrilateral ABEF?

(A) 1 : 2 (B) 1 : 3 (C) 1 : 5 (D) 1 : 6 (E) 1 : 7

23. Two numbers are such that their difference, their sum, and their product are to one another as 1 : 7 : 24. The product of the two numbers is:

(A) 6 (B) 12 (C) 24 (D) 48 (E) 96

24. Let $y = (x - a)^2 + (x - b)^2$, a, b constants. For what value of x is y a minimum?

(A) $\dfrac{a + b}{2}$ (B) $a + b$ (C) $\sqrt{ab}$ (D) $\sqrt{\dfrac{a^2 + b^2}{2}}$ (E) $\dfrac{a + b}{2ab}$

25. The set of values of m for which $x^2 + 3xy + x + my - m$ has two factors, with integer coefficients, which are linear in x and y, is precisely:

(A) 0, 12, -12 (B) 0, 12 (C) 12, -12 (D) 12 (E) 0

26. In a ten-mile race First beats Second by 2 miles and First beats Third by 4 miles. If the runners maintain constant speeds throughout the race, by how many miles does Second beat Third?

(A) 2 (B) $2\frac{1}{4}$ (C) $2\frac{1}{2}$ (D) $2\frac{3}{4}$ (E) 3

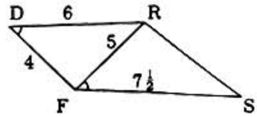
27. If x is a real number and $|x - 4| + |x - 3| < a$ where $a > 0$, then:

(A) $0 < a < .01$ (B) $.01 < a < 1$ (C) $0 < a < 1$ (D) $0 < a \leq 1$ (E) $a > 1$

28. The sum of n terms of an arithmetic progression is 153, and the common difference is 2. If the first term is an integer, and $n > 1$, then the number of possible values for n is:

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

29. In this figure $\angle RFS = \angle FDR$, $\overline{FD} = 4$ inches, $\overline{DR} = 6$ inches, $\overline{FR} = 5$ inches, $FS = 7\frac{1}{2}$ inches. The length of $\overline{RS}$, in inches is:



(A) undetermined (B) 4 (C) $5\frac{1}{2}$ (D) 6 (E) $6\frac{1}{4}$

30. If $(7 + 4\sqrt{3})x^2 + (2 + \sqrt{3})x - 2 = 0$, the larger root minus the smaller root is:

(A) $-2 + 3\sqrt{3}$ (B) $2 - \sqrt{3}$ (C) $6 + 3\sqrt{3}$ (D) $6 - 3\sqrt{3}$ (E) $3\sqrt{3} + 2$

PART III (5 credits each)

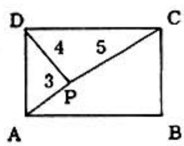
31. Let $f(n) = \frac{5 + 3\sqrt{5}}{10}\left(\frac{1 + \sqrt{5}}{2}\right)^n + \frac{5 - 3\sqrt{5}}{10}\left(\frac{1 - \sqrt{5}}{2}\right)^n$. Then $f(n + 1) - f(n-1)$, expressed in terms of $f(n)$, equals:

(A) $\frac{1}{2} f(n)$ (B) $f(n)$ (C) $2f(n) + 1$ (D) $f^2(n)$ (E) $\frac{1}{2}(f^2(n) - 1)$

32. If $\frac{a + b}{b + c} = \frac{c + d}{d + a}$, then:

(A) a must equal c (B) $a + b + c + d$ must equal zero
(C) either $a = c$ or $a + b + c + d = 0$, or both
(D) $a + b + c + d \neq 0$ if $a = c$ (E) $a(b + c + d) = c(a + b + d)$

33. P is a point interior to rectangle ABCD and such that $\overline{PA} = 3$ inches, $\overline{PD} = 4$ inches, and $\overline{PC} = 5$ inches. Then $\overline{PB}$, in inches, equals:



(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $3\sqrt{3}$ (D) $4\sqrt{2}$ (E) 2

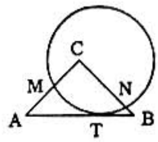
34. If n is a multiple of 4, the sum $s = 1 + 2i + 3i^2 + \ldots + (n + 1)i^n$, where $i = \sqrt{-1}$, equals:

(A) $1 + i$ (B) $\frac{1}{2}(n + 2)$ (C) $\frac{1}{2}(n + 2 - ni)$ (D) $\frac{1}{2}[(n + 1)(1 - i) + 2]$
(E) $\frac{1}{8}(n^2 + 8 - 4ni)$

35. The sides of a triangle are of lengths 13, 14, and 15. The altitudes of the triangle meet at point H. If AD is the altitude to side of length 14, the ratio $\overline{HD} : \overline{HA}$ is:

(A) 3 : 11 (B) 5 : 11 (C) 1 : 2 (D) 2 : 3 (E) 25 : 33

36. In this figure the radius of the circle is equal to the altitude of the equilateral triangle ABC. The circle is made to roll along the side AB, remaining tangent to it at a variable point T and intersecting sides AC and BC in variable points M and N, respectively. Let n be the number of degrees in arc MTN. Then n, for all permissible positions of the circle:



(A) varies from 30 to 90 (B) varies from 30 to 60
(C) varies from 60 to 90 (D) remains constant at 30
(E) remains constant at 60.

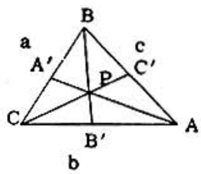
37. Given two positive numbers a, b such that $a < b$. Let A.M. be their arithmetic mean and let G.M. be their positive geometric mean. Then A.M. minus G.M. is always less than:

(A) $\frac{(b+a)^2}{ab}$ (B) $\frac{(b+a)^2}{8b}$ (C) $\frac{(b-a)^2}{ab}$ (D) $\frac{(b-a)^2}{8a}$ (E) $\frac{(b-a)^2}{8b}$

38. The sides PQ and PR of triangle PQR are respectively of lengths 4 inches and 7 inches. The median $\overline{PM}$ is $3\frac{1}{2}$ inches. Then $\overline{QR}$, in inches, is:

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

39. The magnitudes of the sides of triangle ABC are a, b, and c, as shown, with $c \leq b \leq a$. Through interior point P and the vertices A, B, C lines are drawn meeting the opposite sides in A′, B′, C′, respectively. Let $s = \overline{AA'} + \overline{BB'} + \overline{CC'}$. Then, for all positions of point P, s is less than:



(A) 2a + b (B) 2a + c (C) 2b + c (D) a + 2b (E) a + b + c

40. A watch loses $2\frac{1}{2}$ minutes per day. It is set right at 1 P.M. on March 15. Let n be the positive correction, in minutes, to be added to the time shown by the watch at a given time. When the watch shows 9 A.M. on March 21, n equals:

(A) $14\frac{14}{23}$ (B) $14\frac{1}{14}$ (C) $13\frac{101}{115}$ (D) $13\frac{83}{115}$ (E) $13\frac{13}{23}$

COMMITTEE ON HIGH SCHOOL CONTESTS

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SIXTEENTH ANNUAL MATHEMATICS EXAMINATION 1965

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers, marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box on the margin of your answer sheet directly under the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly under No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off the cover of this booklet along the dotted line and turn the cover over. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you fill in your name and the name of your school on it.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

*This instruction may be modified if machine-scoring is used.

THURSDAY MORNING, MARCH 4, 1965

To be filled in by the student

PRINT ______________________________

last name middle name first name

____

school number street

____

city zone county state

PART I

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

PART II

21	22	23	24	25	26	27	28	29	30

PART III

31	32	33	34	35	36	37	38	39	40

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
	TOTALS	C =	W =
	SCORE = $C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

PART I (3 credits each)

1. The number of real values of x satisfying the equation $2^{2x^2 - 7x + 5} = 1$ is:

   (A) 0 (B) 1 (C) 2 (D) 4 (E) more than 4

2. A regular hexagon is inscribed in a circle. The ratio of the length of a side of the hexagon to the length of the shorter of the arcs intercepted by the side, is:

   (A) 1:1 (B) 1:6 (C) $1:\pi$ (D) $3:\pi$ (E) $6:\pi$

3. The expression $(81)^{-(2^{-2})}$ has the same value as:

   (A) $\frac{1}{81}$ (B) $\frac{1}{3}$ (C) 3 (D) 81 (E) 81^4

4. Line l_2 intersects line l_1 and line l_3 is parallel to l_1. The three lines are distinct and lie in a plane. The number of points equidistant from all three lines is:

   (A) 0 (B) 1 (C) 2 (D) 4 (E) 8

5. When the repeating decimal 0.363636. . . is written in simplest fractional form, the sum of the numerator and denominator is:

   (A) 15 (B) 45 (C) 114 (D) 135 (E) 150

6. If $10^{\log_{10} 9} = 8x + 5$ then x equals:

   (A) 0 (B) $\frac{1}{2}$ (C) $\frac{5}{8}$ (D) $\frac{9}{8}$ (E) $\frac{2 \log_{10} 3 - 5}{8}$

7. The sum of the reciprocals of the roots of the equation $ax^2 + bx + c = 0$ is:

   (A) $\frac{1}{a} + \frac{1}{b}$ (B) $-\frac{c}{b}$ (C) $\frac{b}{c}$ (D) $-\frac{a}{b}$ (E) $-\frac{b}{c}$

8. One side of a given triangle is 18 inches. Inside the triangle a line segment is drawn parallel to this side forming a trapezoid whose area is one-third of that of the triangle. The length of this segment, in inches, is:

   (A) $6\sqrt{6}$ (B) $9\sqrt{2}$ (C) 12 (D) $6\sqrt{3}$ (E) 9

9. The vertex of the parabola $y = x^2 - 8x + c$ will be a point on the x-axis if the value of c is:

   (A) -16 (B) -4 (C) 4 (D) 8 (E) 16

10. The statement $x^2 - x - 6 < 0$ is equivalent to the statement:

    (A) $-2 < x < 3$ (B) $x > -2$ (C) $x < 3$ (D) $x > 3$ and $x < -2$
    (E) $x > 3$ or $x < -2$

11. Consider the statements: I: $(\sqrt{-4})(\sqrt{-16}) = \sqrt{(-4)(-16)}$, II: $\sqrt{(-4)(-16)} = \sqrt{64}$, and III: $\sqrt{64} = 8$. Of these the following are incorrect:

(A) none (B) I only (C) II only (D) III only (E) I and III only

12. A rhombus is inscribed in triangle ABC in such a way that one of its vertices is A and two of its sides lie along AB and AC. If $\overline{AC} = 6$ inches, $\overline{AB} = 12$ inches, and $\overline{BC} = 8$ inches, the side of the rhombus, in inches, is:

(A) 2 (B) 3 (C) $3\frac{1}{2}$ (D) 4 (E) 5

13. Let n be the number of number-pairs (x,y) which satisfy $5y - 3x = 15$ and $x^2 + y^2 \leqq 16$. Then n is:

(A) 0 (B) 1 (C) 2 (D) more than two, but finite
(E) greater than any finite number

14. The sum of the numerical coefficients in the complete expansion of $(x^2 - 2xy + y^2)^7$ is:

(A) 0 (B) 7 (C) 14 (D) 128 (E) 128^2

15. The symbol 25_b represents a two-digit number in the base b. If the number 52_b is double the number 25_b, then b is:

(A) 7 (B) 8 (C) 9 (D) 11 (E) 12

16. Let line AC be perpendicular to line CE. Connect A to D, the midpoint of CE and connect E to B, the midpoint of AC. If AD and EB intersect in point F, and $\overline{BC} = \overline{CD} = 15$ inches, then the area of triangle DFE, in square inches, is:

(A) 50 (B) $50\sqrt{2}$ (C) 75 (D) $\frac{15}{2}\sqrt{105}$ (E) 100

17. Given the true statement: The picnic on Sunday will not be held only if the weather is not fair. We can then conclude that:

(A) If the picnic is held, Sunday's weather is undoubtedly fair.
(B) If the picnic is not held, Sunday's weather is possibly unfair.
(C) If it is not fair Sunday, the picnic will not be held.
(D) If it is fair Sunday, the picnic may be held.
(E) If it is fair Sunday, the picnic must be held.

18. If $1 - y$ is used as an approximation to the value of $\frac{1}{1+y}$, $|y| < 1$, the ratio of the error made to the correct value is:

(A) y (B) y^2 (C) $\frac{1}{1+y}$ (D) $\frac{y}{1+y}$ (E) $\frac{y^2}{1+y}$

19. If $x^4 + 4x^3 + 6px^2 + 4qx + r$ is exactly divisible by $x^3 + 3x^2 + 9x + 3$, the value of $(p + q)r$ is:

(A) -18 (B) 12 (C) 15 (D) 27 (E) 45

20. For every n the sum of n terms of an arithmetic progression is $2n + 3n^2$. The r^{th} term is:

(A) $3r^2$ (B) $3r^2 + 2r$ (C) $6r - 1$ (D) $5r + 5$ (E) $6r + 2$

PART II (4 credits each)

21. It is possible to choose $x > \frac{2}{3}$ in such a way that the value of $\log_{10}(x^2 + 3) - 2\log_{10} x$ is:

(A) negative (B) zero (C) one
(D) smaller than any positive number that might be specified
(E) greater than any positive number that might be specified

22. If $a_2 \neq 0$ and r and s are the roots of $a_0 + a_1x + a_2x^2 = 0$, then the equality $a_0 + a_1x + a_2x^2 = a_0\left(1 - \frac{x}{r}\right)\left(1 - \frac{x}{s}\right)$ holds:

(A) for all values of x, $a_0 \neq 0$ (B) for all values of x (C) only when $x = 0$
(D) only when $x = r$ or $x = s$ (E) only when $x = r$ or $x = s$, $a_0 \neq 0$

23. If we write $|x^2 - 4| < N$ for all x such that $|x - 2| < 0.01$, the smallest value we can use for N is:

(A) .0301 (B) .0349 (C) .0399 (D) .0401 (E) .0499

24. Given the sequence $10^{\frac{1}{11}}, 10^{\frac{2}{11}}, 10^{\frac{3}{11}}, \ldots, 10^{\frac{n}{11}}$, the smallest value of n such that the product of the first n members of this sequence exceeds 100,000 is:

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

25. Let ABCD be a quadrilateral with AB extended to E so that $\overline{AB} = \overline{BE}$. Lines AC and CE are drawn to form angle ACE. For this angle to be a right angle it is necessary that quadrilateral ABCD have:

(A) all angles equal (B) all sides equal (C) two pairs of equal sides
(D) one pair of equal sides (E) one pair of equal angles.

26. For the numbers a, b, c, d, e define m to be the arithmetic mean of all five numbers; k to be the arithmetic mean of a and b; l to be the arithmetic mean of c, d, and e; and p to be the arithmetic mean of k and l. Then, no matter how a, b, c, d, e are chosen, we shall always have:

(A) $m = p$ (B) $m \geqq p$ (C) $m > p$ (D) $m < p$ (E) none of these

27. When $y^2 + my + 2$ is divided by $y - 1$ the quotient is $f(y)$ and the remainder is R_1. When $y^2 + my + 2$ is divided by $y + 1$ the quotient is $g(y)$ and the remainder is R_2. If $R_1 = R_2$ then m is:

    (A) 0 (B) 1 (C) 2 (D) -1 (E) an undetermined constant

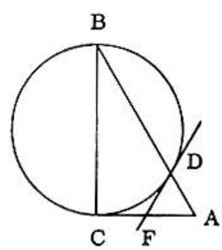
28. An escalator (moving staircase) of n uniform steps visible at all times descends at constant speed. Two boys, A and Z, walk down the escalator steadily as it moves, A negotiating twice as many escalator steps per minute as Z. A reaches the bottom after taking 27 steps while Z reaches the bottom after taking 18 steps. Then n is:

    (A) 63 (B) 54 (C) 45 (D) 36 (E) 30

29. Of 28 students taking at least one subject the number taking Mathematics and English only equals the number taking Mathematics only. No student takes English only or History only, and six students take Mathematics and History, but no English. The number taking English and History only is five times the number taking all three subjects. If the number taking all three subjects is even and non-zero, the number taking English and Mathematics only is:

    (A) 5 (B) 6 (C) 7 (D) 8 (E) 9

30. Let BC of right triangle ABC be the diameter of a circle intersecting hypotenuse AB in D. At D a tangent is drawn cutting leg CA in F. This information is <u>not</u> sufficient to prove that



    (A) DF bisects CA (B) DF bisects $\angle CDA$
    (C) $DF = FA$ (D) $\angle A = \angle BCD$
    (E) $\angle CFD = 2\angle A$

PART III (5 credits each)

31. The number of real values of x satisfying the equality $(\log_a x)(\log_b x) = \log_a b$, where $a > 0$, $b > 0$, $a \neq 1$, $b \neq 1$, is:

    (A) 0 (B) 1 (C) 2 (D) a finite integer greater than 2 (E) not finite

32. An article costing C dollars is sold for \$100 at a loss of x percent of the selling price. It is then resold at a profit of x percent of the new selling price S′. If the difference between S′ and C is $1\frac{1}{9}$ dollars, then x is:

    (A) undetermined (B) $\frac{80}{9}$ (C) 10 (D) $\frac{95}{9}$ (E) $\frac{100}{9}$

33. If the number 15!, that is, $15 \cdot 14 \cdot 13 \cdots 1$, ends with k zeros when given to the base 12 and ends with h zeros when given to the base 10, then $k + h$ equals:

    (A) 5 (B) 6 (C) 7 (D) 8 (E) 9

34. For $x \geqq 0$ the smallest value of $\frac{4x^2 + 8x + 13}{6(1 + x)}$ is:

(A) 1 (B) 2 (C) $\frac{25}{12}$ (D) $\frac{13}{6}$ (E) $\frac{34}{5}$

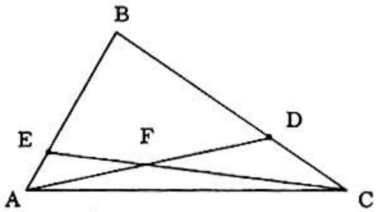
35. The length of a rectangle is 5 inches and its width is less than 4 inches. The rectangle is folded so that two diagonally opposite vertices coincide. If the length of the crease if $\sqrt{6}$, then the width is:

(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) 2 (D) $\sqrt{5}$ (E) $\sqrt{11/2}$

36. Given distinct straight lines OA and OB. From a point in OA a perpendicular is drawn to OB; from the foot of this perpendicular a line is drawn perpendicular to OA. From the foot of this second perpendicular a line is drawn perpendicular to OB; and so on indefinitely. The lengths of the first and second perpendiculars are a and b, respectively. Then the sum of the lengths of the perpendiculars approaches a limit as the number of perpendiculars grows beyond all bounds. This limit is:

(A) $\frac{b}{a - b}$ (B) $\frac{a}{a - b}$ (C) $\frac{ab}{a - b}$ (D) $\frac{b^2}{a - b}$ (E) $\frac{a^2}{a - b}$

37. Point E is selected on side AB of triangle ABC in such a way that $\overline{AE}:\overline{EB} = 1:3$ and point D is selected on side BC so that $\overline{CD}:\overline{DB} = 1:2$. The point of intersection of AD and CE is F. Then $\frac{\overline{EF}}{\overline{FC}} + \frac{\overline{AF}}{\overline{FD}}$ is:

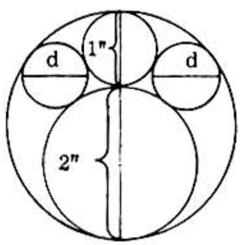


(A) $\frac{4}{5}$ (B) $\frac{5}{4}$ (C) $\frac{3}{2}$ (D) 2 (E) $\frac{5}{2}$

38. A takes m times as long to do a piece of work as B and C together; B takes n times as long as C and A together; and C takes x times as long as A and B together. Then x, in terms of m and n, is:

(A) $\frac{2mn}{m + n}$ (B) $\frac{1}{2(m + n)}$ (C) $\frac{1}{m + n - mn}$ (D) $\frac{1 - mn}{m + n + 2mn}$ (E) $\frac{m + n + 2}{mn - 1}$

39. A foreman noticed an inspector checking a 3"-hole with a 2"-plug and a 1"-plug and suggested that two more gauges be inserted to be sure that the fit was snug. If the new gauges are alike, then the diameter, d, of each, to the nearest hundredth of an inch, is:



(A) .87 (B) .86 (C) .83 (D) .75 (E) .71

40. Let n be the number of integer values of x such that $P = x^4 + 6x^3 + 11x^2 + 3x + 31$ is the square of an integer. Then n is:

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

COMMITTEE ON HIGH SCHOOL CONTESTS

National Office: Pan American College, Edinburg, Texas 78539
New York Office: Polytechnic Institute of Brooklyn, Brooklyn, N.Y. 11201

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SEVENTEENTH ANNUAL MATHEMATICS EXAMINATION 1966

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers, marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box on the margin of your answer sheet directly under the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly under No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off the cover of this booklet along the dotted line and turn the cover over. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you fill in your name and the name of your school on it.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

*This instruction may be modified if machine-scoring is used.

THURSDAY MORNING, MARCH 10, 1966

To be filled in by the student

PRINT ____

last name first name middle name or initial

____

school number street

____

city county state zip code

PART I

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

PART II

21	22	23	24	25	26	27	28	29	30

PART III

31	32	33	34	35	36	37	38	39	40

Not to be filled in by the student

Part	Value	Points Correct, <u>C</u>	Points Wrong, <u>W</u> (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
	TOTALS	C =	W =
	SCORE = $C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

PART I (3 credits each)

1. Given that the ratio of $3x - 4$ to $y + 15$ is constant, and $y = 3$ when $x = 2$, then, when $y = 12$, x equals:

   (A) $\frac{1}{8}$ (B) $\frac{3}{7}$ (C) $\frac{7}{3}$ (D) $\frac{7}{2}$ (E) 8

2. When the base of a triangle is increased 10% and the altitude to this base is decreased 10%, the change in area is:

   (A) 1% increase (B) $\frac{1}{2}$% increase (C) 0%
   (D) $\frac{1}{2}$% decrease (E) 1% decrease

3. If the arithmetic mean of two numbers is 6 and their geometric mean is 10, then an equation with the given two numbers as roots is:

   (A) $x^2 + 12x + 100 = 0$ (B) $x^2 + 6x + 100 = 0$ (C) $x^2 - 12x - 10 = 0$
   (D) $x^2 - 12x + 100 = 0$ (E) $x^2 - 6x + 100 = 0$

4. Circle I is circumscribed about a given square and circle II is inscribed in the given square. If r is the ratio of the area of circle I to that of circle II, then r equals:

   (A) $\sqrt{2}$ (B) 2 (C) $\sqrt{3}$ (D) $2\sqrt{2}$ (E) $2\sqrt{3}$

5. The number of values of x satisfying the equation $\frac{2x^2 - 10x}{x^2 - 5x} = x - 3$ is:

   (A) zero (B) one (C) two (D) three (E) an integer greater than 3

6. AB is a diameter of a circle centered at O. C is a point on the circle such that angle BOC is 60°. If the diameter of the circle is 5 inches, the length of chord AC, expressed in inches, is:

   (A) 3 (B) $\frac{5\sqrt{2}}{2}$ (C) $\frac{5\sqrt{3}}{2}$ (D) $3\sqrt{3}$ (E) none of these

7. Let $\frac{35x - 29}{x^2 - 3x + 2} = \frac{N_1}{x - 1} + \frac{N_2}{x - 2}$ be an identity in x. The numerical value of N_1N_2 is:

   (A) -246 (B) -210 (C) -29 (D) 210 (E) 246

8. The length of the common chord of two intersecting circles is 16 feet. If the radii are 10 feet and 17 feet, the distance between the centers of the circles, expressed in feet, is:

   (A) 27 (B) 21 (C) $\sqrt{389}$ (D) 15 (E) undetermined

9. If $x = (\log_8 2)^{(\log_2 8)}$, then $\log_3 x$ equals:

(A) -3 (B) $-\frac{1}{3}$ (C) $\frac{1}{3}$ (D) 3 (E) 9

10. If the sum of two numbers is 1 and their product is 1, then the sum of their cubes is ($i = \sqrt{-1}$):

(A) 2 (B) $-2 - \frac{3\sqrt{3}\,i}{4}$ (C) 0 (D) $-\frac{3\sqrt{3}\,i}{4}$ (E) -2

11. The sides of triangle BAC are in the ratio $2:3:4$. BD is the angle-bisector drawn to the shortest side AC, dividing it into segments AD and CD. If the length of AC is 10, then the length of the longer segment of AC is:

(A) $3\frac{1}{2}$ (B) 5 (C) $5\frac{5}{7}$ (D) 6 (E) $7\frac{1}{2}$

12. The number of real values of x that satisfy the equation $(2^{6x+3})(4^{3x+6}) = 8^{4x+5}$ is:

(A) zero (B) one (C) two (D) three (E) greater than 3

13. The number of points with positive rational coordinates selected from the set of points in the xy-plane such that $x + y \leqq 5$, is:

(A) 9 (B) 10 (C) 14 (D) 15 (E) infinite

14. The length of rectangle ABCD is 5 inches and its width is 3 inches. Diagonal AC is divided into three equal segments by points E and F. The area of triangle BEF, expressed in square inches, is:

(A) $\frac{3}{2}$ (B) $\frac{5}{3}$ (C) $\frac{5}{2}$ (D) $\frac{1}{3}\sqrt{34}$ (E) $\frac{1}{3}\sqrt{68}$

15. If $x - y > x$ and $x + y < y$, then

(A) $y < x$ (B) $x < y$ (C) $x < y < 0$ (D) $x < 0, y < 0$ (E) $x < 0, y > 0$

16. If $\frac{4^x}{2^{x+y}} = 8$ and $\frac{9^{x+y}}{3^{5y}} = 243$, x and y real numbers, then xy equals:

(A) $\frac{12}{5}$ (B) 4 (C) 6 (D) 12 (E) -4

17. The number of distinct points common to the curves $x^2 + 4y^2 = 1$ and $4x^2 + y^2 = 4$ is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

18. In a given arithmetic sequence the first term is 2, the last term is 29, and the sum of all the terms is 155. The common difference is:

(A) 3 (B) 2 (C) $\frac{27}{19}$ (D) $\frac{13}{9}$ (E) $\frac{23}{38}$

19. Let s_1 be the sum of the first n terms of the arithmetic sequence 8, 12, $\cdots$ and let s_2 be the sum of the first n terms of the arithmetic sequence 17, 19, $\cdots$. Then $s_1 = s_2$ for:

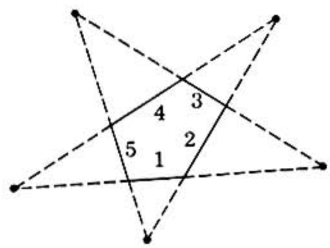
(A) no value of n (B) one value of n (C) two values of n
(D) four values of n (E) a value of n greater than four

20. The negation of the proposition "For real values of a and b, if $a = 0$, then $ab = 0$" is:

(A) If $a \neq 0$, then $ab \neq 0$ (B) If $a \neq 0$, then $ab = 0$
(C) If $a = 0$, then $ab \neq 0$ (D) If $ab \neq 0$, then $a = 0$
(E) If $ab = 0$, then $a \neq 0$

PART II (4 credits each)

21. An "n-pointed star" is formed as follows: the sides of a convex polygon are numbered consecutively 1, 2, $\cdots$, k, $\cdots$, n, $n \geqq 5$; for all n values of k, sides k and $k + 2$ are non-parallel, sides $n + 1$ and $n + 2$ being respectively identical with sides 1 and 2; prolong the n pairs of sides numbered k and $k + 2$ until they meet. (A figure is shown for the case $n = 5$).
Let S be the degree-sum of the interior angles at the n points of the star; then S equals:



(A) 180 (B) 360 (C) $180(n + 2)$ (D) $180(n - 2)$ (E) $180(n - 4)$

22. Consider the statements: (I) $\sqrt{a^2 + b^2} = 0$ (II) $\sqrt{a^2 + b^2} = ab$ (III) $\sqrt{a^2 + b^2} = a + b$ (IV) $\sqrt{a^2 + b^2} = a - b$, where we allow a and b to be real or complex numbers. Those statements for which there exist solutions other than $a = 0$ and $b = 0$, are:

(A) (I), (II), (III), (IV) (B) (II), (III), (IV) only (C) (I), (III), (IV) only
(D) (III), (IV) only (E) (I) only

23. If x is real and $4y^2 + 4xy + x + 6 = 0$, then the complete set of values of x for which y is real, is:

(A) $x \leqq -2$ and $x \geqq 3$ (B) $x \leqq 2$ and $x \geqq 3$ (C) $x \leqq -3$ and $x \geqq 2$
(D) $-3 \leqq x \leqq 2$ (E) $-2 \leqq x \leqq 3$

24. If $\log_M N = \log_N M$, $M \neq N$, $MN > 0$, $M \neq 1$, $N \neq 1$, then MN equals:

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 10
(E) a number greater than 2 and less than 10

25. If $F(n + 1) = \frac{2F(n) + 1}{2}$ and $F(1) = 2$, then $F(101)$ equals:

(A) 49 (B) 50 (C) 51 (D) 52 (E) 53

26. Let m be a positive integer and let the lines $13x + 11y = 700$ and $y = mx - 1$ intersect in a point whose coordinates are integers. Then m can be:

(A) 4 only (B) 5 only (C) 6 only (D) 7 only
(E) one of the integers 4, 5, 6, 7 and one other positive integer

27. At his usual rate a man rows 15 miles downstream in five hours less time than it takes him to return. If he doubles his usual rate, the time downstream is only one hour less than the time upstream. In miles per hour, the rate of the stream's current is:

(A) 2 (B) $\frac{5}{2}$ (C) 3 (D) $\frac{7}{2}$ (E) 4

28. Five points O, A, B, C, D are taken in order on a straight line with distances $OA = a$, $OB = b$, $OC = c$, and $OD = d$. P is a point on the line between B and C and such that $AP : PD = BP : PC$. Then OP equals:

(A) $\frac{b^2 - bc}{a - b + c - d}$ (B) $\frac{ac - bd}{a - b + c - d}$ (C) $-\frac{bd + ac}{a - b + c - d}$

(D) $\frac{bc + ad}{a + b + c + d}$ (E) $\frac{ac - bd}{a + b + c + d}$

29. The number of positive integers less than 1000 divisible by neither 5 nor 7, is:

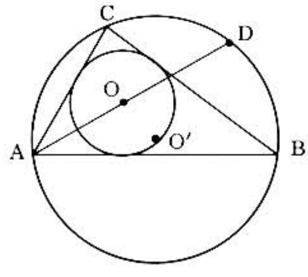
(A) 688 (B) 686 (C) 684 (D) 658 (E) 630

30. If three of the roots of $x^4 + ax^2 + bx + c = 0$ are 1, 2, and 3, then the value of $a + c$ is:

(A) 35 (B) 24 (C) −12 (D) −61 (E) −63

PART III (5 credits each)

31. Triangle ABC is inscribed in a circle with center O′. A circle with center O is inscribed in triangle ABC. AO is drawn, and extended to intersect the larger circle in D. Then we must have:



(A) $CD = BD = O'D$ (B) $AO = CO = OD$
(C) $CD = CO = BD$ (D) $CD = OD = BD$
(E) $O'B = O'C = OD$

32. Let M be the midpoint of side AB of triangle ABC. Let P be a point in AB between A and M, and let MD be drawn parallel to PC and intersecting BC at D. If the ratio of the area of triangle BPD to that of triangle ABC is designated by r, then

(A) $\frac{1}{2} < r < 1$ depending upon the position of P (B) $r = \frac{1}{2}$ independent of the position of P (C) $\frac{1}{2} \leqq r < 1$ depending upon the position of P (D) $\frac{1}{3} < r < \frac{2}{3}$ depending upon the position of P (E) $r = \frac{1}{3}$ independent of the position of P

33. If $ab \neq 0$ and $|a| \neq |b|$ the number of distinct values of x satisfying the equation $\frac{x-a}{b} + \frac{x-b}{a} = \frac{b}{x-a} + \frac{a}{x-b}$, is:

(A) zero (B) one (C) two (D) three (E) four

34. Let r be the speed in miles per hour at which a wheel, 11 feet in circumference, travels. If the time for a complete rotation of the wheel is shortened by $\frac{1}{4}$ of a second, the speed r is increased by 5 miles per hour. Then r is:

(A) 9 (B) 10 (C) $10\frac{1}{2}$ (D) 11 (E) 12

35. Let O be an interior point of triangle ABC and let $s_1 = OA + OB + OC$. If $s_2 = AB + BC + CA$, then

(A) for every triangle $s_1 > \frac{1}{2}s_2$, $s_1 \leqq s_2$ (B) for every triangle $s_1 \geqq \frac{1}{2}s_2$, $s_1 < s_2$ (C) for every triangle $s_1 > \frac{1}{2}s_2$, $s_1 < s_2$ (D) for every triangle $s_1 \geqq \frac{1}{2}s_2$, $s_1 \leqq s_2$ (E) neither (A) nor (B) nor (C) nor (D) applies to every triangle

36. Let $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$ be an identity in x. If we let $s = a_0 + a_2 + a_4 + \cdots + a_{2n}$, then s equals:

(A) 2^n (B) $2^n + 1$ (C) $\frac{3^n - 1}{2}$ (D) $\frac{3^n}{2}$ (E) $\frac{3^n + 1}{2}$

37. Three men, Alpha, Beta, and Gamma, working together, do a job in 6 hours less time than Alpha alone, in 1 hour less time than Beta alone, and in one-half the time needed by Gamma when working alone. Let h be the number of hours needed by Alpha and Beta, working together, to do the job. Then h equals:

(A) $\frac{5}{2}$ (B) $\frac{3}{2}$ (C) $\frac{4}{3}$ (D) $\frac{5}{4}$ (E) $\frac{3}{4}$

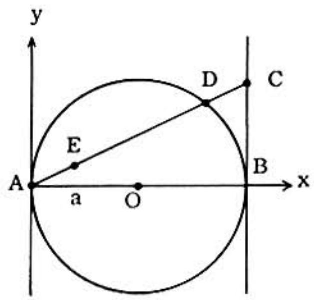
38. In triangle ABC the medians AM and CN to sides BC and AB, respectively intersect in point O. P is the midpoint of side AC and MP intersects CN in Q. If the area of triangle OMQ is n, then the area of triangle ABC is:

(A) 16n (B) 18n (C) 21n (D) 24n (E) 27n

39. In base R_1 the expanded fraction F_1 becomes $.373737\cdots$ and the expanded fraction F_2 becomes $.737373\cdots$. In base R_2 fraction F_1, when expanded, becomes $.252525\cdots$ while fraction F_2 becomes $.525252\cdots$. The sum of R_1 and R_2, each written in the base ten, is:

(A) 24 (B) 22 (C) 21 (D) 20 (E) 19

40. In this figure AB is a diameter of a circle, centered at O, with radius a. A chord AD is drawn and extended to meet the tangent to the circle at B, in point C. Point E is taken on AC so that AE = DC. If the coordinates of E are (x, y), then:



(A) $y^2 = \frac{x^3}{2a - x}$ (B) $y^2 = \frac{x^3}{2a + x}$

(C) $y^4 = \frac{x^2}{2a - x}$ (D) $x^2 = \frac{y^2}{2a - x}$

(E) $x^2 = \frac{y^2}{2a + x}$

COMMITTEE ON HIGH SCHOOL CONTESTS

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EIGHTEENTH ANNUAL MATHEMATICS EXAMINATION 1967

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box of your answer sheet directly above the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly above No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off the cover of this booklet along the dotted line and turn the cover over. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you fill in your name and the name of your school on it.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

*This instruction may be modified if machine-scoring is used.

THURSDAY MORNING, MARCH 9, 1967

To be filled in by the student

PRINT ____

last name | first name | middle name or initial

____

school | number | street

____

city | county | state | zip code

PART I

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

PART II

21	22	23	24	25	26	27	28	29	30

PART III

31	32	33	34	35	36	37	38	39	40

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
	TOTALS	C =	W =
	SCORE = $C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

EIGHTEENTH ANNUAL H. S. MATHEMATICS EXAMINATION—1967

PART I (3 credits each)

1. The three-digit number 2a3 is added to the number 326 to give the three-digit number 5b9. If 5b9 is divisible by 9, then a + b equals:

   (A) 2 (B) 4 (C) 6 (D) 8 (E) 9

2. An equivalent of the expression $\left(\frac{x^2+1}{x}\right)\left(\frac{y^2+1}{y}\right)+\left(\frac{x^2-1}{y}\right)\left(\frac{y^2-1}{x}\right)$, $xy \neq 0$, is:

   (A) 1 (B) $2xy$ (C) $2x^2y^2+2$ (D) $2xy+\frac{2}{xy}$ (E) $\frac{2x}{y}+\frac{2y}{x}$

3. The side of an equilateral triangle is $\underline{s}$. A circle is inscribed in the triangle and a square is inscribed in the circle. The area of the square is:

   (A) $\frac{s^2}{24}$ (B) $\frac{s^2}{6}$ (C) $\frac{s^2\sqrt{2}}{6}$ (D) $\frac{s^2\sqrt{3}}{6}$ (E) $\frac{s^2}{3}$

4. Given $\frac{\log a}{p}=\frac{\log b}{q}=\frac{\log c}{r}=\log x$, all logarithms to the same base and $x \neq 1$. If $\frac{b^2}{ac}=x^y$, then y is:

   (A) $\frac{q^2}{p+r}$ (B) $\frac{p+r}{2q}$ (C) $2q-p-r$ (D) $2q-pr$ (E) q^2-pr

5. A triangle is circumscribed about a circle of radius r inches. If the perimeter of the triangle is P inches and the area is K square inches, then P/K is:

   (A) independent of the value of r (B) $\sqrt{2}/r$ (C) $2/\sqrt{r}$ (D) $2/r$ (E) $r/2$

6. If $f(x) = 4^x$ then $f(x+1) - f(x)$ equals:

   (A) 4 (B) f(x) (C) 2f(x) (D) 3f(x) (E) 4f(x)

7. If $\frac{a}{b} < \frac{-c}{d}$ where a,b,c,d are real numbers and $bd \neq 0$, then:

   (A) a must be negative (B) a must be positive (C) a must not be zero
   (D) a can be negative or zero, but not positive
   (E) a can be positive, negative, or zero

8. To m ounces of an m% solution of acid, x ounces of water are added to yield an (m −10)% solution. If m > 25, then x is:

   (A) $\frac{10m}{m-10}$ (B) $\frac{5m}{m-10}$ (C) $\frac{m}{m-10}$ (D) $\frac{5m}{m-20}$
   (E) not determined by the given information

9. Let K, in square units, be the area of a trapezoid such that the shorter base, the altitude, and the longer base, in that order, are in arithmetic progression. Then:

(A) K must be an integer (B) K must be a rational fraction (C) K must be an irrational number (D) K must be an integer or a rational fraction
(E) taken alone neither (A) nor (B) nor (C) nor (D) is true

10. If $\frac{a}{10^x - 1} + \frac{b}{10^x + 2} = \frac{2 \cdot 10^x + 3}{(10^x - 1)(10^x + 2)}$ is an identity for positive rational values of x, then the value of $a - b$ is:

(A) 4/3 (B) 5/3 (C) 2 (D) 11/4 (E) 3

11. If the perimeter of rectangle ABCD is 20 inches, the least value of diagonal AC, in inches, is:

(A) 0 (B) $\sqrt{50}$ (C) 10 (D) $\sqrt{200}$ (E) none of these

12. If the (convex) area bounded by the x-axis and the lines $y = mx + 4$, $x = 1$, and $x = 4$ is 7, then m equals:

(A) $-1/2$ (B) $-2/3$ (C) $-3/2$ (D) -2 (E) none of these

13. A triangle ABC is to be constructed given side a (opposite angle A), angle B, and h_c, the altitude from C. If N is the number of noncongruent solutions, then N

(A) is 1 (B) is 2 (C) must be zero (D) must be infinite
(E) must be zero or infinite

14. Let $f(t) = \frac{t}{1-t}$, $t \neq 1$. If $y = f(x)$, then x can be expressed as:

(A) $f\left(\frac{1}{y}\right)$ (B) $-f(y)$ (C) $-f(-y)$ (D) $f(-y)$ (E) $f(y)$

15. The difference in the areas of two similar triangles is 18 square feet, and the ratio of the larger area to the smaller is the square of an integer. The area of the smaller triangle, in square feet, is an integer, and one of its sides is 3 feet. The corresponding side of the larger triangle, in feet, is:

(A) 12 (B) 9 (C) $6\sqrt{2}$ (D) 6 (E) $3\sqrt{2}$

16. Let the product (12)(15)(16), each factor written in base b, equal 3146 in base b. Let $s = 12 + 15 + 16$, each term expressed in base b. Then s, in base b, is:

(A) 43 (B) 44 (C) 45 (D) 46 (E) 47

17. If r_1 and r_2 are the distinct real roots of $x^2 + px + 8 = 0$, then it must follow that:

(A) $|r_1 + r_2| > 4\sqrt{2}$ (B) $|r_1| > 3$ or $|r_2| > 3$ (C) $|r_1| > 2$ and $|r_2| > 2$
(D) $r_1 < 0$ and $r_2 < 0$ (E) $|r_1 + r_2| < 4\sqrt{2}$

18. If $x^2 - 5x + 6 < 0$ and $P = x^2 + 5x + 6$ then

(A) P can take any real value (B) $20 < P < 30$ (C) $0 < P < 20$
(D) $P < 0$ (E) $P > 30$

19. The area of a rectangle remains unchanged when it is made $2\frac{1}{2}$ inches longer and $\frac{2}{3}$ inch narrower, or when it is made $2\frac{1}{2}$ inches shorter and $\frac{4}{3}$ inch wider. Its area, in square inches, is:

(A) 30 (B) 80/3 (C) 24 (D) 45/2 (E) 20

20. A circle is inscribed in a square of side m, then a square is inscribed in that circle, then a circle is inscribed in the latter square, and so on. If S_n is the sum of the areas of the first n circles so inscribed, then, as n grows beyond all bounds, S_n approaches:

(A) $\frac{\pi m^2}{2}$ (B) $\frac{3\pi m^2}{8}$ (C) $\frac{\pi m^2}{3}$ (D) $\frac{\pi m^2}{4}$ (E) $\frac{\pi m^2}{8}$

PART II (4 credits each)

21. In right triangle ABC the hypotenuse AB = 5 and leg AC = 3. The bisector of angle A meets the opposite side in A_1. A second right triangle PQR is then constructed with hypotenuse $PQ = A_1B$ and leg $PR = A_1C$. If the bisector of angle P meets the opposite side in P_1, the length of PP_1 is:

(A) $\frac{3\sqrt{6}}{4}$ (B) $\frac{3\sqrt{5}}{4}$ (C) $\frac{3\sqrt{3}}{4}$ (D) $\frac{3\sqrt{2}}{2}$ (E) $\frac{15\sqrt{2}}{16}$

22. For the natural numbers, when P is divided by D, the quotient is Q and the remainder is R. When Q is divided by D′, the quotient is Q′ and the remainder is R′. Then, when P is divided by DD′, the remainder is:

(A) $R + R'D$ (B) $R' + RD$ (C) RR' (D) R (E) R'

23. If x is real and positive and grows beyond all bounds, then $\log_3(6x - 5) - \log_3(2x + 1)$ approaches:

(A) 0 (B) 1 (C) 3 (D) 4 (E) no finite number

24. The number of solution-pairs in positive integers of the equation $3x + 5y = 501$ is:

(A) 33 (B) 34 (C) 35 (D) 100 (E) none of these.

25. For every odd number $p > 1$ we have:

(A) $(p-1)^{\frac{1}{2}(p-1)} - 1$ is divisible by $p-2$
(B) $(p-1)^{\frac{1}{2}(p-1)} + 1$ is divisible by p (C) $(p-1)^{\frac{1}{2}(p-1)}$ is divisible by p
(D) $(p-1)^{\frac{1}{2}(p-1)} + 1$ is divisible by $p+1$
(E) $(p-1)^{\frac{1}{2}(p-1)} - 1$ is divisible by $p-1$

26. If one uses only the tabular information $10^3 = 1000$, $10^4 = 10{,}000$, $2^{10} = 1024$, $2^{11} = 2048$, $2^{12} = 4096$, $2^{13} = 8192$, then the strongest statement one can make for $\log_{10} 2$ is that it lies between:

(A) $\frac{3}{10}$ and $\frac{4}{11}$ (B) $\frac{3}{10}$ and $\frac{4}{12}$ (C) $\frac{3}{10}$ and $\frac{4}{13}$ (D) $\frac{3}{10}$ and $\frac{40}{132}$ (E) $\frac{3}{11}$ and $\frac{40}{132}$

27. Two candles of the same length are made of different materials so that one burns out completely at a uniform rate in 3 hours and the other, in 4 hours. At what time P.M. should the candles be lighted so that, at 4 P.M., one stub is twice the length of the other?

(A) 1:24 (B) 1:28 (C) 1:36 (D) 1:40 (E) 1:48

28. Given the two hypotheses: I Some Mems are not Ens and II No Ens are Vees. If "some" means "at least one", we can conclude that:

(A) Some Mems are not Vees (B) Some Vees are not Mems
(C) No Mem is a Vee (D) Some Mems are Vees
(E) Neither (A) nor (B) nor (C) nor (D) is deducible from the given statements.

29. AB is a diameter of a circle. Tangents AD and BC are drawn so that AC and BD intersect in a point on the circle. If $AD = a$ and $BC = b$, $a \neq b$, the diameter of the circle is:

(A) $|a - b|$ (B) $\frac{1}{2}(a + b)$ (C) $\sqrt{ab}$ (D) $\frac{ab}{a + b}$ (E) $\frac{1}{2}\frac{ab}{a + b}$

30. A dealer bought n radios for d dollars, d, a positive integer. He contributed two radios to a community bazaar at half their cost. The rest he sold at a profit of \$8 on each radio sold. If the overall profit was \$72, then the least possible value of n for the given information is:

(A) 18 (B) 16 (C) 15 (D) 12 (E) 11

PART III (5 credits each)

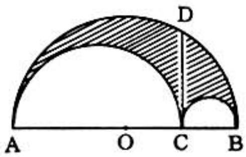
31. Let $D = a^2 + b^2 + c^2$ where a,b are consecutive integers and $c = ab$. Then $\sqrt{D}$ is:

(A) always an even integer (B) sometimes an odd integer, sometimes not
(C) always an odd integer (D) sometimes rational, sometimes not
(E) always irrational

32. In quadrilateral ABCD with diagonals AC and BD, intersecting at O, BO = 4, OD = 6, AO = 8, OC = 3, and AB = 6. The length of AD is:

(A) 9 (B) 10 (C) $6\sqrt{3}$ (D) $8\sqrt{2}$ (E) $\sqrt{166}$

33. In this diagram semi-circles are constructed on diameters AB, AC, and CB, so that they are mutually tangent. If CD ⊥ AB, then the ratio of the shaded area to the area of a circle with CD as radius is:



(A) 1:2 (B) 1:3 (C) $\sqrt{3}$:7 (D) 1:4 (E) $\sqrt{2}$:6

34. Points D,E,F are taken respectively on sides AB, BC, and CA of triangle ABC so that AD:DB = BE:CE = CF:FA = 1:n. The ratio of the area of triangle DEF to that of triangle ABC is:

(A) $\frac{n^2 - n + 1}{(n+1)^2}$ (B) $\frac{1}{(n+1)^2}$ (C) $\frac{2n^3}{(n+1)^3}$ (D) $\frac{n^3}{(n+1)^3}$ (E) $\frac{n(n-1)}{n+1}$

35. The roots of $64x^3 - 144x^2 + 92x - 15 = 0$ are in arithmetic progression. The difference between the largest and smallest roots is:

(A) 2 (B) 1 (C) 1/2 (D) 3/8 (E) 1/4

36. Given a geometric progression of five terms, each a positive integer less than 100. The sum of the five terms is 211. If S is the sum of those terms in the progression which are squares of integers, then S is:

(A) 0 (B) 91 (C) 133 (D) 195 (E) 211

37. Segments AD = 10, BE = 6, CF = 24 are drawn from the vertices of triangle ABC, each perpendicular to a straight line RS, not intersecting the triangle. Points D,E,F are the intersection points of RS with the perpendiculars. If x is the length of the perpendicular segment drawn to RS from the intersection point of the medians of the triangle, then x is:

(A) 40/3 (B) 16 (C) 56/3 (D) 80/3 (E) undetermined

38. Given a set S consisting of two undefined elements "pib" and "maa", and the four postulates: P_1: Every pib is a collection of maas, P_2: Any two distinct pibs have one and only one maa in common, P_3: Every maa belongs to two and only two pibs, P_4: There are exactly four pibs.

Consider the three theorems: T_1: There are exactly six maas, T_2: There are exactly three maas in each pib, T_3: For each maa there is exactly one other maa not in the same pib with it. The theorems which are deducible from the postulates are:

(A) T_3 only (B) T_2 and T_3 only (C) T_1 and T_2 only (D) T_1 and T_3 only (E) all

39. Given the sets of consecutive integers $\{1\}, \{2,3\}, \{4,5,6\}, \{7,8,9,10\}, \cdots$, where each set contains one more element than the preceding one, and where the first element of each succeeding set is one more than the last element of the preceding set. Let S_n be the sum of the elements in the nth set. Then S_{21} equals:

(A) 1113 (B) 4641 (C) 5082 (D) 53361 (E) none of these

40. Located inside equilateral triangle ABC is point P such that $PA = 6$, $PB = 8$, and $PC = 10$. To the nearest integer the area of triangle ABC is:

(A) 159 (B) 131 (C) 95 (D) 79 (E) 50

COMMITTEE ON HIGH SCHOOL CONTESTS

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19

NINETEENTH ANNUAL MATHEMATICS EXAMINATION 1968

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box of your answer sheet directly above the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly above No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off this cover along the dotted line inside. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you write the information required in the first three lines.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

*This instruction may be modified if machine-scoring is used.

TUESDAY, MARCH 12, 1968

To be filled in by the student

PRINT ____

last name   first name   middle initial

____

school   street address

____

city   state   zip code

PART I

1	2	3	4	5	6	7	8	9	10

PART II

11	12	13	14	15	16	17	18	19	20

PART III

21	22	23	24	25	26	27	28	29	30

PART IV

31	32	33	34	35

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times
II	4 points each	4 times =	4 times
III	5 points each	5 times =	5 times
IV	6 points each	6 times =	6 times
	TOTALS	C =	W =
	SCORE = $C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

NINETEENTH ANNUAL H. S. MATHEMATICS EXAMINATION–1968

PART I (3 credits each)

1. Let P units be the increase in the circumference of a circle resulting from an increase of π units in the diameter. Then P equals:

   (A) $\frac{1}{\pi}$ (B) π (C) $\frac{\pi^2}{2}$ (D) π^2 (E) 2π

2. The real value of x such that 64^{x-1} divided by 4^{x-1} equals 256^{2x} is:

   (A) $-\frac{2}{3}$ (B) $-\frac{1}{3}$ (C) 0 (D) $\frac{1}{4}$ (E) $\frac{3}{8}$

3. A straight line passing through the point (0, 4) is perpendicular to the line $x - 3y - 7 = 0$. Its equation is:

   (A) $y + 3x - 4 = 0$ (B) $y + 3x + 4 = 0$ (C) $y - 3x - 4 = 0$
   (D) $3y + x - 12 = 0$ (E) $3y - x - 12 = 0$

4. Define an operation * for positive real numbers as $a * b = \frac{ab}{a+b}$. Then $4 * (4 * 4)$ equals:

   (A) $\frac{3}{4}$ (B) 1 (C) $\frac{4}{3}$ (D) 2 (E) $\frac{16}{3}$

5. If $f(n) = \frac{1}{3}n(n+1)(n+2)$, then $f(r) - f(r-1)$ equals:

   (A) $r(r+1)$ (B) $(r+1)(r+2)$ (C) $\frac{1}{3}r(r+1)$
   (D) $\frac{1}{3}(r+1)(r+2)$ (E) $\frac{1}{3}r(r+1)(2r+1)$

6. Let side AD of convex quadrilateral ABCD be extended through D and let side BC be extended through C, to meet in point E. Let S represent the degree-sum of angles CDE and DCE and let S′ represent the degree-sum of angles BAD and ABC. If $r = S/S'$, then:

   (A) $r = 1$ sometimes, $r > 1$ sometimes
   (B) $r = 1$ sometimes, $r < 1$ sometimes
   (C) $0 < r < 1$ (D) $r > 1$ (E) $r = 1$

7. Let O be the intersection point of medians AP and CQ of triangle ABC. If OQ is 3 inches, then OP, in inches, is:

   (A) 3 (B) $\frac{9}{2}$ (C) 6 (D) 9 (E) undetermined

8. A positive number is mistakenly divided by 6 instead of being multiplied by 6. Based on the correct answer, the error thus committed, to the nearest percent, is:

(A) 100 (B) 97 (C) 83 (D) 17 (E) 3

9. The sum of the real values of x satisfying the equality $|x + 2| = 2|x - 2|$ is:

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) 6 (D) $6\frac{1}{3}$ (E) $6\frac{2}{3}$

10. Assume that, for a certain school, it is true that

I: Some students are not honest.

II: All fraternity members are honest.

A necessary conclusion is:

(A) Some students are fraternity members. (B) Some fraternity members are not students (C) Some students are not fraternity members (D) No fraternity member is a student (E) No student is a fraternity member.

PART II (4 credits each)

11. If an arc of $60°$ on circle I has the same length as an arc of $45°$ on circle II, the ratio of the area of circle I to that of circle II is:

(A) 16 : 9 (B) 9 : 16 (C) 4 : 3 (D) 3 : 4
(E) none of these

12. A circle passes through the vertices of a triangle with side-lengths $7\frac{1}{2}$, 10, $12\frac{1}{2}$. The radius of the circle is:

(A) $\frac{15}{4}$ (B) 5 (C) $\frac{25}{4}$ (D) $\frac{35}{4}$ (E) $\frac{15\sqrt{2}}{2}$

13. If m and n are the roots of $x^2 + mx + n = 0$, $m \neq 0$, $n \neq 0$, then the sum of the roots is:

(A) $-\frac{1}{2}$ (B) -1 (C) $\frac{1}{2}$ (D) 1 (E) undetermined

14. If x and y are non-zero numbers such that $x = 1 + \frac{1}{y}$ and $y = 1 + \frac{1}{x}$, then y equals

(A) $x - 1$ (B) $1 - x$ (C) $1 + x$ (D) $-x$ (E) x

15. Let P equal the product of any three consecutive positive odd integers. The largest integer dividing all such P is:

(A) 15 (B) 6. (C) 5 (D) 3 (E) 1

16. If x is such that $\frac{1}{x} < 2$ and $\frac{1}{x} > -3$, then:

(A) $-\frac{1}{3} < x < \frac{1}{2}$ (B) $-\frac{1}{2} < x < 3$ (C) $x > \frac{1}{2}$
(D) $x > \frac{1}{2}$ or $-\frac{1}{3} < x < 0$ (E) $x > \frac{1}{2}$ or $x < -\frac{1}{3}$

17. Let $f(n) = \frac{x_1 + x_2 + \ldots + x_n}{n}$, where n is a positive integer. If $x_k = (-1)^k$, $k = 1, 2, 3, \ldots, n$, the set of possible values of $f(n)$ is:

(A) $\{0\}$ (B) $\{\frac{1}{n}\}$ (C) $\{0, -\frac{1}{n}\}$ (D) $\{0, \frac{1}{n}\}$ (E) $\{1, \frac{1}{n}\}$

18. Side AB of triangle ABC has length 8 inches. Line DEF is drawn parallel to AB so that D is on segment AC and E is on segment BC. Line AE extended bisects angle FEC. If DE has length 5 inches, then the length of CE, in inches, is:

(A) $\frac{51}{4}$ (B) 13 (C) $\frac{53}{4}$ (D) $\frac{40}{3}$ (E) $\frac{27}{2}$

19. Let n be the number of ways that 10 dollars can be changed into dimes and quarters, with at least one of each coin being used. Then n equals:

(A) 40 (B) 38 (C) 21 (D) 20 (E) 19

20. The measures of the interior angles of a convex polygon of n sides are in arithmetic progression. If the common difference is 5° and the largest angle is 160°, then n equals:

(A) 9 (B) 10 (C) 12 (D) 16 (E) 32

PART III (5 credits each)

21. If all the operations in $S = 1! + 2! + 3! + \ldots + 99!$ are correctly performed, the units digit in the value of S is:

(A) 9 (B) 8 (C) 5 (D) 3 (E) 0

22. A segment of length 1 is divided into four segments. Then there exists a simple quadrilateral with the four segments as sides if and only if each segment is:

(A) equal to $\frac{1}{4}$ (B) equal to or greater than $\frac{1}{8}$ and less than $\frac{1}{2}$ (C) greater than $\frac{1}{8}$ and less than $\frac{1}{2}$ (D) greater than $\frac{1}{8}$ and less than $\frac{1}{4}$ (E) less than $\frac{1}{2}$

23. If all the logarithms are real numbers, the equality $\log(x+3) + \log(x-1) = \log(x^2 - 2x - 3)$ is satisfied for:

(A) all real values of x (B) no real values of x (C) all real values of x except $x = 0$ (D) no real values of x except $x = 0$ (E) all real values of x except $x = 1$

24. A painting $18'' \times 24''$ is to be placed into a wooden frame with the longer dimension vertical. The wood at the top and bottom is twice as wide as the wood on the sides. If the frame area equals that of the painting itself, the ratio of the smaller to the larger dimension of the framed painting is:

(A) 1 : 3 (B) 1 : 2 (C) 2 : 3 (D) 3 : 4 (E) 1 : 1

25. Ace runs with constant speed and Flash runs x times as fast, $x > 1$. Flash gives Ace a head start of y yards, and, at a given signal, they start off in the same direction. Then the number of yards Flash must run to catch Ace is:

(A) xy (B) $\frac{y}{x+y}$ (C) $\frac{xy}{x-1}$ (D) $\frac{x+y}{x+1}$ (E) $\frac{x+y}{x-1}$

26. Let $S = 2 + 4 + 6 + \ldots + 2N$, where N is the smallest positive integer such that $S > 1{,}000{,}000$. Then the sum of the digits of N is:

(A) 27 (B) 12 (C) 6 (D) 2 (E) 1

27. Let $S_n = 1 - 2 + 3 - 4 + \ldots + (-1)^{n-1}n$, $n = 1, 2, \ldots$ Then $S_{17} + S_{33} + S_{50}$ equals:

(A) 0 (B) 1 (C) 2 (D) -1 (E) -2

28. If the arithmetic mean of a and b is double their geometric mean, with $a > b > 0$, then a possible value for the ratio $\frac{a}{b}$, to the nearest integer, is

(A) 5 (B) 8 (C) 11 (D) 14 (E) none of these

29. Given the three numbers x, $y = x^x$, $z = x^{(x^x)}$ with $.9 < x < 1.0$. Arranged in order of increasing magnitude, they are:

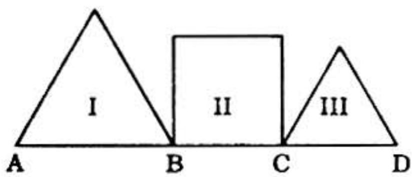
(A) x, z, y (B) x, y, z (C) y, x, z (D) y, z, x (E) z, x, y

30. Convex polygons P_1 and P_2 are drawn in the same plane with n_1 and n_2 sides, respectively, $n_1 \le n_2$. If P_1 and P_2 do not have any line segment in common, then the maximum number of intersections of P_1 and P_2 is:

(A) $2n_1$ (B) $2n_2$ (C) n_1n_2 (D) $n_1 + n_2$ (E) none of these

PART IV (6 credits each)

31. In this diagram, not drawn to scale, figures I and III are equilateral triangular regions with respective areas of $32\sqrt{3}$ and $8\sqrt{3}$, square inches. Figure II is a square region with area 32 sq. in. Let the length of segment AD be decreased by $12\frac{1}{2}\%$ of itself, while the lengths of AB and CD remain unchanged. The percent decrease in the area of the square is:



(A) $12\frac{1}{2}$ (B) 25 (C) 50 (D) 75 (E) $87\frac{1}{2}$

32. A and B move uniformly along two straight paths intersecting at right angles in point O. When A is at O, B is 500 yards short of O. In 2 minutes they are equidistant from O, and in 8 minutes more they are again equidistant from O. Then the ratio of A's speed to B's speed is:

(A) 4 : 5 (B) 5 : 6 (C) 2 : 3 (D) 5 : 8 (E) 1 : 2

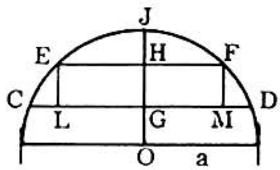
33. A number N has three digits when expressed in base 7. When N is expressed in base 9 the digits are reversed. Then the middle digit is:

(A) 0 (B) 1 (C) 3 (D) 4 (E) 5

34. With 400 members voting the House of Representatives defeated a bill. A re-vote, with the same members voting, resulted in passage of the bill by twice the margin by which it was originally defeated. The number voting for the bill on the re-vote was $\frac{12}{11}$ of the number voting against it originally. How many more members voted for the bill the second time than voted for it the first time?

(A) 75 (B) 60 (C) 50 (D) 45 (E) 20

35. In this diagram the center of the circle is O, the radius is a inches, chord EF is parallel to chord CD, O, G, H, J are collinear, and G is the midpoint of CD. Let K (sq. in.) represent the area of trapezoid CDFE and let R (sq. in.) represent the area of rectangle ELMF. Then, as CD and EF are translated upward so that OG increases toward the value a, while JH always equals HG, the ratio K:R becomes arbitrarily close to:



(A) 0 (B) 1 (C) $\sqrt{2}$ (D) $\frac{1}{\sqrt{2}}+\frac{1}{2}$ (E) $\frac{1}{\sqrt{2}}+1$

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20

TWENTIETH ANNUAL MATHEMATICS EXAMINATION 1969

INSTRUCTIONS

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5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
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*This instruction may be modified if machine-scoring is used.

TUESDAY, MARCH 11, 1969

To be filled in by the student

PRINT

last name | first name | middle initial

school (full name) | street address

city | state | zip code

PART I

1	2	3	4	5	6	7	8	9	10

PART II

11	12	13	14	15	16	17	18	19	20

PART III

21	22	23	24	25	26	27	28	29	30

PART IV

31	32	33	34	35

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
IV	6 points each	6 times =	6 times =
	TOTALS	C =	W =
	SCORE = $C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

TWENTIETH ANNUAL H. S. MATHEMATICS EXAMINATION—1969

PART I (3 credits each)

1. When x is added to both the numerator and the denominator of the fraction $\frac{a}{b}$, $a \neq b$, $b \neq 0$, the value of the fraction is changed to $\frac{c}{d}$. Then x equals:

   (A) $\frac{1}{c-d}$ (B) $\frac{ad-bc}{c-d}$ (C) $\frac{ad-bc}{c+d}$ (D) $\frac{bc-ad}{c-d}$ (E) $\frac{bc-ad}{c+d}$

2. If an item is sold for x dollars, there is a loss of 15% based on the cost. If, however, the same item is sold for y dollars, there is a profit of 15% based on the cost. The ratio $y:x$ is:

   (A) $23:17$ (B) $17y:23$ (C) $23x:17$ (D) dependent upon the cost (E) none of these.

3. If N, written in base 2, is 11000, the integer immediately preceding N, written in base 2, is:

   (A) 10001 (B) 10010 (C) 10011 (D) 10110 (E) 10111

4. Let a binary operation $*$ on ordered pairs of integers be defined as $(a,b)*(c,d) = (a-c, b+d)$. Then, if $(3,2)*(0,0)$ and $(x,y)*(3,2)$ represent identical pairs, x equals:

   (A) -3 (B) 0 (C) 2 (D) 3 (E) 6

5. If a number N, $N \neq 0$, diminished by four times its reciprocal, equals a given real constant R, then, for this given R, the sum of all such possible values of N is:

   (A) $\frac{1}{R}$ (B) R (C) 4 (D) $\frac{1}{4}$ (E) $-R$

6. The area of the ring between two concentric circles is $12\frac{1}{2}\pi$ square inches. The length of a chord of the larger circle tangent to the smaller circle, in inches, is:

   (A) $\frac{5}{\sqrt{2}}$ (B) 5 (C) $5\sqrt{2}$ (D) 10 (E) $10\sqrt{2}$

7. If the points $(1,y_1)$ and $(-1,y_2)$ lie on the graph of $y = ax^2 + bx + c$, and $y_1 - y_2 = -6$, then b equals:

   (A) -3 (B) 0 (C) 3 (D) $\sqrt{ac}$ (E) $\frac{a+c}{2}$

8. Triangle ABC is inscribed in a circle. The measures of the non-overlapping minor arcs AB, BC, and CA are, respectively, $x + 75°$, $2x + 25°$, $3x - 22°$. Then one interior angle of the triangle, in degrees, is:

(A) $57\frac{1}{2}$ (B) 59 (C) 60 (D) 61 (E) 122

9. The arithmetic mean (ordinary average) of the fifty-two successive positive integers beginning with 2, is:

(A) 27 (B) $27\frac{1}{4}$ (C) $27\frac{1}{2}$ (D) 28 (E) $28\frac{1}{2}$

10. The number of points equidistant from a circle and two parallel tangents to the circle, is:

(A) 0 (B) 2 (C) 3 (D) 4 (E) infinite

PART II (4 credits each)

11. Given points $P(-1,-2)$ and $Q(4,2)$ in the xy-plane; point $R(1,m)$ is taken so that $PR + RQ$ is a minimum. Then m equals:

(A) $-\frac{3}{5}$ (B) $-\frac{2}{5}$ (C) $-\frac{1}{5}$ (D) $\frac{1}{5}$ (E) either $-\frac{1}{5}$ or $\frac{1}{5}$.

12. Let $F = \dfrac{6x^2 + 16x + 3m}{6}$ be the square of an expression which is linear in x. Then m has a particular value between:

(A) 3 and 4 (B) 4 and 5 (C) 5 and 6 (D) -4 and -3
(E) -6 and -5

13. A circle with radius r is contained within the region bounded by a circle with radius R. The area bounded by the larger circle is $\dfrac{a}{b}$ times the area of the region outside the smaller circle and inside the larger circle. Then $R : r$ equals:

(A) $\sqrt{a} : \sqrt{b}$ (B) $\sqrt{a} : \sqrt{a-b}$ (C) $\sqrt{b} : \sqrt{a-b}$ (D) $a : \sqrt{a-b}$
(E) $b : \sqrt{a-b}$

14. The complete set of x-values satisfying the inequality $\dfrac{x^2-4}{x^2-1} > 0$ is the set of all x such that:

(A) $x > 2$ or $x < -2$ or $-1 < x < 1$ (B) $x > 2$ or $x < -2$
(C) $x > 1$ or $x < -2$ (D) $x > 1$ or $x < -1$
(E) x is any real number except 1 or -1

15. In a circle with center at O and radius r, chord AB is drawn with length equal to r (units). From O a perpendicular to AB meets AB at M. From M a perpendicular to OA meets OA at D. In terms of r the area of triangle MDA, in appropriate square units, is:

(A) $\frac{3r^2}{16}$ (B) $\frac{\pi r^2}{16}$ (C) $\frac{\pi r^2\sqrt{2}}{8}$ (D) $\frac{r^2\sqrt{3}}{32}$ (E) $\frac{r^2\sqrt{6}}{48}$

16. When $(a-b)^n$, $n \geq 2$, $ab \neq 0$, is expanded by the binomial theorem, it is found that, when $a = kb$, where k is a positive integer, the sum of the second and third terms is zero. Then n equals:

(A) $\frac{1}{2}k(k-1)$ (B) $\frac{1}{2}k(k+1)$ (C) $2k-1$ (D) $2k$ (E) $2k+1$

17. The equation $2^{2x} - 8 \cdot 2^x + 12 = 0$ is satisfied by:

(A) $\log 3$ (B) $\frac{1}{2}\log 6$ (C) $1 + \log\frac{3}{2}$ (D) $1 + \frac{\log 3}{\log 2}$
(E) none of these

18. The number of points common to the graphs of $(x - y + 2)(3x + y - 4) = 0$ and $(x + y - 2)(2x - 5y + 7) = 0$ is:

(A) 2 (B) 4 (C) 6 (D) 16 (E) infinite

19. The number of distinct ordered pairs (x, y) where x and y have positive integral values satisfying the equation $x^4y^4 - 10x^2y^2 + 9 = 0$ is:

(A) 0 (B) 3 (C) 4 (D) 12 (E) infinite

20. Let P equal the product of 3,659,893,456,789,325,678 and 342,973,489,379,256. The number of digits in P is:

(A) 36 (B) 35 (C) 34 (D) 33 (E) 32

PART III (5 credits each)

21. If the graph of $x^2 + y^2 = m$ is tangent to the graph of $x + y = \sqrt{2m}$, then:

(A) m must equal $\frac{1}{2}$ (B) m must equal $\frac{1}{\sqrt{2}}$ (C) m must equal $\sqrt{2}$
(D) m must equal 2 (E) m may be any non-negative real number

22. Let K be the measure of the area bounded by the x-axis, the line $x = 8$, and the curve defined by $f = \{(x, y) \mid y = x \text{ when } 0 \leqq x \leqq 5,\ y = 2x - 5 \text{ when } 5 \leqq x \leqq 8\}$. Then K is:

(A) 21.5 (B) 36.4 (C) 36.5 (D) 44
(E) less than 44 but arbitrarily close to it.

23. For $n \geq 50$ the number of prime numbers greater than $n! + 1$ and less than $n! + n$, is: $[n! = 1 \cdot 2 \cdots (n-1) \cdot n$; thus: $3! = 1 \cdot 2 \cdot 3 = 6$; $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120]$

(A) 0 (B) 1 (C) $\frac{n}{2}$ for n even, $\frac{n+1}{2}$ for n odd
(D) $n - 1$ (E) n

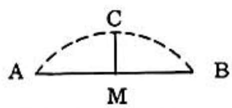
24. When the natural numbers P and P′, with $P > P'$, are divided by the natural number D, the remainders are R and R′, respectively. When PP′ and RR′ are divided by D, the remainders are r and r′, respectively. Then:

(A) $r > r'$ always (B) $r < r'$ always
(C) $r > r'$ sometimes and $r < r'$ sometimes
(D) $r > r'$ sometimes and $r = r'$ sometimes (E) $r = r'$ always

25. If it is known that $\log_2 a + \log_2 b \leqq 6$ and that ab is a maximum, then the least value that can be taken on by $a + b$ is:

(A) $2\sqrt{6}$ (B) 6 (C) $8\sqrt{2}$ (D) 16 (E) none of these.

26. A parabolic arch has a height of 16 inches and a span of 40 inches. The height, in inches, of the arch at a point 5 inches from the center M, is:



M center AB span
MC height C vertex

(A) 1 (B) 15 (C) $15\frac{1}{3}$ (D) $15\frac{1}{2}$ (E) $15\frac{3}{4}$

27. A particle moves so that its speed for the second and subsequent miles varies inversely as the integral number of miles already traveled. For each subsequent mile the speed is constant. If the second mile is traversed in 2 hours, then the time, in hours, needed to traverse the nth mile is:

(A) $\frac{2}{n-1}$ (B) $\frac{n-1}{2}$ (C) $\frac{2}{n}$ (D) 2n (E) $2(n-1)$

28. Let n be the number of points P interior to the region bounded by a circle with radius 1, such that the sum of the squares of the distances from P to the endpoints of a given diameter is 3. Then n is:

(A) 0 (B) 1 (C) 2 (D) 4 (E) infinite

29. If $x = t^{\frac{1}{t-1}}$ and $y = t^{\frac{t}{t-1}}$, $t > 0$, $t \neq 1$, a relation between x and y is:

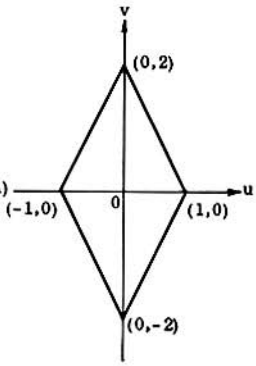
(A) $y^x = x^{\frac{1}{y}}$ (B) $y^{\frac{1}{x}} = x^y$ (C) $y^x = x^y$ (D) $x^x = y^y$
(E) none of these

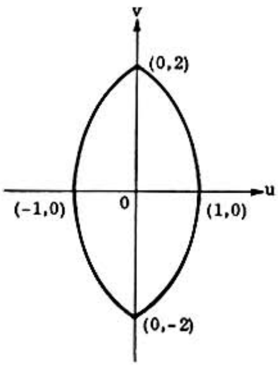
30. Let P be a point of hypotenuse AB (or its extension) of isosceles right triangle ABC. Let $s = AP^2 + PB^2$. Then:

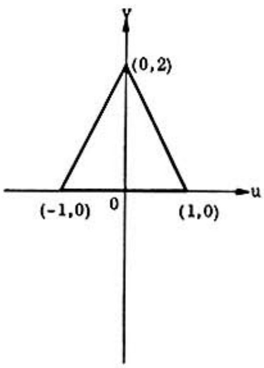
(A) $s < 2CP^2$ for a finite number of positions of P
(B) $s < 2CP^2$ for an infinite number of positions of P
(C) $s = 2CP^2$ only if P is the midpoint of AB or an endpoint of AB
(D) $s = 2CP^2$ always (E) $s > 2\ CP^2$ if P is a trisection point of AB

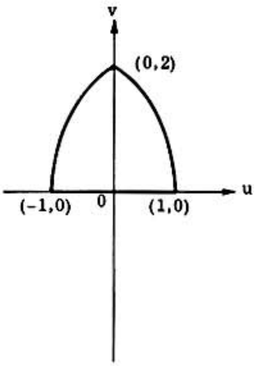
PART IV (6 points each)

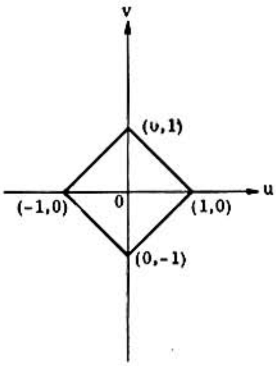
31. Let OABC be a unit square in the xy-plane with O(0,0), A(1,0), B(1,1) and C(0,1). Let $u = x^2 - y^2$ and $v = 2xy$ be a transformation of the xy-plane into the uv-plane. The transform (or image) of the square is:

(A)


(B)


(C)


(D)


(E)


32. Let a sequence $\{u_n\}$ be defined by the relation $u_n + 1 - u_n = 3 + 4(n-1)$, $n = 1, 2, 3, \ldots$, and $u_1 = 5$. If u_n is expressed as a polynomial in n, the algebraic sum of its coefficients is:

(A) 3 (B) 4 (C) 5 (D) 6 (E) 11

33. Let S_n and T_n be the respective sums of the first n terms of two arithmetic series. If $S_n : T_n = (7n + 1) : (4n + 27)$ for all n, the ratio of the eleventh term of the first series to the eleventh term of the second series, is:

(A) 4 : 3 (B) 3 : 2 (C) 7 : 4 (D) 78 : 71 (E) undetermined

34. The remainder R obtained by dividing x^{100} by $x^2 - 3x + 2$ is a polynomial of degree less than 2. Then R may be written as:

(A) $2^{100} - 1$ (B) $2^{100}(x-1) - (x-2)$ (C) $2^{100}(x-3)$
(D) $x(2^{100} - 1) + 2(2^{99} - 1)$ (E) $2^{100}(x+1) - (x+2)$

35. Let L(m) be the x - coordinate of the left end point of the intersection of the graphs of $y = x^2 - 6$ and $y = m$ where $-6 < m < 6$. Let $r = \dfrac{L(-m) - L(m)}{m}$. Then, as m is made arbitrarily close to zero, the value of r is:

(A) arbitrarily close to zero (B) arbitrarily close to $\dfrac{1}{\sqrt{6}}$
(C) arbitrarily close to $\dfrac{2}{\sqrt{6}}$ (D) arbitrarily large
(E) undetermined

M.A.A. COMMITTEE ON HIGH SCHOOL CONTESTS

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TWENTY FIRST ANNUAL MATHEMATICS EXAMINATION 1970

21

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box of your answer sheet directly above the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly above No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off this cover along the dotted line inside. Page 2 is your answer sheet.
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TUESDAY, MARCH 10, 1970

To be filled in by the student

PRINT ____

last name first name middle initial

____

school (full name) street address

____

city state zip code

PART I

1	2	3	4	5	6	7	8	9	10

PART II

11	12	13	14	15	16	17	18	19	20

PART III

21	22	23	24	25	26	27	28	29	30

PART IV

31	32	33	34	35

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
IV	6 points each	6 times =	6 times =
	TOTALS	C =	W =
	SCORE $= C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

PART I (3 credits each)

1. The fourth power of $\sqrt{1 + \sqrt{1 + \sqrt{1}}}$ is

   (a) $\sqrt{2} + \sqrt{3}$ (B) $\frac{1}{2}(7 + 3\sqrt{5})$ (C) $1 + 2\sqrt{3}$ (D) 3 (E) $3 + 2\sqrt{2}$

2. A square and a circle have equal perimeters. The ratio of the area of the circle to the area of the square is

   (A) $4/\pi$ (B) $\pi/\sqrt{2}$ (C) $4/1$ (D) $\sqrt{2}/\pi$ (E) $\pi/4$

3. If $x = 1 + 2^p$ and $y = 1 + 2^{-p}$, then y in terms of x is

   (A) $\frac{x+1}{x-1}$ (B) $\frac{x+2}{x-1}$ (C) $\frac{x}{x-1}$ (D) $2 - x$ (E) $\frac{x-1}{x}$

4. Let S be the set of all numbers which are the sum of the squares of three consecutive integers. Then we can say that

   (A) No member of S is divisible by 2
   (B) No member of S is divisible by 3 but some member is divisible by 11
   (C) No member of S is divisible by 3 or by 5
   (D) No member of S is divisible by 3 or by 7
   (E) None of these

5. If $f(x) = \frac{x^4 + x^2}{x + 1}$, then $f(i)$, where $i = \sqrt{-1}$, is equal to

   (A) $1 + i$ (B) 1 (C) -1 (D) 0 (E) $-1 - i$

6. The smallest value of $x^2 + 8x$ for real values of x is

   (A) -16.25 (B) -16 (C) -15 (D) -8 (E) None of these

7. Inside square ABCD with side s, quarter-circle arcs with radii s and centers at A and B are drawn. These arcs intersect at a point X inside the square. How far is X from side CD?

   (A) $\frac{1}{2}s(\sqrt{3} + 4)$ (B) $\frac{1}{2}s\sqrt{3}$ (C) $\frac{1}{2}s(1 + \sqrt{3})$ (D) $\frac{1}{2}s(\sqrt{3} - 1)$ (E) $\frac{1}{2}s(2 - \sqrt{3})$

8. If $a = \log_8 225$ and $b = \log_2 15$, then

   (A) $a = \frac{1}{2}b$ (B) $a = 2b/3$ (c) $a = b$ (D) $b = \frac{1}{2}a$ (E) $a = 3b/2$

9. Points P and Q are on line segment AB, and both points are on the same side of the midpoint of AB. Point P divides AB in the ratio 2:3 and Q divides AB in the ratio 3:4. If PQ = 2, then the length of segment AB is

(A) 12 (B) 28 (C) 70 (D) 75 (E) 105

10. Let F = .48181 ··· be an infinite repeating decimal with the digits 8 and 1 repeating. When F is written as a fraction in lowest terms, the denominator exceeds the numerator by

(A) 13 (B) 14 (C) 29 (D) 57 (E) 126

PART II (4 credits each)

11. If two factors of $2x^3 - hx + k$ are $x + 2$ and $x - 1$, the value of $|2h - 3k|$ is

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

12. A circle with radius r is tangent to sides AB, AD, and CD of rectangle ABCD and passes through the midpoint of diagonal AC. The area of the rectangle, in terms of r, is

(A) $4r^2$ (B) $6r^2$ (C) $8r^2$ (D) $12r^2$ (E) $20r^2$

13. Given the binary operation* defined by $a*b = a^b$ for all positive numbers a and b. Then for all positive a, b, c, n, we have

(A) $a*b = b*a$ (B) $a*(b*c) = (a*b)*c$ (C) $(a*b^n) = (a*n)*b$
(D) $(a*b)^n = a*(bn)$ (E) None of these

14. Consider $x^2 + px + q = 0$ where p and q are positive numbers. If the roots of this equation differ by 1, then p equals

(A) $\sqrt{4q+1}$ (B) $q - 1$ (C) $-\sqrt{4q+1}$ (D) $q + 1$ (E) $\sqrt{4q-1}$

15. Lines in the xy-plane are drawn through the point (3, 4) and the trisection points of the line segment joining the points (−4, 5) and (5, −1). One of these lines has the equation

(A) $3x - 2y - 1 = 0$ (B) $4x - 5y + 8 = 0$ (C) $5x + 2y - 23 = 0$
(D) $x + 7y - 31 = 0$ (E) $x - 4y + 13 = 0$

16. If $F(n)$ is a function such that $F(1) = F(2) = F(3) = 1$, and such that $F(n+1) = \frac{F(n) \cdot F(n-1) + 1}{F(n-2)}$ for $n \geqq 3$, then $F(6)$ is equal to

(A) 2 (B) 3 (C) 7 (D) 11 (E) 26

17. If $r > 0$, then for all p and q such that $pq \neq 0$ and $pr > qr$, we have

(A) $-p > -q$ (B) $-p > q$ (C) $1 > -q/p$ (D) $1 < q/p$ (E) None of the:

18. $\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}$ is equal to

(A) 2 (B) $2\sqrt{3}$ (C) $4\sqrt{2}$ (D) $\sqrt{6}$ (E) $2\sqrt{2}$

19. The sum of an infinite geometric series with common ratio r such that $|r| < 1$, is 15, and the sum of the squares of the terms of this series is 45. The first term of the series is

(A) 12 (B) 10 (C) 5 (D) 3 (E) 2

20. Lines HK and BC lie in a plane. M is the midpoint of line segment BC, and BH and CK are perpendicular to HK. Then we

(A) always have $MH = MK$ (B) always have $MH > BK$
(C) sometimes have $MH = MK$ but not always
(D) always have $MH > MB$ (E) always have $BH < BC$

PART III (5 credits each)

21. On an auto trip, the distance read from the instrument panel was 450 miles. With snow tires on for the return trip over the same route, the reading was 440 miles. Find, to the nearest hundredth of an inch, the increase in radius of the wheels if the original radius was 15 inches.

(A) .33 (B) .34 (C) .35 (D) .38 (E) .66

22. If the sum of the first 3n positive integers is 150 more than the sum of the first n positive integers, then the sum of the first 4n positive integers is

(A) 300 (B) 350 (C) 400 (D) 450 (E) 600

23. The number 10! (10 is written in base 10), when written in the base 12 system, ends with exactly k zeros. The value of k is

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

24. An equilateral triangle and a regular hexagon have equal perimeters. If the area of the triangle is 2, then the area of the hexagon is

(A) 2 (B) 3 (C) 4 (D) 6 (E) 12

25. For every real number x, let $[x]$ be the greatest integer which is less than or equal to x. If the postal rate for first class mail is six cents for every ounce or portion thereof, then the cost in cents of first-class postage on a letter weighing W ounces is always

(A) $6W$ (B) $6[W]$ (C) $6([W]-1)$ (D) $6([W]+1)$ (E) $-6[-W]$

26. The number of distinct points in the xy-plane common to the graphs of $(x+y-5)(2x-3y+5)=0$ and $(x-y+1)(3x+2y-12)=0$ is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4 (F) infinite

27. In a triangle, the area is numerically equal to the perimeter. What is the radius of the inscribed circle?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

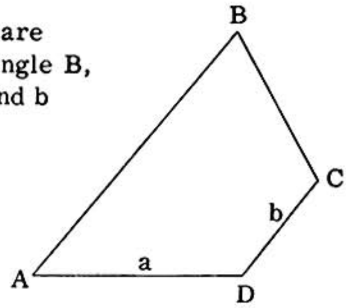
28. In triangle ABC, the median from vertex A is perpendicular to the median from vertex B. If the lengths of sides AC and BC are 6 and 7 respectively, then the length of side AB is

(A) $\sqrt{17}$ (B) 4 (C) $4\frac{1}{2}$ (D) $2\sqrt{5}$ (E) $4\frac{1}{4}$

29. It is now between 10:00 and 11:00 o'clock, and six minutes from now, the minute hand of a watch will be exactly opposite the place where the hour hand was three minutes ago. What is the exact time now?

(A) $10{:}05\frac{5}{11}$ (B) $10{:}07\frac{1}{2}$ (C) 10:10 (D) 10:15 (E) $10{:}17\frac{1}{2}$

30. In the accompanying figure, segments AB and CD are parallel, the measure of angle D is twice that of angle B, and the measures of segments AD and CD are a and b respectively. Then the measure of AB is equal to



(A) $\frac{1}{2}a + 2b$ (B) $\frac{3}{2}b + \frac{3}{4}a$ (C) $2a - b$
(D) $4b - \frac{1}{2}a$ (E) $a + b$

PART IV (6 credits each)

31. If a number is selected at random from the set of all five-digit numbers in which the sum of the digits is equal to 43, what is the probability that this number will be divisible by 11?

(A) 2/5 (B) 1/5 (C) 1/6 (D) 1/11 (E) 1/15

32. A and B travel around a circular track at uniform speeds in opposite directions, starting from diametrically opposite points. If they start at the same time, meet first after B has travelled 100 yards, and meet a second time 60 yards before A completes one lap, then the circumference of the track in yards is

(A) 400 (B) 440 (C) 480 (D) 560 (E) 880

33. Find the sum of the digits of all the numerals in the sequence $1, 2, 3, 4, \cdots, 10000$.

(A) 180,001 (B) 154,756 (C) 45,001 (D) 154,755 (E) 270,001

34. The greatest integer that will divide 13,511, 13,903 and 14,589 and leave the same remainder is

(A) 28 (B) 49 (C) 98 (D) an odd multiple of 7 greater than 49
(E) an even multiple of 7 greater than 98

35. A retiring employee receives an annual pension proportional to the square root of the number of years of his service. Had he served a years more, his pension would have been p dollars greater, whereas, had he served b years more ($b \neq a$), his pension would have been q dollars greater than the original annual pension. Find his annual pension in terms of a, b, p, and q.

(A) $\frac{p^2-q^2}{2(a-b)}$ (B) $\frac{(p-q)^2}{2\sqrt{ab}}$ (C) $\frac{ap^2-bq^2}{2(ap-bq)}$ (D) $\frac{aq^2-bp^2}{2(bp-aq)}$ (E) $\sqrt{(a-b)(p-q)}$

M.A.A. COMMITTEE ON HIGH SCHOOL CONTESTS

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22

TWENTY SECOND ANNUAL MATHEMATICS EXAMINATION 1971

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last name first name middle initial

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school (full name) street address

____

city state zip code

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11	12	13	14	15	16	17	18	19	20

PART III

21	22	23	24	25	26	27	28	29	30

PART IV

31	32	33	34	35

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
IV	6 points each	6 times =	6 times =
	TOTALS	C =	W =
	SCORE = $C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

PART I (3 credits each)

1. The number of digits in the number $N = 2^{12} \times 5^{8}$ is

   (A) 9 (B) 10 (C) 11 (D) 12 (E) 20

2. If b men take c days to lay f bricks, then the number of days it will take c men working at the same rate to lay b bricks, is

   (A) fb^2 (B) b/f^2 (C) f^2/b (D) b^2/f (E) f/b^2

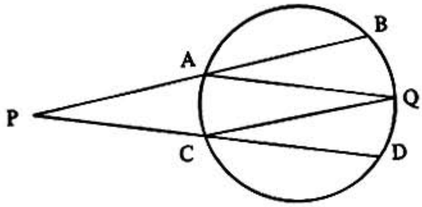
3. If the point $(x,-4)$ lies on the straight line joining the points (0,8) and (–4,0) in the xy-plane, then x is equal to

   (A) –2 (B) 2 (C) –8 (D) 6 (E) –6

4. After simple interest for two months at 5% per annum was credited, a Boy Scout Troop had a total of $255.31 in the Council Treasury. The interest credited was a number of dollars plus the following number of cents

   (A) 11 (B) 12 (C) 13 (D) 21 (E) 31

5. Points A,B,Q,D, and C lie on the circle shown and the measures of arcs $\widehat{BQ}$ and $\widehat{QD}$ are 42° and 38° respectively. The sum of the measures of angles P and Q is



   (A) 80° (B) 62° (C) 40° (D) 46°
   (E) None of these

6. Let $*$ be a symbol denoting the binary operation on the set S of all non-zero real numbers as follows: For any two numbers a and b of S, $a * b = 2ab$. Then the one of the following statements which is not true, is

   (A) $*$ is commutative over S
   (B) $*$ is associative over S
   (C) ½ is an identity element for $*$ in S
   (D) Every element of S has an inverse for $*$
   (E) $\frac{1}{2a}$ is an inverse for $*$ of the element a of S

7. $2^{-(2k+1)} - 2^{-(2k-1)} + 2^{-2k}$ is equal to

   (A) 2^{-2k} (B) $2^{-(2k-1)}$ (C) $-2^{-(2k+1)}$ (D) 0 (E) 2

8. The solution set of $6x^2 + 5x < 4$ is the set of all values of x such that

(A) $-2 < x < 1$ (B) $-\frac{4}{3} < x < \frac{1}{2}$ (C) $-\frac{1}{2} < x < \frac{4}{3}$ (D) $x < \frac{1}{2}$ or $x > -\frac{4}{3}$
(E) $x < -\frac{4}{3}$ or $x > \frac{1}{2}$

9. An uncrossed belt is fitted without slack around two circular pulleys with radii of 14 inches and 4 inches. If the distance between the points of contact of the belt with the pulleys is 24 inches, then the distance between the centers of the pulleys in inches is

(A) 24 (B) $2\sqrt{119}$ (C) 25 (D) 26 (E) $4\sqrt{35}$

10. Each of a group of 50 girls is blonde or brunette and is blue or brown eyed. If 14 are blue-eyed blondes, 31 are brunettes, and 18 are brown-eyed, then the number of brown-eyed brunettes is

(A) 5 (B) 7 (C) 9 (D) 11 (E) 13

PART II (4 credits each)

11. The numeral 47 in base a represents the same number as 74 in base b. Assuming that both bases are positive integers, the least possible value for a + b written as a Roman numeral, is

(A) XIII (B) XV (C) XXI (D) XXIV (E) XVI

12. For each integer $N > 1$, there is a mathematical system in which two or more integers are defined to be congruent if they leave the same non-negative remainder when divided by N. If 69, 90, and 125 are congruent in one such system, then in that same system, 81 is congruent to

(A) 3 (B) 4 (C) 5 (D) 7 (E) 8

13. If $(1.0025)^{10}$ is evaluated correct to 5 decimal places, then the digit in the fifth decimal place is

(A) 0 (B) 1 (C) 2 (D) 5 (E) 8

14. The number $(2^{48} - 1)$ is exactly divisible by two numbers between 60 and 70. These numbers are

(A) 61, 63 (B) 61, 65 (C) 63, 65 (D) 63, 67 (E) 67, 69

15. An aquarium on a level table has rectangular faces and is 10 inches wide and 8 inches high. When it was tilted, the water in it just covered an 8″ × 10″ end but only three-fourths of the rectangular bottom. The depth of the water when the bottom was again made level was

(A) $2\frac{1}{2}''$ (B) $3''$ (C) $3\frac{1}{4}''$ (D) $3\frac{1}{2}''$ (E) $4''$

16. After finding the average of 35 scores, a student carelessly included the average with the 35 scores and found the average of these 36 numbers. The ratio of the second average to the true average was

(A) 1 : 1 (B) 35 : 36 (C) 36 : 35 (D) 2 : 1 (E) None of these

17. A circular disk is divided by 2n equally spaced radii ($n > 0$) and one secant line. The maximum number of non-overlapping areas into which the disk can be divided is

(A) $2n + 1$ (B) $2n + 2$ (C) $3n - 1$ (D) $3n$ (E) $3n + 1$

18. The current in a river is flowing steadily at 3 miles per hour. A motor boat which travels at a constant rate in still water goes downstream 4 miles and then returns to its starting point. The trip takes one hour, excluding the time spent in turning the boat around. The ratio of the downstream to the upstream rate is .

(A) 4 : 3 (B) 3 : 2 (C) 5 : 3 (D) 2 : 1 (E) 5 : 2

19. If the line $y = mx + 1$ intersects the ellipse $x^2 + 4y^2 = 1$ exactly once, then the value of m^2 is

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{3}{4}$ (D) $\frac{4}{5}$ (E) $\frac{5}{6}$

20. The sum of the squares of the roots of the equation $x^2 + 2hx = 3$ is 10. The absolute value of h is equal to

(A) -1 (B) $\frac{1}{2}$ (C) $\frac{3}{2}$ (D) 2 (E) None of these

PART III (5 credits each)

21. If $\log_2(\log_3(\log_4 x)) = \log_3(\log_4(\log_2 y)) = \log_4(\log_2(\log_3 z)) = 0$, then the sum $x + y + z$ is equal to

(A) 50 (B) 58 (C) 89 (D) 111 (E) 1296

22. If w is one of the imaginary roots of the equation $x^3 = 1$, then the product $(1 - w + w^2)(1 + w - w^2)$ is equal to

(A) 4 (B) w (C) 2 (D) w^2 (E) 1

23. Teams A and B are playing a series of games. If the odds for either team to win any game are even and Team A must win two or Team B three games to win the series, then the odds favoring Team A to win the series are

(A) 11 to 5 (B) 5 to 2 (C) 8 to 3 (D) 3 to 2 (E) 13 to 6

24. Pascal's triangle is an array of positive integers (See figure), in which the first row is 1, the second row is two 1's, each row begins and ends with 1, and the kth number in any row when it is not 1, is the sum of the kth and (k − 1)th numbers in the immediately preceding row. The quotient of the number of numbers in the first n rows which are not 1's and the number of 1's is

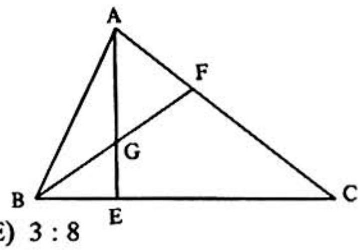
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
etc.

(A) $\frac{n^2 - n}{2n - 1}$ (B) $\frac{n^2 - n}{4n - 2}$ (C) $\frac{n^2 - 2n}{2n - 1}$ (D) $\frac{n^2 - 3n + 2}{4n - 2}$ (E) None of these

25. A teen age boy wrote his own age after his father's. From this new four place number he subtracted the absolute value of the difference of their ages to get 4,289. The sum of their ages was

(A) 48 (B) 52 (C) 56 (D) 59 (E) 64

26. In triangle ABC, point F divides side AC in the ratio 1 : 2. Let E be the point of intersection of side BC and AG where G is the midpoint of BF. Then point E divides side BC in the ratio



(A) 1 : 4 (B) 1 : 3 (C) 2 : 5 (D) 4 : 11 (E) 3 : 8

27. A box contains chips, each of which is red, white, or blue. The number of blue chips is at least half the number of white chips, and at most one third the number of red chips. The number which are white or blue is at least 55. The minimum number of red chips is

(A) 24 (B) 33 (C) 45 (D) 54 (E) 57

28. Nine lines parallel to the base of a triangle divide the other sides each into 10 equal segments and the area into 10 distinct parts. If the area of the largest of these parts is 38, then the area of the original triangle is

(A) 180 (B) 190 (C) 200 (D) 210 (E) 240

29. Given the progression $10^{\frac{1}{11}}, 10^{\frac{2}{11}}, 10^{\frac{3}{11}}, 10^{\frac{4}{11}}, \cdots, 10^{\frac{n}{11}}$. The least positive integer n such that the product of the first n terms of the progression exceeds 100,000 is

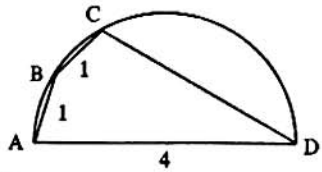
(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

30. Given the linear fractional transformation of x into $f_1(x) = \frac{2x-1}{x+1}$. Define $f_{n+1}(x) = f_1(f_n(x))$ for $n = 1, 2, 3, \cdots$. Assuming that $f_{35}(x) = f_5(x)$, it follows that $f_{28}(x)$ is equal to

(A) x (B) $\frac{1}{x}$ (C) $\frac{x-1}{x}$ (D) $\frac{1}{1-x}$ (E) None of these

PART IV (6 credits each)

31. Quadrilateral ABCD is inscribed in a circle with side AD, a diameter of length 4. If sides AB and BC each have length 1, then side CD has length



(A) $\frac{7}{2}$ (B) $\frac{5\sqrt{2}}{2}$ (C) $\sqrt{11}$ (D) $\sqrt{13}$ (E) $2\sqrt{3}$

32. If $s = (1 + 2^{-\frac{1}{32}})(1 + 2^{-\frac{1}{16}})(1 + 2^{-\frac{1}{8}})(1 + 2^{-\frac{1}{4}})(1 + 2^{-\frac{1}{2}})$, then s is equal to

(A) $\frac{1}{2}(1 - 2^{-\frac{1}{32}})^{-1}$ (B) $(1 - 2^{-\frac{1}{32}})^{-1}$ (C) $1 - 2^{-\frac{1}{32}}$ (D) $\frac{1}{2}(1 - 2^{-\frac{1}{32}})$ (E) $\frac{1}{2}$

33. If P is the product of n quantities in Geometric Progression, S their sum, and S′ the sum of their reciprocals, then P in terms of S, S′, and n is

(A) $(SS')^{\frac{1}{2}n}$ (B) $(S/S')^{\frac{1}{2}n}$ (C) $(SS')^{n-2}$ (D) $(S/S')^{n}$ (E) $(S'/S)^{\frac{1}{2}(n-1)}$

34. An ordinary clock in a factory is running slow so that the minute hand passes the hour hand at the usual dial positions (12 o'clock, etc.) but only every 69 minutes. At time and one-half for overtime, the extra pay to which a \$4.00 per hour worker should be entitled after working a normal 8 hour day by that slow running clock, is

(A) \$2.30 (B) \$2.60 (C) \$2.80 (D) \$3.00 (E) \$3.30

35. Each circle in an infinite sequence with decreasing radii is tangent externally to the one following it and to both sides of a given right angle. The ratio of the area of the first circle to the sum of areas of all other circles in the sequence, is

(A) $(4+3\sqrt{2}):4$ (B) $9\sqrt{2}:2$ (C) $(16+12\sqrt{2}):1$ (D) $(2+2\sqrt{2}):1$
(E) $(3+2\sqrt{2}):1$

M.A.A. COMMITTEE ON HIGH SCHOOL CONTESTS

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Omaha Office: Univ. of Nebraska at Omaha, Omaha, Nebr. 68101
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23

TWENTY THIRD ANNUAL MATHEMATICS EXAMINATION 1972

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box of your answer sheet directly above the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly above No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off this cover along the dotted line inside. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you write the information required in the first three lines.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

*This instruction may be modified if machine-scoring is used.

TUESDAY, MARCH 14, 1972

M.A.A. COMMITTEE ON HIGH SCHOOL CONTESTS

National Office: Univ. of Nebraska, Lincoln, Nebr. 68508
Omaha Office: Univ. of Nebraska at Omaha, Omaha, Nebr. 68101
New York Office: Fred F. Kuhn, 270 Madison Avenue, New York, N.Y. 10016

To be filled in by the student

PRINT

last name ___ first name ___ middle initial ___

school (full name) ___ street address ___

city ___ state ___ zip code ___

PART I

1	2	3	4	5	6	7	8	9	10

PART II

11	12	13	14	15	16	17	18	19	20

PART III

21	22	23	24	25	26	27	28	29	30

PART IV

31	32	33	34	35

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
IV	6 points each	6 times =	6 times =
	TOTALS	C =	W =
	SCORE = $C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

TWENTY THIRD ANNUAL H.S. MATHEMATICS EXAMINATION 1972

PART I (3 credits each)

1. The lengths in inches of the three sides of each of four triangles I, II, III, and IV are as follows:

I	3, 4, and 5	III	7, 24, and 25
II	4, $7\frac{1}{2}$, and $8\frac{1}{2}$	IV	$3\frac{1}{2}$, $4\frac{1}{2}$, and $5\frac{1}{2}$

Of these four given triangles, the only right triangles are

(A) I and II (B) I and III (C) I and IV
(D) I, II, and III (E) I, II, and IV

2. If a dealer could get his goods for 8% less while keeping his selling price fixed, his profit, based on cost, would be increased to $(x + 10)\%$ from his present profit of $x\%$ which is

(A) 12% (B) 15% (C) 30% (D) 50% (E) 75%

3. If $x = \frac{1 - i\sqrt{3}}{2}$ where $i = \sqrt{-1}$, then $\frac{1}{x^2 - x}$ is equal to

(A) -2 (B) -1 (C) $1 + i\sqrt{3}$ (D) 1 (E) 2

4. The number of solutions to $\{1,2\} \subseteq X \subseteq \{1,2,3,4,5\}$ where X is a subset of $\{1,2,3,4,5\}$ is

(A) 2 (B) 4 (C) 6 (D) 8 (E) None of these

5. From among $2^{1/2}$, $3^{1/3}$, $8^{1/8}$, $9^{1/9}$ those which have the greatest and the next to the greatest values in that order, are

(A) $3^{1/3}$, $2^{1/2}$ (B) $3^{1/3}$, $8^{1/8}$ (C) $3^{1/3}$, $9^{1/9}$ (D) $8^{1/8}$, $9^{1/9}$
(E) None of these

6. If $3^{2x} + 9 = 10(3^x)$, then the value of $(x^2 + 1)$ is

(A) 1 only (B) 5 only (C) 1 or 5 (D) 2 (E) 10

7. If $yz:zx:xy = 1:2:3$, then $\frac{x}{yz} : \frac{y}{zx}$ is equal to

(A) 3:2 (B) 1:2 (C) 1:4 (D) 2:1 (E) 4:1

8. If $|x - \log y| = x + \log y$ where x and $\log y$ are real, then

(A) $x = 0$ (B) $y = 1$ (C) $x = 0$ and $y = 1$
(D) $x(y-1) = 0$ (E) None of these

9. Ann and Sue bought identical boxes of stationery. Ann used hers to write 1-sheet letters and Sue used hers to write 3-sheet letters. Ann used all the envelopes and had 50 sheets of paper left, while Sue used all of the sheets of paper and had 50 envelopes left. The number of sheets of paper in each box was

(A) 150 (B) 125 (C) 120 (D) 100 (E) 80

10. For x real, the inequality $1 \le |x - 2| \le 7$ is equivalent to

(A) $x \le 1$ or $x \ge 3$ (B) $1 \le x \le 3$ (C) $-5 \le x \le 9$
(D) $-5 \le x \le 1$ or $3 \le x \le 9$ (E) $-6 \le x \le 1$ or $3 \le x \le 10$

Part II (4 credits each)

11. The value(s) of y for which the following pair of equations

$$x^2 + y^2 - 16 = 0 \text{ and } x^2 - 3y + 12 = 0$$

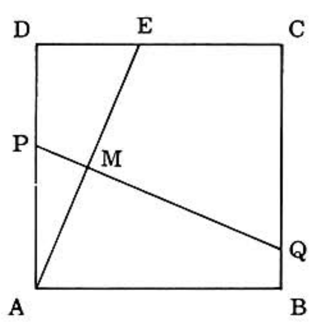
may have a real common solution, are

(A) 4 only (B) $-7, 4$ (C) $0, 4$ (D) no y (E) all y

12. The number of cubic feet in the volume of a cube is the same as the number of square inches in its surface area. The length of the edge expressed as a number of feet is

(A) 6 (B) 864 (C) 1728 (D) 6×1728 (E) 2304

13. Inside square ABCD (See figure) with sides of length 12 inches, segment AE is drawn where E is the point on DC which is 5 inches from D. The perpendicular bisector of AE is drawn and intersects AE, AD, and BC at points M, P, and Q respectively. The ratio of segment PM to MQ is



(A) 5:12 (B) 5:13 (C) 5:19
(D) 1:4 (E) 5:21

14. A triangle has angles of 30° and 45°. If the side opposite the 45° angle has length 8, then the side opposite the 30° angle has length

    (A) 4 (B) $4\sqrt{2}$ (C) $4\sqrt{3}$ (D) $4\sqrt{6}$ (E) 6

15. A contractor estimated that one of his two bricklayers would take 9 hours to build a certain wall and the other 10 hours. However, he knew from experience that when they worked together, their combined output fell by 10 bricks per hour. Being in a hurry, he put both men on the job and found that it took exactly 5 hours to build the wall. The number of bricks in the wall was

    (A) 500 (B) 550 (C) 900 (D) 950 (E) 960

16. There are two positive numbers that may be inserted between 3 and 9 such that the first three are in geometric progression while the last three are in arithmetic progression. The sum of those two positive numbers is

    (A) $13\frac{1}{2}$ (B) $11\frac{1}{4}$ (C) $10\frac{1}{2}$ (D) 10 (E) $9\frac{1}{2}$

17. A piece of string is cut in two at a point selected at random. The probability that the longer piece is at least x times as large as the shorter piece is

    (A) $\frac{1}{2}$ (B) $\frac{2}{x}$ (C) $\frac{1}{x+1}$ (D) $\frac{1}{x}$ (E) $\frac{2}{x+1}$

18. Let ABCD be a trapezoid with the measure of base AB twice that of base DC, and let E be the point of intersection of the diagonals. If the measure of diagonal AC is 11, then that of segment EC is equal to

    (A) $3\frac{2}{3}$ (B) $3\frac{3}{4}$ (C) 4 (D) $3\frac{1}{2}$ (E) 3

19. The sum of the first n terms of the sequence

    $$1,\ (1+2),\ (1+2+2^2),\ \ldots\ (1+2+2^2+\ldots+2^{n-1})$$

    in terms of n is

    (A) 2^n (B) $2^n - n$ (C) $2^{n+1} - n$

    (D) $2^{n+1} - n - 2$ (E) $n \cdot 2^n$

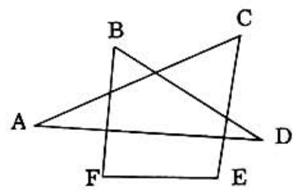
20. If $\tan x = \dfrac{2ab}{a^2 - b^2}$ where $a > b > 0$ and $0° < x < 90°$, then $\sin x$ is equal to

(A) $\dfrac{a}{b}$ (B) $\dfrac{b}{a}$ (C) $\dfrac{\sqrt{a^2 - b^2}}{2a}$ (D) $\dfrac{\sqrt{a^2 - b^2}}{2ab}$ (E) $\dfrac{2ab}{a^2 + b^2}$

PART III (5 credits each)

21. If the sum of the measures in degrees of angles A, B, C, D, E, and F in the figure to the right is 90n, then n is equal to

(A) 2 (B) 3 (C) 4
(D) 5 (E) 6

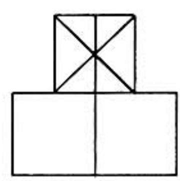


22. If $a \pm bi$ $(b \neq 0)$ are imaginary roots of the equation $x^3 + qx + r = 0$ where a, b, q, and r are real numbers, then q in terms of a and b is

(A) $a^2 + b^2$ (B) $2a^2 - b^2$ (C) $b^2 - a^2$
(D) $b^2 - 2a^2$ (E) $b^2 - 3a^2$

23. The radius of the smallest circle containing the symmetric figure composed of the 3 unit squares shown at the right is

(A) $\sqrt{2}$ (B) $\sqrt{1.25}$ (C) 1.25
(D) $\dfrac{5\sqrt{17}}{16}$ (E) None of these



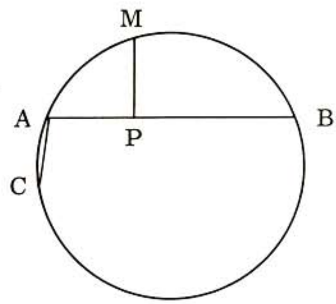
24. A man walked a certain distance at a constant rate. If he had gone $\frac{1}{2}$ mile per hour faster, he would have walked the distance in four-fifths of the time; if he had gone $\frac{1}{2}$ mile per hour slower, he would have been $2\frac{1}{2}$ hours longer on the road. The distance in miles he walked was

(A) $13\frac{1}{2}$ (B) 15 (C) $17\frac{1}{2}$ (D) 20 (E) 25

25. Inscribed in a circle is a quadrilateral having sides of lengths 25, 39, 52, and 60 taken consecutively. The diameter of this circle has length

(A) 62 (B) 63 (C) 65 (D) 66 (E) 69

26. In the circle to the right, M is the mid-point of arc CAB and segment MP is perpendicular to chord AB at P. If the measure of chord AC is x and that of segment AP is (x + 1), then segment PB has measure equal to



(A) $3x + 2$ (B) $3x + 1$
(C) $2x + 3$ (D) $2x + 2$
(E) $2x + 1$

27. If the area of ΔABC is 64 square inches and the geometric mean (mean proportional) between sides AB and AC is 12 inches, then sin A is equal to

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{3}{5}$ (C) $\frac{4}{5}$ (D) $\frac{8}{9}$ (E) $\frac{15}{17}$

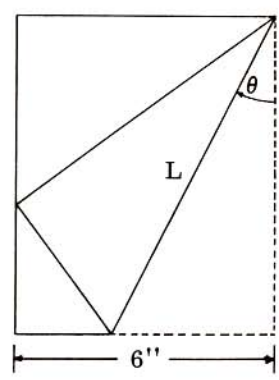
28. A circular disc with diameter D is placed on an 8×8 checkerboard with width D so that the centers coincide. The number of checkerboard squares which are completely covered by the disc is

(A) 48 (B) 44 (C) 40 (D) 36 (E) 32

29. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ for $-1 < x < 1$, then $f\left(\frac{3x + x^3}{1 + 3x^2}\right)$ in terms of $f(x)$ is

(A) $-f(x)$ (B) $2f(x)$ (C) $3f(x)$ (D) $[f(x)]^2$
(E) $[f(x)]^3 - f(x)$

30. A rectangular piece of paper 6 inches wide is folded as in the diagram so that one corner touches the opposite side. The length in inches of the crease L in terms of angle θ is



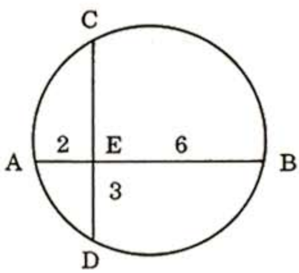
(A) $3 \sec^2 \theta \csc \theta$
(B) $6 \sin \theta \sec \theta$
(C) $3 \sec \theta \csc \theta$
(D) $6 \sec \theta \csc^2 \theta$
(E) None of these

PART IV (6 credits each)

31. When the number 2^{1000} is divided by 13, the remainder in the division is

(A) 1 (B) 2 (C) 3 (D) 7 (E) 11

32. Chords AB and CD in the circle to the right intersect at E and are perpendicular to each other. If segments AE, EB, and ED have measures 2, 6, and 3 respectively, then the length of the diameter of the circle is



(A) $4\sqrt{5}$ (B) $\sqrt{65}$ (C) $2\sqrt{17}$
(D) $3\sqrt{7}$ (E) $6\sqrt{2}$

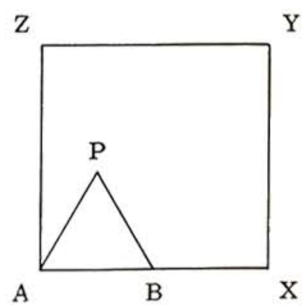
33. The minimum value of the quotient of a (base ten) number of three different nonzero digits divided by the sum of its digits is

(A) 9.7 (B) 10.1 (C) 10.5 (D) 10.9 (E) 20.5

34. Three times Dick's age plus Tom's age equals twice Harry's age. Double the cube of Harry's age is equal to three times the cube of Dick's age added to the cube of Tom's age. Their respective ages are relatively prime to each other. The sum of the squares of their ages is

(A) 42 (B) 46 (C) 122 (D) 290 (E) 326

35. Equilateral triangle ABP (See figure) with side AB of length 2 inches is placed inside square AXYZ with side of length 4 inches so that B is on side AX. The triangle is rotated clockwise about B, then P, and so on along the sides of the square until P returns to its original position. The length of the path in inches traversed by vertex P is equal to



(A) $20\pi/3$ (B) $32\pi/3$ (C) 12π
(D) $40\pi/3$ (E) 15π

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TWENTY FOURTH ANNUAL MATHEMATICS EXAMINATION 1973

24

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box of your answer sheet directly above the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly above No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off this cover along the dotted line inside. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you write the information required in the first three lines.
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*This instruction may be modified if machine-scoring is used.

TUESDAY, MARCH 13, 1973

M.A.A. COMMITTEE ON HIGH SCHOOL CONTESTS

National Office: Univ. of Nebraska, Lincoln, Nebr. 68508
Omaha Office: Univ. of Nebraska at Omaha, Omaha, Nebr. 68101
New York Office: Fred F. Kuhn, 270 Madison Avenue, New York, N.Y. 10016

To be filled in by the student

PRINT

last name | first name | middle initial

school (full name) | street address

city | state | zip code

PART I

1	2	3	4	5	6	7	8	9	10

PART II

11	12	13	14	15	16	17	18	19	20

PART III

21	22	23	24	25	26	27	28	29	30

PART IV

31	32	33	34	35

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
IV	6 points each	6 times =	6 times =
	TOTALS	C =	W =
	SCORE = $C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

TWENTY FOURTH ANNUAL H.S. MATHEMATICS EXAMINATION 1973

PART I (3 credits each)

1. A chord which is the perpendicular bisector of a radius of length 12 in a circle, has length

   (A) $3\sqrt{3}$ (B) 27 (C) $6\sqrt{3}$ (D) $12\sqrt{3}$ (E) None of these

2. One thousand unit cubes are fastened together to form a large cube with edges of 10 units which is painted then separated into the original cubes. The number of these unit cubes which have at least one face painted is

   (A) 600 (B) 520 (C) 488 (D) 480 (E) 400

3. The stronger Goldbach conjecture states that any even whole number greater than 7 can be written as the sum of exactly two different prime numbers. For such representation of even number 126, the largest difference between the two primes is

   (A) 112 (B) 100 (C) 92 (D) 88 (E) 80

4. Two congruent $30°-60°-90°$ triangles are placed so that they overlap partly and their hypotenuses coincide. If the hypotenuse of each triangle is 12, the area common to both triangles is

   (A) $6\sqrt{3}$ (B) $8\sqrt{3}$ (C) $9\sqrt{3}$ (D) $12\sqrt{3}$ (E) 24

5. Of the following five statements, I to V, about the binary operation of averaging (arithmetic mean),

   I. Averaging is associative
   II. Averaging is commutative
   III. Averaging distributes over addition
   IV. Addition distributes over averaging
   V. Averaging has an identity element

   those which are always true, are

   (A) All (B) I and II only (C) II and III only (D) II and IV only (E) II and V only.

6. In a certain scale of notation, the square of 24 is represented by 554. The base of this scale is

   (A) 6 (B) 8 (C) 12 (D) 14 (E) 16

7. The sum of all the integers between 50 and 350 which end in 1, is

   (A) 5880 (B) 5539 (C) 5208 (D) 4877 (E) 4566

8. If 1 pint of paint is needed to paint a statue 6 ft. high, then the number of pints it will take to paint (to the same thickness) 540 copies 1 ft. high, is

(A) 90 (B) 72 (C) 45 (D) 30 (E) 15

9. In Δ ABC with right angle at C, altitude CH and median CM trisect the right angle. If the area of Δ CHM is K, then the area of Δ ABC is

(A) 6K (B) $4\sqrt{3}$ K (C) $3\sqrt{3}$ K (D) 3K (E) 4K

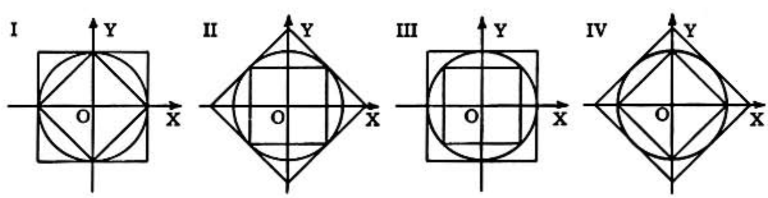
10. If n is a real number, then the simultaneous system to the right has no solution if and only if n is equal to

$$\begin{cases} nx + y = 1 \\ ny + z = 1 \\ x + nz = 1 \end{cases}$$

(A) -1 (B) 0 (C) 1 (D) 0 or 1 (E) $\frac{1}{2}$

PART II (4 credits each)

11. A circle with a circumscribed and an inscribed square centered at the origin O of a rectangular coordinate system with positive x and y axes OX and OY, is shown in each figure I to IV below



The inequalities

$$|x| + |y| \leq \sqrt{2(x^2 + y^2)} \leq 2\,\text{Max}\,(|x|, |y|)$$

are represented geometrically by the figure numbered

(A) I (B) II (C) III (D) IV (E) None of these.

12. The average (arithmetic mean) age of a group consisting of doctors and lawyers is 40. If the doctors average 35 and the lawyers 50 years old, then the ratio of the number of doctors to the number of lawyers is

(A) 3:2 (B) 3:1 (C) 2:3 (D) 2:1 (E) 1:2

13. The fraction $\dfrac{2(\sqrt{2} + \sqrt{6})}{3\sqrt{2 + \sqrt{3}}}$ is equal to

(A) $\frac{2\sqrt{2}}{3}$ (B) 1 (C) $\frac{2\sqrt{3}}{3}$ (D) $\frac{4}{3}$ (E) $\frac{16}{9}$

14. Each valve A, B, and C, when open, releases water into a tank at its own constant rate. With all three valves open, the tank fills in 1 hour, with only valves A and C open it takes 1.5 hour, and with only valves B and C open it takes 2 hours. The number of hours required with only valves A and B open is

(A) 1.1 (B) 1.15 (C) 1.2 (D) 1.25 (E) 1.75

15. A sector with acute central angle θ is cut from a circle of radius 6. The radius of the circle circumscribed about the sector is

(A) $3\cos\theta$ (B) $3\sec\theta$ (C) $3\cos\frac{1}{2}\theta$ (D) $3\sec\frac{1}{2}\theta$ (E) 3

16. If the sum of all the angles except one of a convex polygon is 2190°, then the number of sides of the polygon must be

(A) 13 (B) 15 (C) 17 (D) 19 (E) 21

17. If θ is an acute angle and $\sin\frac{1}{2}\theta = \sqrt{\frac{x-1}{2x}}$, then $\tan\theta$ equals,

(A) x (B) $\frac{1}{x}$ (C) $\frac{\sqrt{x-1}}{x+1}$ (D) $\frac{\sqrt{x^2-1}}{x}$ (E) $\sqrt{x^2-1}$

18. If $p \geq 5$ is a prime number, then 24 divides $p^2 - 1$ without remainder

(A) Never (B) Sometimes only (C) Always (D) only if $p = 5$
(E) None of these

19. Define $n_a!$ for n and a positive to be

$$n_a! = n(n-a)(n-2a)(n-3a)\ldots(n-ka)$$

where k is the greatest integer for which $n > ka$.
Then the quotient $72_8!/18_2!$ is equal to

(A) 4^5 (B) 4^6 (C) 4^8 (D) 4^9 (E) 4^{12}

20. A cowboy is 4 miles south of a stream which flows due east. He is also 8 miles west and 7 miles north of his cabin. He wishes to water his horse at the stream and return home. The shortest distance (in miles) he can travel and accomplish this is

(A) $4+\sqrt{185}$ (B) 16 (C) 17 (D) 18 (E) $\sqrt{32}+\sqrt{137}$

PART III (5 credits each)

21. The number of sets of two or more consecutive positive integers whose sum is 100, is

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

22. The set of all real solutions of the inequality $|x-1| + |x+2| < 5$ is

(A) $\{x: -3 < x < 2\}$ (B) $\{x: -1 < x < 2\}$ (C) $\{x: -2 < x < 1\}$
(D) $\{x: -\frac{3}{2} < x < \frac{7}{2}\}$ (E) ϕ (empty)

23. There are two cards; one is red on both sides and the other is red on one side and blue on the other. The cards have the same probability ($\frac{1}{2}$) of being chosen, and one is chosen and placed on the table. If the upper-side of the card on the table is red, then the probability that the under-side is also red, is

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) $\frac{3}{4}$

24. The check for a luncheon of 3 sandwiches, 7 cups of coffee and one piece of pie came to \$3.15. The check for a luncheon consisting of 4 sandwiches, 10 cups of coffee and one piece of pie came to \$4.20 at the same place. The cost of a luncheon consisting of one sandwich, one cup of coffee and one piece of pie at the same place will come to

(A) \$1.70 (B) \$1.65 (C) \$1.20 (D) \$1.05 (E) \$.95

25. A circular grass plot 12 feet in diameter is cut by a straight gravel path 3 feet wide, one edge of which passes through the center of the plot. The number of square feet in the remaining grass area is

(A) $36\pi - 34$ (B) $30\pi - 15$ (C) $36\pi - 33$ (D) $35\pi - 9\sqrt{3}$
(E) $30\pi - 9\sqrt{3}$

26. The number of terms in an A.P. (Arithmetic Progression) is even. The sums of the odd and even numbered terms are 24 and 30 respectively. If the last term exceeds the first by 10.5, the number of terms in the A.P. is

(A) 20 (B) 18 (C) 12 (D) 10 (E) 8

27. Cars A and B travel the same distance. Car A travels half that *distance* at u miles per hour and half at v miles per hour. Car B travels half the *time* at u miles per hour and half at v miles per hour. The average speed of Car A is x miles per hour and that of Car B is y miles per hour: Then we always have

(A) $x \le y$ (B) $x \ge y$ (C) $x = y$ (D) $x < y$ (E) $x > y$

28. If a, b, and c are in geometric progression (G.P.) with $0 < a < b < c$ and $n > 1$ is an integer, then $\log_a n, \log_b n, \log_c n$ form a sequence

(A) which is a G.P. (B) which is an arithmetic progression (A.P.)
(C) in which the reciprocals of the terms forms an A.P.
(D) in which the second and third terms are the n^{th} powers of the first and second respectively (E) None of these.

29. Two boys start moving from the same point A on a circular track but in opposite directions. Their speeds are 5 ft. per sec. and 9 ft. per sec. If they start at the same time and finish when they first meet at the point A again, then the number of times they meet between the start and finish is

(A) 13 (B) 25 (C) 44 (D) infinitely many
(E) None of these.

30. Let $[t]$ denote the greatest integer $\leq t$ where $t \geq 0$ and $S = \{(x, y) : (x - T)^2 + y^2 \leq T^2 \text{ where } T = t - [t]\}$. Then we have

(A) The point (0, 0) does not belong to S for any t.
(B) The area of S is bounded by 0 and π.
(C) S is contained in the first quadrant for all $t \geq 5$.
(D) The center of S for any t is on the line $y = x$.
(E) None of the other statements is true.

PART IV (6 credits each)

31. In the following equation, each of the letters represents uniquely a different digit and in base ten,
(YE) · (ME) = TTT with the number YE less than ME in the product on the left.
The sum E + M + T + Y equals

(A) 19 (B) 20 (C) 21 (D) 22 (E) 24

32. The volume of a regular pyramid whose base is an equilateral triangle of side length 6 and whose other edges are each of length $\sqrt{15}$, is

(A) 9 (B) 9/2 (C) 27/2 (D) $\frac{9\sqrt{3}}{2}$ (E) None of these.

33. When one ounce of water is added to a mixture of acid and water, the new mixture is 20% acid. When one ounce of acid is added to the new mixture, the result is $33\frac{1}{3}\%$ acid. The percentage of acid in the original mixture is

(A) 22% (B) 24% (C) 25% (D) 30% (E) $33\frac{1}{3}\%$

34. A plane flew straight against a wind between two towns in 84 minutes and returned with that wind in 9 minutes less than it would take in still air. The number of minutes (2 answers) for the return trip was

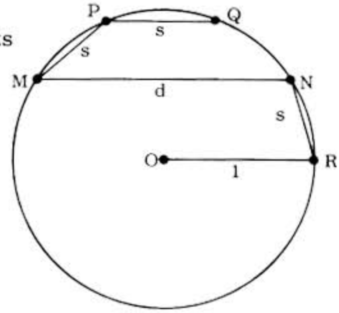
(A) 54 or 18 (B) 60 or 15 (C) 63 or 12 (D) 72 or 36
(E) 75 or 20

35. In the unit circle shown in the figure to the right, chords PQ and MN are parallel to the unit radius OR of the circle with center at 0. Chords MP, PQ and NR are each s units long and chord MN is d units long. Of the three equations

I. $d - s = 1$, II. $ds = 1$, III. $d^2 - s^2 = \sqrt{5}$

those which are necessarily true are

(A) I only (B) II only (C) III only
(D) I and II only (E) I, II, and III.



THE SECOND U. S. A. MATHEMATICAL OLYMPIAD WILL BE HELD ON TUESDAY, MAY 1, 1973. PARTICIPATION IS BY INVITATION, BASED TO A LARGE EXTENT ON YOUR PERFORMANCE ON THE 1973 ANNUAL HIGH SCHOOL MATHEMATICS EXAMINATION.

TWENTY FIFTH ANNUAL MATHEMATICS EXAMINATION 1974

Sponsored Jointly by the

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TUESDAY, MARCH 12, 1974

M.A.A. COMMITTEE ON HIGH SCHOOL CONTESTS

Chairman: City College, CUNY, 138th St. at Convent Ave., New York, N.Y. 10031

Executive Director: Univ. of Nebraska, Lincoln, Nebr. 68508

Publisher: Fred F. Kuhn Associates, 3690 N.W. 50th St., Miami, Fla. 33142

To be filled in by the student

PRINT ____

last name | first name | middle initial

____

school (full name) | street address

____

city | state | zip code

1	2	3	4	5	6	7	8	9	10

11	12	13	14	15	16	17	18	19	20

21	22	23	24	25	26	27	28	29	30

Not to be filled in by student

Each question has a value of *five* points. *One* point is to be deducted for each wrong answer (do *not* include omissions). If the number of points deducted for incorrect answers exceeds the number of points for correct answers the student is to receive a score of 0.

Number Correct [ ] X 5 Points Correct [ ]

− Number Wrong [ ]

Total Score [ ]

TWENTY FIFTH ANNUAL H.S. MATHEMATICS EXAMINATION 1974

1. If $x \neq 0$ or 4 and $y \neq 0$ or 6, then $\frac{2}{x} + \frac{3}{y} = \frac{1}{2}$ is equivalent to

(A) $4x + 3y = xy$

(B) $y = \frac{4x}{6-y}$

(C) $\frac{x}{2} + \frac{y}{3} = 2$

(D) $\frac{4y}{y-6} = x$

(E) none of these

2. Let x_1 and x_2 be such that $x_1 \neq x_2$ and $3x_i^2 - hx_i = b, i = 1,2$. Then $x_1 + x_2$ equals

(A) $-\frac{h}{3}$ (C) $\frac{b}{3}$ (E) $-\frac{b}{3}$

(B) $\frac{h}{3}$ (D) 2b

3. The coefficient of x^7 in the polynomial expansion of $(1 + 2x - x^2)^4$ is

(A) −8 (C) 6 (E) none of these

(B) 12 (D) −12

4. What is the remainder when $x^{51} + 51$ is divided by $x + 1$?

(A) 0 (C) 49 (E) 51

(B) 1 (D) 50

5. Given a quadrilateral ABCD inscribed in a circle with side AB extended beyond B to point E, if $\measuredangle BAD = 92^\circ$ and $\measuredangle ADC = 68^\circ$, find $\measuredangle EBC$

(A) 66° (C) 70° (E) 92°

(B) 68° (D) 88°

6. For positive real numbers x and y define $x * y = \frac{x \cdot y}{x+y}$; then

(A) "*" is commutative but not associative

(B) "*" is associative but not commutative

(C) "*" is neither commutative nor associative

(D) "*" is commutative and associative

(E) none of these

7. A town's population increased by 1,200 people, and then this new population decreased by 11%. The town now had 32 less people than it did before the 1,200 increase. What is the original population?

(A) 1,200
(B) 11,200
(C) 9,968
(D) 10,000
(E) none of these

8. What is the smallest prime number dividing the sum $3^{11} + 5^{13}$?

(A) 2
(B) 3
(C) 5
(D) $3^{11} + 5^{13}$
(E) none of these

9. The integers greater than one are arranged in five columns as follows:

	2	3	4	5
9	8	7	6	
	10	11	12	13
17	16	15	14	
	.	.	.	.

(Four consecutive integers appear in each row; in the first, third and other odd numbered rows, the integers appear in the last four columns and increase from left to right; in the second, fourth and other even numbered rows, the integers appear in the first four columns and increase from right to left.)

In which column will the number 1,000 fall?

(A) first
(B) second
(C) third
(D) fourth
(E) fifth

10. What is the smallest integral value of k such that $2x(kx - 4) - x^2 + 6 = 0$ has no real roots?

(A) -1
(B) 2
(C) 3
(D) 4
(E) 5

11. If (a,b) and (c,d) are two points on the line whose equation is $y = mx + k$, then the distance between (a,b) and (c,d), in terms of a,c and m, is

(A) $|a - c| \sqrt{1 + m^2}$
(B) $|a + c| \sqrt{1 + m^2}$
(C) $\frac{|a - c|}{\sqrt{1 + m^2}}$
(D) $|a - c| (1 + m^2)$
(E) $|a - c| \, |m|$

12. If $g(x) = 1 - x^2$ and $f(g(x)) = \frac{1-x^2}{x^2}$ when $x \neq 0$, then $f(1/2)$ equals

(A) 3/4 (B) 1 (C) 3 (D) $\sqrt{2}/2$ (E) $\sqrt{2}$

13. Which of the following is equivalent to "If P is true then Q is false."?

(A) "P is true or Q is false."
(B) "If Q is false then P is true."
(C) "If P is false then Q is true."
(D) "If Q is true then P is false."
(E) "If Q is true then P is true."

14. Which statement is correct?

(A) If $x < 0$, then $x^2 > x$.
(B) If $x^2 > 0$, then $x > 0$.
(C) If $x^2 > x$, then $x > 0$.
(D) If $x^2 > x$, then $x < 0$.
(E) If $x < 1$, then $x^2 < x$.

15. If $x < -2$ then $|1 - |1 + x||$ equals

(A) $2 + x$ (B) $-2 - x$ (C) x (D) $-x$ (E) -2

16. A circle of radius r is inscribed in a right isosceles triangle, and a circle of radius R is circumscribed about the triangle. Then $\frac{R}{r}$ equals

(A) $1 + \sqrt{2}$ (B) $\frac{2+\sqrt{2}}{2}$ (C) $\frac{\sqrt{2}-1}{2}$ (D) $\frac{1+\sqrt{2}}{2}$ (E) $2(2 - \sqrt{2})$

17. If $i^2 = -1$, then $(1 + i)^{20} - (1 - i)^{20}$ equals

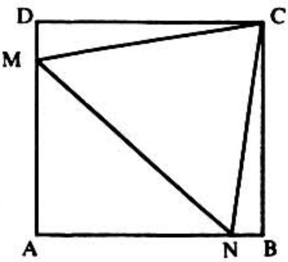
(A) -1024 (B) $-1024i$ (C) 0 (D) 1024 (E) $1024i$

18. If $\log_8 3 = p$ and $\log_3 5 = q$, then, in terms of p and q, $\log_{10} 5$ equals

(A) pq (C) $\frac{1+3pq}{p+q}$ (E) $p^2 + q^2$

(B) $\frac{3p+q}{5}$ (D) $\frac{3pq}{1+3pq}$

19. In the adjoining figure ABCD is a square and CMN is an equilateral triangle. If the area of ABCD is one square inch, then the area of CMN in square inches is



(A) $2\sqrt{3} - 3$ (D) $\sqrt{2}/3$
(B) $1 - \sqrt{3}/3$ (E) $4 - 2\sqrt{3}$
(C) $\sqrt{3}/4$

20. Let $T = \frac{1}{3-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-2}$;

then

(A) $T < 1$
(B) $T = 1$
(C) $1 < T < 2$
(D) $T > 2$
(E) $T = \frac{1}{(3-\sqrt{8})(\sqrt{8}-\sqrt{7})(\sqrt{7}-\sqrt{6})(\sqrt{6}-\sqrt{5})(\sqrt{5}-2)}$

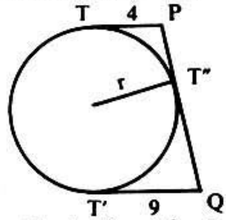
21. In a geometric series of positive terms the difference between the fifth and fourth terms is 576, and the difference between the second and first terms is 9. What is the sum of the first five terms of this series?

(A) 1061 (C) 1024 (E) none of these
(B) 1023 (D) 768

22. The minimum value of $\sin\frac{A}{2} - \sqrt{3}\cos\frac{A}{2}$ is attained when A is

(A) $-180°$ (C) $120°$ (E) none of these
(B) $60°$ (D) $0°$

23. In the adjoining figure TP and T′Q are parallel tangents to a circle of radius r, with T and T′ the points of tangency. PT″Q is a third tangent with T″ as point of tangency. If TP = 4 and T′Q = 9 then r is

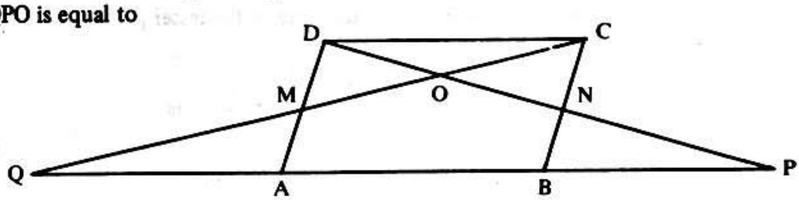


(A) 25/6
(B) 6
(C) 25/4
(D) a number other than 25/6, 6, 25/4
(E) not determinable from the given information

24. A fair die is rolled six times. The probability of rolling at least a five at least five times is

(A) 13/729
(B) 12/729
(C) 2/729
(D) 3/729
(E) none of these

25. In parallelogram ABCD of the accompanying diagram, line DP is drawn bisecting BC at N and meeting AB (extended) at P. From vertex C, line CQ is drawn bisecting side AD at M and meeting AB (extended) at Q. Lines DP and CQ meet at O. If the area of parallelogram ABCD is k, then the area of triangle QPO is equal to



(A) k
(B) 6k/5
(C) 9k/8
(D) 5k/4
(E) 2k

26. The number of distinct positive integral divisors of $(30)^4$ excluding 1 and $(30)^4$ is

(A) 100
(B) 125
(C) 123
(D) 30
(E) none of these

27. If $f(x) = 3x + 2$ for all real x, then the statement:
"$|f(x) + 4| < a$ whenever $|x + 2| < b$ and $a > 0$ and $b > 0$" is true when

(A) $b \leq a/3$
(B) $b > a/3$
(C) $a \leq b/3$
(D) $a > b/3$
(E) The statement is never true

28. Which of the following is satisfied by all numbers x of the form

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \ldots + \frac{a_{25}}{3^{25}} \quad ,$$

where a_1 is 0 or 2, a_2 is 0 or 2, . . . , a_{25} is 0 or 2?

(A) $0 \leqslant x < 1/3$
(B) $1/3 \leqslant x < 2/3$
(C) $2/3 \leqslant x < 1$
(D) $0 \leqslant x < 1/3$ or $2/3 \leqslant x < 1$
(E) $1/2 \leqslant x \leqslant 3/4$

29. For p = 1, 2, . . . , 10 let S_p be the sum of the first 40 terms of the arithmetic progression whose first term is p and whose common difference is $2p-1$; then

$S_1 + S_2 + \ldots + S_{10}$ is

(A) 80,000 (B) 80,200 (C) 80,400 (D) 80,600 (E) 80,800

30. A line segment is divided so that the lesser part is to the greater part as the greater part is to the whole. If R is the ratio of the lesser part to the greater part, then the value of

$$R^{[R^{(R^2 + \frac{1}{R})} + \frac{1}{R}]} + \frac{1}{R} \quad \text{is}$$

(A) 2 (B) 2R (C) 1/R (D) 2 + 1/R (E) 2 + R

THE THIRD U. S. A. MATHEMATICAL OLYMPIAD WILL BE HELD ON TUESDAY, MAY 7, 1974. PARTICIPATION IS BY INVITATION, BASED TO A LARGE EXTENT ON YOUR PERFORMANCE ON THE 1974 ANNUAL HIGH SCHOOL MATHEMATICS EXAMINATION.

Printed in U.S.A.

Sponsored Jointly by

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TWENTY SIXTH ANNUAL MATHEMATICS EXAMINATION

26

1975

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box of your answer sheet directly above the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly above No. 3. Fill in the answers as you find them.
4. *There is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted. *Slide rules and calculators are not permitted.*
6. *When your teacher gives the signal tear off this cover along the dotted line inside. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you write the information required in the first three lines.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

*This instruction may be modified if machine-scoring is used.

TUESDAY, MARCH 11, 1975

COMMITTEE ON HIGH SCHOOL CONTESTS

Chairman: Mathematics Department, City College, CUNY, 138th St. at Convent Ave., New York, N.Y. 10031

Executive Director: Oldfather Hall, Univ. of Nebraska, Lincoln, Nebr. 68508

Olympiad Subcommittee Chairman: Hill Mathematical Center, Rutgers University, New Brunswick, N.J. 08903

Publisher: Fred F. Kuhn Associates, 3690 N.W. 50th St., Miami, Fla. 33142

To be filled in by the student

PRINT ____

last name | first name | middle initial

____

school (full name) | street address

____

city | state | zip code

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30

Not to be filled in by the student

Each question has a value of *five points. One* point is to be deducted for each wrong answer (do *not* deduct points for questions left unanswered). If the number of points deducted for incorrect answers exceeds the number of points for correct answers the student is to receive a score of 0.

Number Correct [ ] X 5 Points Correct [ ]

− Number Wrong [ ]

Total Score [ ]

1. The value of $\dfrac{1}{2-\dfrac{1}{2-\dfrac{1}{2-\dfrac{1}{2}}}}$ is

(A) 3/4 (B) 4/5 (C) 5/6 (D) 6/7 (E) 6/5

2. For which real values of m are the simultaneous equations

$$y = mx + 3$$
$$y = (2m - 1)x + 4$$

satisfied by at least one pair of real numbers (x, y)?

(A) all m (C) all $m \neq 1/2$ (E) no values of m
(B) all $m \neq 0$ (D) all $m \neq 1$

3. Which of the following inequalities are satisfied for all real numbers a, b, c, x, y, z which satisfy the conditions $x<a, y<b$, and $z<c$?

I. $xy + yz + zx < ab + bc + ca$
II. $x^2 + y^2 + z^2 < a^2 + b^2 + c^2$
III. $xyz < abc$

(A) None are satisfied. (C) II only (E) All are satisfied.
(B) I only (D) III only

4. If the side of one square is the diagonal of a second square, what is the ratio of the area of the first square to the area of the second?

(A) 2 (B) $\sqrt{2}$ (C) 1/2 (D) $2\sqrt{2}$ (E) 4

5. The polynomial $(x + y)^9$ is expanded in decreasing powers of x. The second and third terms have equal value when evaluated at $x = p$ and $y = q$, where p and q are positive numbers whose sum is one. What is the value of p?

(A) 1/5 (B) 4/5 (C) 1/4 (D) 3/4 (E) 8/9

6. The sum of the first eighty positive odd integers subtracted from the sum of the first eighty positive even integers is

(A) 0 (B) 20 (C) 40 (D) 60 (E) 80

7. For which non-zero real numbers x is $\frac{|x - |x||}{x}$ a positive integer?

(A) for negative x only
(B) for positive x only
(C) only for x an even integer
(D) for all non-zero real numbers x
(E) for no non-zero real numbers x

8. If the statement "All shirts in this store are on sale." is false, then which of the following statements must be true?

I. All shirts in this store are not on sale.
II. There is some shirt in this store not on sale.
III. No shirt in this store is on sale.
IV. Not all shirts in this store are on sale.

(A) II only
(B) IV only
(C) I and III only
(D) II and IV only
(E) I, II and IV only

9. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be arithmetic progressions such that $a_1 = 25$, $b_1 = 75$ and $a_{100} + b_{100} = 100$. Find the sum of the first one hundred terms of the progression $a_1 + b_1, a_2 + b_2, \ldots$.

(A) 0
(B) 100
(C) 10,000
(D) 505,000
(E) not enough information given to solve the problem

10. The sum of the digits in base ten of $(10^{4n^2+8} + 1)^2$, where n is a positive integer, is

(A) 4 (B) $4n$ (C) $2 + 2n$ (D) $4n^2$ (E) $n^2 + n + 2$

11. Let P be an interior point of circle K other than the center of K. Form all chords of K which pass through P, and determine their midpoints. The locus of these midpoints is

(A) a circle with one point deleted
(B) a circle if the distance from P to the center of K is less than one half the radius of K; otherwise a circular arc of less than 360°
(C) a semicircle with one point deleted
(D) a semicircle
(E) a circle

12. If $a \neq b, a^3 - b^3 = 19x^3$ and $a - b = x$, which of the following conclusions is correct?

(A) $a = 3x$ (C) $a = -3x$ or $a = 2x$ (E) $a = 2x$
(B) $a = 3x$ or $a = -2x$ (D) $a = 3x$ or $a = 2x$

13. The equation $x^6 - 3x^5 - 6x^3 - x + 8 = 0$ has

(A) no real roots
(B) exactly two distinct negative roots
(C) exactly one negative root
(D) no negative roots, but at least one positive root
(E) none of these

14. If the *whatsis* is *so* when the *whosis* is *is* and the *so* and *so* is *is·so*, what is the *whosis·whatsis* when the *whosis* is *so*, the *so* and *so* is *so·so*, and the *is* is two (*whatsis*, *whosis*, *is* and *so* are variables taking positive values)?

(A) *whosis·is·so* (B) *whosis* (C) *is* (D) *so* (E) *so* and *so*

15. In the sequence of numbers 1, 3, 2, . . . each term after the first two is equal to the term preceding it minus the term preceding that. The sum of the first one hundred terms of the sequence is

(A) 5 (B) 4 (C) 2 (D) 1 (E) −1

16. If the first term of an infinite geometric series is a positive integer, the common ratio is the reciprocal of a positive integer, and the sum of the series is 3, then the sum of the first two terms of the series is

(A) 1/3 (B) 2/3 (C) 8/3 (D) 2 (E) 9/2

17. A man can commute either by train or by bus. If he goes to work on the train in the morning, he comes home on the bus in the afternoon; and if he comes home in the afternoon on the train, he took the bus in the morning. During x days the man rode the train 9 times and rode the bus 8 times in the morning and 15 times in the afternoon. Find x.

(A) 19 (C) 17 (E) not enough information given to solve the problem
(B) 18 (D) 16

18. A positive integer N with three digits in its base ten representation is chosen at random, with each three digit number having an equal chance of being chosen. The probability that $\log_2 N$ is an integer is

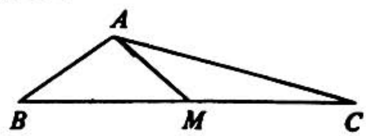
(A) 0 (B) 3/899 (C) 1/225 (D) 1/300 (E) 1/450

19. Which positive numbers x satisfy the equation $(\log_3 x)(\log_x 5) = \log_3 5$?

(A) 3 and 5 only
(B) 3, 5 and 15 only
(C) only numbers of the form $5^n \cdot 3^m$, where n and m are positive integers
(D) all positive $x \neq 1$
(E) none of these

20. In the adjoining figure triangle ABC is such that $AB = 4$ and $AC = 8$. If M is the midpoint of BC and $AM = 3$, what is the length of BC?

(A) $2\sqrt{26}$
(B) $2\sqrt{31}$
(C) 9
(D) $4 + 2\sqrt{13}$
(E) not enough information given to solve the problem



21. Suppose $f(x)$ is defined for all real numbers x; $f(x) > 0$ for all x; and $f(a)f(b) = f(a + b)$ for all a and b. Which of the following statements are true?

I. $f(0) = 1$
II. $f(-a) = 1/f(a)$ for all a
III. $f(a) = \sqrt[3]{f(3a)}$ for all a
IV. $f(b) > f(a)$ if $b > a$

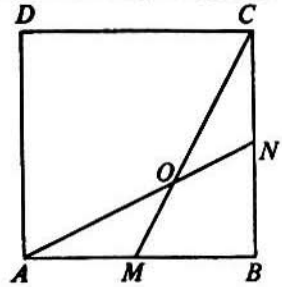
(A) III and IV only
(B) I, III and IV only
(C) I, II and IV only
(D) I, II and III only
(E) All are true.

22. If p and q are prime integers and $x^2 - px + q = 0$ has distinct positive integral roots, then which of the following statements are true?

I. The difference of the roots is odd.
II. At least one root is prime.
III. $p^2 - q$ is prime.
IV. $p + q$ is prime.

(A) I only
(B) II only
(C) II and III only
(D) I, II and IV only
(E) All are true.

23. In the adjoining figure AB and BC are adjacent sides of square $ABCD$; M is the midpoint of AB; N is the midpoint of BC; and AN and CM intersect at O. The ratio of the area of $AOCD$ to the area of $ABCD$ is



(A) 5/6 (B) 3/4 (C) 2/3 (D) $\sqrt{3}/2$ (E) $(\sqrt{3}-1)/2$

24. In triangle ABC, $\measuredangle C = \theta$ and $\measuredangle B = 2\theta$, where $0^\circ < \theta < 60^\circ$. The circle with center A and radius AB intersects AC at D and intersects BC, extended if necessary, at B and at E (E may coincide with B). Then $EC = AD$

(A) for no values of θ
(B) only if $\theta = 45^\circ$
(C) only if $0^\circ < \theta \leq 45^\circ$
(D) only if $45^\circ \leq \theta < 60^\circ$
(E) for all θ such that $0^\circ < \theta < 60^\circ$

25. A woman, her brother, her son and her daughter (all relations by birth) are chess players. The worst player's twin (who is one of the four players) and the best player are of opposite sex. The worst player and the best player are the same age. Who is the worst player?

(A) the woman
(B) her son
(C) her brother
(D) her daughter
(E) No solution is consistent with the given information.

26. In acute triangle ABC the bisector of $\measuredangle A$ meets side BC at D. The circle with center B and radius BD intersects side AB at M; and the circle with center C and radius CD intersects side AC at N. Then it is always true that

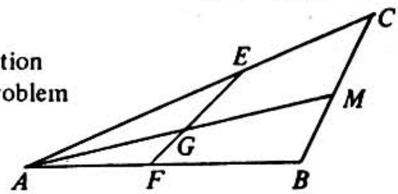
(A) $\measuredangle CND + \measuredangle BMD - \measuredangle DAC = 120^\circ$
(B) $AMDN$ is a trapezoid
(C) BC is parallel to MN
(D) $AM - AN = \dfrac{3(DB - DC)}{2}$
(E) $AB - AC = \dfrac{3(DB - DC)}{2}$

27. If p, q and r are distinct roots of $x^3 - x^2 + x - 2 = 0$, then $p^3 + q^3 + r^3$ equals

(A) -1 (B) 1 (C) 3 (D) 5 (E) none of these

28. In triangle ABC shown in the adjoining figure, M is the midpoint of side BC, $AB = 12$ and $AC = 16$. Points E and F are taken on AC and AB, respectively, and lines EF and AM intersect at G. If $AE = 2AF$ then EG/GF equals

(A) 3/2 (D) 6/5
(B) 4/3 (E) not enough information given to solve the problem
(C) 5/4



29. What is the smallest integer larger than $(\sqrt{3}+\sqrt{2})^6$?

(A) 972 (B) 971 (C) 970 (D) 969 (E) 968

30. Let $x = \cos 36° - \cos 72°$. Then x equals

(A) 1/3 (B) 1/2 (C) $3-\sqrt{6}$ (D) $2\sqrt{3}-3$ (E) none of these

THE FOURTH U.S.A. MATHEMATICAL OLYMPIAD WILL BE HELD ON TUESDAY, MAY 6, 1975. PARTICIPATION IS BY INVITATION, BASED TO A LARGE EXTENT ON PERFORMANCE ON THE 1975 ANNUAL HIGH SCHOOL MATHEMATICS EXAMINATION.

THE XVII INTERNATIONAL MATHEMATICAL OLYMPIAD WILL BEGIN JULY 7, 1975.

Printed in U.S.A.

Sponsored Jointly by
MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
and
CASUALTY ACTUARIAL SOCIETY

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27

TWENTY SEVENTH ANNUAL MATHEMATICS EXAMINATION 1976

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box of your answer sheet directly above the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly above No. 3. Fill in the answers as you find them.
4. *There is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted. *Slide rules and calculators are not permitted.*
6. *When your teacher gives the signal tear off this cover along the dotted line inside. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you write the information required in the first three lines.
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*This instruction may be modified if machine-scoring is used.

TUESDAY, MARCH 9, 1976

COMMITTEE ON HIGH SCHOOL CONTESTS

Chairman: Mathematics Department, City College, CUNY,
138th St. at Convent Ave., New York, N.Y. 10031
Executive Director: Oldfather Hall, Univ. of Nebraska, Lincoln, Nebr. 68508
Olympiad Subcommittee Chairman: Hill Mathematical Center,
Rutgers University, New Brunswick, N.J. 08903
Publisher: Kuhn Associates, Inc. P.O. Box 6878, Hollywood, Fla. 33020

TWENTY SEVENTH ANNUAL MATHEMATICS EXAMINATION 1976

To be filled in by the student

PRINT ____

last name | first name | middle initial

____

school (full name) | street address

____

city | state | zip code

At what date do you anticipate entering college full time? ________ ________

month year

1	2	3	4	5	6	7	8	9	10

11	12	13	14	15	16	17	18	19	20

21	22	23	24	25	26	27	28	29	30

Not to be filled in by the student

Each question has a value of *five points. One* point is to be deducted for each wrong answer (do *not* deduct points for questions left unanswered). If the number of points deducted for incorrect answers exceeds the number of points for correct answers the student is to receive a score of 0.

Number Correct ☐ X 5 Points Correct ☐

–Number Wrong ☐

Total Score ☐

1. If one minus the reciprocal of $(1-x)$ equals the reciprocal of $(1-x)$, then x equals

   (A) -2 (B) -1 (C) $1/2$ (D) 2 (E) 3

2. For how many real numbers x is $\sqrt{-(x+1)^2}$ a real number?

   (A) none (B) one (C) two (D) a finite number greater than two (E) infinitely many

3. The sum of the distances from one vertex of a square with sides of length two to the midpoints of each of the sides of the square is

   (A) $2\sqrt{5}$ (B) $2+\sqrt{3}$ (C) $2+2\sqrt{3}$ (D) $2+\sqrt{5}$ (E) $2+2\sqrt{5}$

4. Let a geometric progression with n terms have first term one, common ratio r and sum s, where r and s are not zero. The sum of the geometric progression formed by replacing each term of the original progression by its reciprocal is

   (A) $\frac{1}{s}$ (B) $\frac{1}{r^n s}$ (C) $\frac{s}{r^{n-1}}$ (D) $\frac{r^n}{s}$ (E) $\frac{r^{n-1}}{s}$

5. How many integers greater than ten and less than one hundred, written in base ten notation, are increased by nine when their digits are reversed?

   (A) 0 (B) 1 (C) 8 (D) 9 (E) 10

6. If c is a real number and the negative of one of the solutions of $x^2-3x+c=0$ is a solution of $x^2+3x-c=0$, then the solutions of $x^2-3x+c=0$ are

   (A) $1, 2$ (B) $-1, -2$ (C) $0, 3$ (D) $0, -3$ (E) $\frac{3}{2}, \frac{3}{2}$

7. If x is a real number, then the quantity $(1-|x|)(1+x)$ is positive if and only if

   (A) $|x|<1$ (B) $x<1$ (C) $|x|>1$ (D) $x<-1$ (E) $x<-1$ or $-1<x<1$

8. A point in the plane, both of whose rectangular coordinates are integers with absolute value less than or equal to four, is chosen at random, with all such points having an equal probability of being chosen. What is the probability that the distance from the point to the origin is at most two units?

   (A) $\frac{13}{81}$ (B) $\frac{15}{81}$ (C) $\frac{13}{64}$ (D) $\frac{\pi}{16}$ (E) the square of a rational number

9. In triangle ABC, D is the midpoint of AB; E is the midpoint of DB; and F is the midpoint of BC. If the area of ΔABC is 96, then the area of ΔAEF is

(A) 16 (B) 24 (C) 32 (D) 36 (E) 48

10. If m, n, p and q are real numbers and $f(x) = mx + n$ and $g(x) = px + q$, then the equation $f(g(x)) = g(f(x))$ has a solution

(A) for all choices of m, n, p and q
(B) if and only if $m = p$ and $n = q$
(C) if and only if $mq - np = 0$
(D) if and only if $n(1 - p) - q(1 - m) = 0$
(E) if and only if $(1 - n)(1 - p) - (1 - q)(1 - m) = 0$

11. Which of the following statements is (are) equivalent to the statement "If the pink elephant on planet alpha has purple eyes, then the wild pig on planet beta does not have a long nose"?

I. "If the wild pig on planet beta has a long nose, then the pink elephant on planet alpha has purple eyes."
II. "If the pink elephant on planet alpha does not have purple eyes, then the wild pig on planet beta does not have a long nose."
III. "If the wild pig on planet beta has a long nose, then the pink elephant on planet alpha does not have purple eyes."
IV. "The pink elephant on planet alpha does not have purple eyes, or the wild pig on planet beta does not have a long nose."

(A) I and III only (C) II and IV only (E) III only
(B) III and IV only (D) II and III only

12. A supermarket has 128 crates of apples. Each crate contains at least 120 apples and at most 144 apples. What is the largest integer n such that there must be at least n crates containing the same number of apples?

(A) 4 (B) 5 (C) 6 (D) 24 (E) 25

13. If x cows give $x + 1$ cans of milk in $x + 2$ days, how many days will it take $x + 3$ cows to give $x + 5$ cans of milk?

(A) $\dfrac{x(x+2)(x+5)}{(x+1)(x+3)}$ (C) $\dfrac{(x+1)(x+3)(x+5)}{x(x+2)}$ (E) none of these

(B) $\dfrac{x(x+1)(x+5)}{(x+2)(x+3)}$ (D) $\dfrac{(x+1)(x+3)}{x(x+2)(x+5)}$

14. The measures of the interior angles of a convex polygon are in arithmetic progression. If the smallest angle is $100°$ and the largest angle is $140°$, then the number of sides the polygon has is

(A) 6 (B) 8 (C) 10 (D) 11 (E) 12

15. If r is the remainder when each of the numbers 1059, 1417 and 2312 is divided by d, where d is an integer greater than one, then $d - r$ equals

(A) 1 (B) 15 (C) 179 (D) $d - 15$ (E) $d - 1$

16. In triangles ABC and DEF, lengths AC, BC, DF and EF are all equal. Length AB is twice the length of the altitude of ΔDEF from F to DE. Which of the following statements is (are) true?

I. $\measuredangle ACB$ and $\measuredangle DFE$ must be complementary.
II. $\measuredangle ACB$ and $\measuredangle DFE$ must be supplementary.
III. The area of ΔABC must equal the area of ΔDEF.
IV. The area of ΔABC must equal twice the area of ΔDEF.

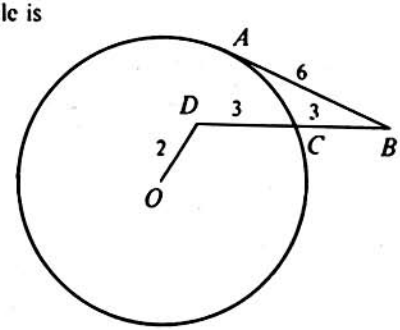
(A) II only (B) III only (C) IV only (D) I and III only (E) II and III only

17. If θ is an acute angle and $\sin 2\theta = a$, then $\sin\theta + \cos\theta$ equals

(A) $\sqrt{a+1}$ (B) $(\sqrt{2}-1)a+1$ (C) $\sqrt{a+1} - \sqrt{a^2-a}$ (D) $\sqrt{a+1} + \sqrt{a^2-a}$ (E) $\sqrt{a+1} + a^2 - a$

18. In the adjoining figure, AB is tangent at A to the circle with center O; point D is interior to the circle; and DB intersects the circle at C. If $BC = DC = 3$, $OD = 2$ and $AB = 6$, then the radius of the circle is

(A) $3+\sqrt{3}$ (B) $15/\pi$ (C) $9/2$ (D) $2\sqrt{6}$ (E) $\sqrt{22}$



19. A polynomial $p(x)$ has remainder three when divided by $x - 1$ and remainder five when divided by $x - 3$. The remainder when $p(x)$ is divided by $(x-1)(x-3)$ is

(A) $x - 2$ (B) $x + 2$ (C) 2 (D) 8 (E) 15

20. Let a, b and x be positive real numbers distinct from one. Then

$$4(\log_a x)^2 + 3(\log_b x)^2 = 8(\log_a x)(\log_b x)$$

(A) for all values of a, b and x
(B) if and only if $a = b^2$
(C) if and only if $b = a^2$
(D) if and only if $x = ab$
(E) for none of these

21. What is the smallest positive odd integer n such that the product

$$2^{1/7}2^{3/7}\cdots 2^{(2n+1)/7}$$

is greater than 1000? (In the product the denominators of the exponents are all sevens, and the numerators are the successive odd integers from 1 to $2n + 1$.)

(A) 7 (B) 9 (C) 11 (D) 17 (E) 19

22. Given an equilateral triange with side of length s, consider the locus of all points P in the plane of the triange such that the sum of the squares of the distances from P to the vertices of the triangle is a fixed number a. This locus

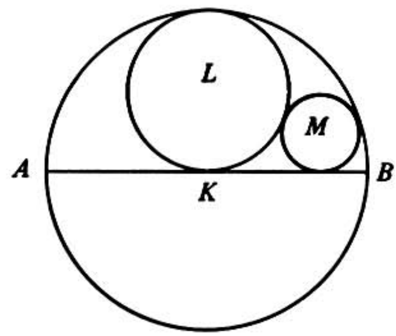
(A) is a circle if $a > s^2$
(B) contains only three points if $a = 2s^2$ and is a circle if $a > 2s^2$
(C) is a circle with positive radius only if $s^2 < a < 2s^2$
(D) contains only a finite number of points for any value of a
(E) is none of these

23. For integers k and n such that $1 \leq k < n$, let $C_k^n = \dfrac{n!}{k!(n-k)!}$.
Then $\left(\dfrac{n-2k-1}{k+1}\right) C_k^n$ is an integer

(A) for all k and n
(B) for all even values of k and n, but not for all k and n
(C) for all odd values of k and n, but not for all k and n
(D) if $k = 1$ or $n - 1$, but not for all odd values of k and n
(E) if n is divisible by k, but not for all even values of k and n

24. In the adjoining figure, circle K has diameter AB; circle L is tangent to circle K and to AB at the center of circle K; and circle M is tangent to circle K, to circle L and to AB. The ratio of the area of circle K to the area of circle M is

(A) 12 (B) 14 (C) 16 (D) 18 (E) not an integer

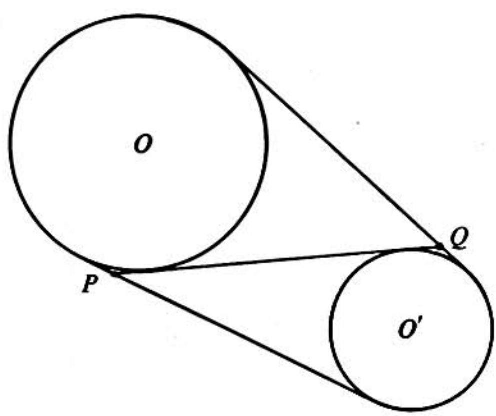


25. For a sequence $u_1, u_2, \ldots$, define $\Delta^1(u_n) = u_{n+1} - u_n$ and, for all integers $k > 1$, $\Delta^k(u_n) = \Delta^1(\Delta^{k-1}(u_n))$. If $u_n = n^3 + n$, then $\Delta^k(u_n) = 0$ for all n

(A) if $k = 1$
(B) if $k = 2$, but not if $k = 1$
(C) if $k = 3$, but not if $k = 2$
(D) if $k = 4$, but not if $k = 3$
(E) for no value of k

26. In the adjoining figure, every point of circle O' is exterior to circle O. Let P and Q be the points of intersection of an internal common tangent with the two external common tangents. Then the length of PQ is

(A) the average of the lengths of the internal and external common tangents
(B) equal to the length of an external common tangent if and only if circles O and O' have equal radii
(C) always equal to the length of an external common tangent
(D) greater than the length of an external common tangent
(E) the geometric mean of the lengths of the internal and external common tangents



27. If

$$N = \frac{\sqrt{\sqrt{5}+2}+\sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}} - \sqrt{3-2\sqrt{2}},$$

then N equals

(A) 1 (B) $2\sqrt{2}-1$ (C) $\frac{\sqrt{5}}{2}$ (D) $\sqrt{\frac{5}{2}}$ (E) none of these

28. Lines $L_1, L_2 \dots, L_{100}$ are distinct. All lines L_{4n}, n a positive integer, are parallel to each other. All lines L_{4n-3}, n a positive integer, pass through a given point A. The maximum number of points of intersection of pairs of lines from the complete set $\{L_1, L_2, \dots, L_{100}\}$ is

(A) 4350 (B) 4351 (C) 4900 (D) 4901 (E) 9851

29. Ann and Barbara were comparing their ages and found that Barbara is as old as Ann was when Barbara was as old as Ann had been when Barbara was half as old as Ann is. If the sum of their present ages is 44 years, then Ann's age is

(A) 22 (B) 24 (C) 25 (D) 26 (E) 28

30. How many distinct ordered triples (x, y, z) satisfy the equations

$$\begin{aligned} x + 2y + 4z &= 12 \\ xy + 4yz + 2xz &= 22 \\ xyz &= 6 \ ? \end{aligned}$$

(A) none (B) 1 (C) 2 (D) 4 (E) 6

THE FIFTH U.S.A. MATHEMATICAL OLYMPIAD WILL BE HELD ON TUESDAY, MAY 4, 1976. PARTICIPATION IS BY INVITATION, BASED TO A LARGE EXTENT ON PERFORMANCE ON THE 1976 ANNUAL HIGH SCHOOL MATHEMATICS EXAMINATION.

THE XVIII INTERNATIONAL MATHEMATICAL OLYMPIAD WILL BEGIN JULY 5, 1976.

Printed in U.S.A.

Sponsored Jointly by
MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
and
CASUALTY ACTUARIAL SOCIETY

MAA | SA | MAΘ | CAS | NCTM

28

TWENTY EIGHTH ANNUAL MATHEMATICS EXAMINATION 1977

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box of your answer sheet directly above the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly above No. 3. Fill in the answers as you find them.
4. *There is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted. *Slide rules and calculators are not permitted.*
6. *When your teacher gives the signal tear off this cover along the dotted line inside. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you write the information required in the first three lines.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

*This instruction may be modified if machine-scoring is used.

TUESDAY, MARCH 8, 1977

COMMITTEE ON HIGH SCHOOL CONTESTS

Chairman: Mathematics Department, City College, CUNY, 138th St. at Convent Ave., New York, N.Y. 10031

Executive Director: Oldfather Hall, Univ. of Nebraska, Lincoln, Nebr. 68508

Olympiad Subcommittee Chairman: Hill Mathematical Center, Rutgers University, New Brunswick, N.J. 08903

TWENTY EIGHTH ANNUAL MATHEMATICS EXAMINATION 1977

To be filled in by the student

PRINT ____

last name first name middle initial

____

school (full name) street address

____

city state zip code

At what date do you anticipate entering college full time? ________ month ________ year

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30

Not to be filled in by the student

Each question has a value of *five points*. *One* point is to be deducted for each wrong answer (do *not* deduct points for questions left unanswered). If the number of points deducted for incorrect answers exceeds the number of points for correct answers the student is to receive a score of 0.

Number Correct [ ] × 5 Points Correct [ ]

−Number Wrong [ ]

Total Score [ ]

1. If $y = 2x$ and $z = 2y$, then $x + y + z$ equals

(A) x (B) $3x$ (C) $5x$ (D) $7x$ (E) $9x$

2. Which one of the following statements is false? All equilateral triangles are

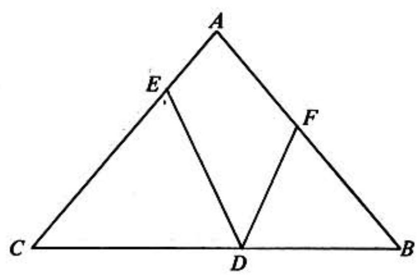
(A) equiangular (B) isosceles (C) regular polygons
(D) congruent to each other (E) similar to each other

3. A man has \$2.73 in pennies, nickels, dimes, quarters and half dollars. If he has an equal number of coins of each kind then the total number of coins he has is

(A) 3 (B) 5 (C) 9 (D) 10 (E) 15

4. In triangle ABC, $AB = AC$ and $\measuredangle A = 80°$. If points D, E and F lie on sides BC, AC and AB, respectively, and $CE = CD$ and $BF = BD$, then $\measuredangle EDF$ equals

(A) 30° (B) 40°
(C) 50° (D) 65°
(E) none of these



5. The set of all points P such that the sum of the (undirected) distances from P to two fixed points A and B equals the distance between A and B is

(A) the line segment from A to B
(B) the line passing through A and B
(C) the perpendicular bisector of the line segment from A to B
(D) an ellipse having positive area
(E) a parabola

6. If x, y and $2x + \frac{y}{2}$ are not zero, then $\left(2x + \frac{y}{2}\right)^{-1}\left[(2x)^{-1} + \left(\frac{y}{2}\right)^{-1}\right]$ equals

(A) 1 (B) xy^{-1} (C) $x^{-1}y$ (D) $(xy)^{-1}$ (E) none of these

7. If $t = \frac{1}{1 - \sqrt[4]{2}}$, then t equals

(A) $(1 - \sqrt[4]{2})(2 - \sqrt{2})$ (B) $(1 - \sqrt[4]{2})(1 + \sqrt{2})$
(C) $(1 + \sqrt[4]{2})(1 - \sqrt{2})$ (D) $(1 + \sqrt[4]{2})(1 + \sqrt{2})$
(E) $-(1 + \sqrt[4]{2})(1 + \sqrt{2})$

8. For every triple (a, b, c) of non-zero real numbers, form the number

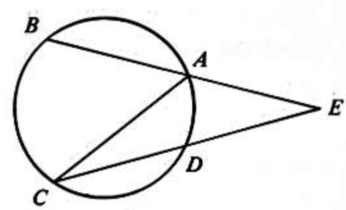
$$\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} + \frac{abc}{|abc|}.$$

The set of all numbers formed is

(A) $\{0\}$ (B) $\{-4, 0, 4\}$ (C) $\{-4, -2, 0, 2, 4\}$
(D) $\{-4, -2, 2, 4\}$ (E) none of these

9. In the adjoining figure $\measuredangle E = 40^\circ$ and arc AB, arc BC and arc CD all have equal length. Find the measure of $\measuredangle ACD$.

(A) 10° (B) 15°
(C) 20° (D) $\left(\frac{45}{2}\right)^\circ$
(E) 30°



10. If $(3x - 1)^7 = a_7x^7 + a_6x^6 + \ldots + a_0$, then $a_7 + a_6 + \ldots + a_0$ equals

(A) 0 (B) 1 (C) 64 (D) -64 (E) 128

11. For each real number x, let $[x]$ be the largest integer not exceeding x (i.e., the integer n such that $n \leqslant x < n + 1$). Which of the following statements is (are) true?

I. $[x + 1] = [x] + 1$ for all x
II. $[x + y] = [x] + [y]$ for all x and y
III. $[xy] = [x][y]$ for all x and y

(A) none (B) I only (C) I and II only
(D) III only (E) all

12. Al's age is 16 more than the sum of Bob's age and Carl's age, and the square of Al's age is 1632 more than the square of the sum of Bob's age and Carl's age. The sum of the ages of Al, Bob and Carl is

(A) 64 (B) 94 (C) 96 (D) 102 (E) 140

13. If $a_1, a_2, a_3, \ldots$ is a sequence of positive numbers such that $a_{n+2} = a_n a_{n+1}$ for all positive integers n, then the sequence $a_1, a_2, a_3, \ldots$ is a geometric progression

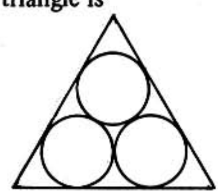
(A) for all positive values of a_1 and a_2
(B) if and only if $a_1 = a_2$
(C) if and only if $a_1 = 1$
(D) if and only if $a_2 = 1$
(E) if and only if $a_1 = a_2 = 1$

14. How many pairs (m, n) of integers satisfy the equation $m + n = mn$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) more than 4

15. Each of the three circles in the adjoining figure is externally tangent to the other two, and each side of the triangle is tangent to two of the circles. If each circle has radius three, then the perimeter of the triangle is

(A) $36 + 9\sqrt{2}$ (B) $36 + 6\sqrt{3}$
(C) $36 + 9\sqrt{3}$ (D) $18 + 18\sqrt{3}$
(E) 45



16. If $i^2 = -1$, then the sum $\cos 45^\circ + i \cos 135^\circ + \ldots + i^n \cos (45 + 90n)^\circ + \ldots + i^{40} \cos 3645^\circ$ equals

(A) $\frac{\sqrt{2}}{2}$ (B) $-10i\sqrt{2}$ (C) $\frac{21\sqrt{2}}{2}$

(D) $\frac{\sqrt{2}}{2}(21 - 20i)$ (E) $\frac{\sqrt{2}}{2}(21 + 20i)$

17. Three fair dice are tossed at random (i.e., all faces have the same probability of coming up). What is the probability that the three numbers turned up can be arranged to form an arithmetic progression with common difference one?

(A) $\frac{1}{6}$ (B) $\frac{1}{9}$ (C) $\frac{1}{27}$ (D) $\frac{1}{54}$ (E) $\frac{7}{36}$

18. If $y = (\log_2 3)(\log_3 4) \cdots (\log_n [n+1]) \cdots (\log_{31} 32)$ then

(A) $4 < y < 5$ (B) $y = 5$ (C) $5 < y < 6$
(D) $y = 6$ (E) $6 < y < 7$

19. Let E be the point of intersection of the diagonals of convex quadrilateral $ABCD$, and let P, Q, R and S be the centers of the circles circumscribing triangles ABE, BCE, CDE and ADE, respectively. Then

(A) $PQRS$ is a parallelogram
(B) $PQRS$ is a parallelogram if and only if $ABCD$ is a rhombus
(C) $PQRS$ is a parallelogram if and only if $ABCD$ is a rectangle
(D) $PQRS$ is a parallelogram if and only if $ABCD$ is a parallelogram
(E) none of the above are true

20. For how many paths consisting of a sequence of horizontal and/or vertical line segments, with each segment connecting a pair of adjacent letters in the diagram below, is the word CONTEST spelled out as the path is traversed from beginning to end?

(A) 63 (B) 128
(C) 129 (D) 255
(E) none of these

```
      C
     COC
    CONOC
   CONTNOC
  CONTETNOC
 CONTESETNOC
CONTESTSETNOC
```

21. For how many values of the coefficient a do the equations

$$x^2 + ax + 1 = 0$$
$$x^2 - x - a = 0$$

have a common real solution?

(A) 0 (B) 1 (C) 2 (D) 3 (E) infinitely many

22. If $f(x)$ is a real valued function of the real variable x, and $f(x)$ is not identically zero, and for all a and b

$$f(a+b) + f(a-b) = 2f(a) + 2f(b),$$

then for all x and y

(A) $f(0) = 1$ (B) $f(-x) = -f(x)$ (C) $f(-x) = f(x)$
(D) $f(x+y) = f(x) + f(y)$ (E) there is a positive number T such that $f(x+T) = f(x)$

23. If the solutions of the equation $x^2 + px + q = 0$ are the cubes of the solutions of the equation $x^2 + mx + n = 0$, then

(A) $p = m^3 + 3mn$ (B) $p = m^3 - 3mn$ (C) $p + q = m^3$
(D) $\left(\frac{m}{n}\right)^3 = \frac{p}{q}$ (E) none of these

24. Find the sum

$$\frac{1}{1(3)} + \frac{1}{3(5)} + \ldots + \frac{1}{(2n-1)(2n+1)} + \ldots + \frac{1}{255(257)}.$$

(A) $\frac{127}{255}$ (B) $\frac{128}{255}$ (C) $\frac{1}{2}$ (D) $\frac{128}{257}$ (E) $\frac{129}{257}$

25. Determine the largest positive integer n such that $1005!$ is divisible by 10^n.

(A) 102 (B) 112 (C) 249 (D) 502 (E) none of these

26. Let a, b, c and d be the lengths of sides MN, NP, PQ and QM, respectively, of quadrilateral $MNPQ$. If A is the area of $MNPQ$, then

(A) $A = \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is convex
(B) $A = \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a rectangle
(C) $A \leqslant \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a rectangle
(D) $A \leqslant \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a parallelogram
(E) $A \geqslant \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a parallelogram

27. There are two spherical balls of different sizes lying in two corners of a rectangular room, each touching two walls and the floor. If there is a point on each ball which is 5 inches from each wall which that ball touches and 10 inches from the floor, then the sum of the diameters of the balls is

(A) 20 inches (B) 30 inches (C) 40 inches
(D) 60 inches (E) not determined by the given information

28. Let $g(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. What is the remainder when the polynomial $g(x^{12})$ is divided by the polynomial $g(x)$?

(A) 6 (B) $5 - x$ (C) $4 - x + x^2$
(D) $3 - x + x^2 - x^3$ (E) $2 - x + x^2 - x^3 + x^4$

29. Find the smallest integer n such that

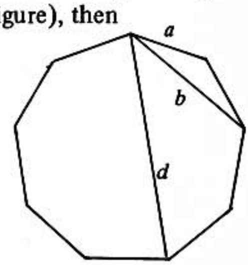
$$(x^2 + y^2 + z^2)^2 \leqslant n(x^4 + y^4 + z^4)$$

for all real numbers x, y and z.

(A) 2 (B) 3 (C) 4 (D) 6 (E) There is no such integer n.

30. If a, b and d are the lengths of a side, a shortest diagonal and a longest diagonal, respectively, of a regular nonagon (see adjoining figure), then

(A) $d = a + b$
(B) $d^2 = a^2 + b^2$
(C) $d^2 = a^2 + ab + b^2$
(D) $b = \frac{a + d}{2}$
(E) $b^2 = ad$



THE SIXTH U.S.A. MATHEMATICAL OLYMPIAD WILL BE HELD ON TUESDAY, MAY 3, 1977. PARTICIPATION IS BY INVITATION, BASED ON PERFORMANCE ON THE 1977 ANNUAL HIGH SCHOOL MATHEMATICS EXAMINATION.

THE XIX INTERNATIONAL MATHEMATICAL OLYMPIAD WILL BE IN YUGOSLAVIA BEGINNING JULY 4, 1977 (DATE TENTATIVE).

Printed in U.S.A.

TWENTY NINTH ANNUAL MATHEMATICS EXAMINATION 1978

29

Sponsored Jointly by
MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
and
CASUALTY ACTUARIAL SOCIETY

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. *The answer sheet on which you are to indicate the correct answer to each question is on the back of this cover.
3. This is a multiple choice test. Each question is followed by answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box to the right of the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box immediately to the right of No. 3. Each question has only one correct answer. Fill in the answers as you find them.
4. *There is a penalty for wrong answers.* On the average random guessing will neither increase nor decrease your score.
5. Use pencil. Scratch paper, graph paper, ruler, compasses, and eraser are permitted. *Slide rules and calculators are not permitted.*
6. *When your teacher gives the signal tear off this cover.
7. Keep the questions covered with the answer sheet while you write the information required in the first six lines.
8. When your teacher gives the signal begin working the problems. You have <u>90 MINUTES</u> working time for the test.

*This instruction may be modified if machine-scoring is used.

TUESDAY, MARCH 14, 1978

COMMITTEE ON HIGH SCHOOL CONTESTS

Chairman: Committee on High School Contests
Metropolitan Life Insurance Company, 1 Madison Avenue, New York, N.Y. 10010.
and
Mathematics Department, The City College of New York,
138th St. at Convent Ave., New York, N.Y. 10031.
Executive Director: Mathematics Department, Univ. of Nebraska
917 Oldfather Hall, Lincoln, Nebr. 68588
Olympiad Subcommittee Chairman: Hill Mathematical Center,
Rutgers University, New Brunswick, N.J. 08903.

TWENTY NINTH ANNUAL MATHEMATICS EXAMINATION 1978

To be filled in by the student

PRINT ____

last name first name middle initial

____

(home address) no. street

____

city state zip

____

school (full name) street address

____

city state zip

At what date if any do you anticipate entering college full time? ______________

month year

1		16	
2		17	
3		18	
4		19	
5		20	
6		21	
7		22	
8		23	
9		24	
10		25	
11		26	
12		27	
13		28	
14		29	
15		30	

Not to be filled in by the student.

Each question has a value of *four points. One* point is to be deducted for each wrong answer (do *not* deduct points for questions left unanswered). The total score is computed by taking four times the number right subtracting the number wrong *and* adding 30.

	Number Correct		Points Correct
New Scoring Procedure →		X 4	
			Number Wrong
		—	
New Scoring Procedure →		+	30
		*Total Score	

*Note: Blank papers should receive a score of 30.

1. If $1 - \frac{4}{x} + \frac{4}{x^2} = 0$, then $\frac{2}{x}$ equals

(A) 1 (B) 1 (C) 2 (D) 1 or 2 (E) 1 or 2

2. If four times the reciprocal of the circumference of a circle equals the diameter of the circle, then the area of the circle is

(A) $\frac{1}{\pi^2}$ (B) $\frac{1}{\pi}$ (C) 1 (D) π (E) π^2

3. For all non-zero numbers x and y such that $x = \frac{1}{y}$, $\left(x - \frac{1}{x}\right)\left(y + \frac{1}{y}\right)$ equals

(A) $2x^2$ (B) $2y^2$ (C) $x^2 + y^2$ (D) $x^2 - y^2$ (E) $y^2 - x^2$

4. If $a = 1$, $b = 10$, $c = 100$ and $d = 1000$, then

$$(a + b + c - d) + (a + b - c + d) + (a - b + c + d) + (-a + b + c + d)$$

is equal to

(A) 1111 (B) 2222 (C) 3333 (D) 1212 (E) 4242

5. Four boys bought a boat for \$60. The first boy paid one half of the sum of the amounts paid by the other boys; the second boy paid one third of the sum of the amounts paid by the other boys; and the third boy paid one fourth of the sum of the amounts paid by the other boys. How much did the fourth boy pay?

(A) \$10 (B) \$12 (C) \$13 (D) \$14 (E) \$15

6. The number of distinct pairs (x,y) of real numbers satisfying both of the following equations:

$$x = x^2 + y^2$$
$$y = 2xy$$

is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

7. Opposite sides of a regular hexagon are 12 inches apart. The length of each side, in inches, is

(A) 7.5 (B) $6\sqrt{2}$ (C) $5\sqrt{2}$ (D) $\frac{9}{2}\sqrt{3}$ (E) $4\sqrt{3}$

8. If $x \neq y$ and the sequences x, a_1, a_2, y and x, b_1, b_2, b_3, y each are in arithmetic progression, then $(a_2 - a_1)/(b_2 - b_1)$ equals

(A) $\frac{2}{3}$ (B) $\frac{3}{4}$ (C) 1 (D) $\frac{4}{3}$ (E) $\frac{3}{2}$

9. If $x < 0$, then $|x - \sqrt{(x-1)^2}|$ equals

(A) 1 (B) $1 - 2x$ (C) $-2x - 1$ (D) $1 + 2x$ (E) $2x - 1$

10. If B is a point on circle C with center P, then the set of all points A in the plane of circle C such that the distance between A and B is less than or equal to the distance between A and any other point on circle C is

(A) the line segment from P to B
(B) the ray beginning at P and passing through B
(C) a ray beginning at B
(D) a circle whose center is P
(E) a circle whose center is B

11. If r is positive and the line whose equation is $x + y = r$ is tangent to the circle whose equation is $x^2 + y^2 = r$, then r equals

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) $\sqrt{2}$ (E) $2\sqrt{2}$

12. In $\triangle ADE$, $\measuredangle ADE = 140^\circ$ and points B and C lie on sides AD and AE, respectively. If lengths AB, BC, CD and DE are all equal, then the measure of $\measuredangle EAD$ is

(A) 5° (B) 6° (C) 7.5° (D) 8° (E) 10°

13. If a, b, c, and d are non-zero numbers such that c and d are solutions of $x^2 + ax + b = 0$ and a and b are solutions of $x^2 + cx + d = 0$, then $a + b + c + d$ equals

(A) 0 (B) -2 (C) 2 (D) 4 (E) $(-1 + \sqrt{5})/2$

14. If an integer n, greater than 8, is a solution of the equation $x^2 - ax + b = 0$ and the representation of a in the base n numeration system is 18, then the base n representation of b is

(A) 18 (B) 28 (C) 80 (D) 81 (E) 280

15. If $\sin x + \cos x = 1/5$ and $0 \leqslant x < \pi$, then $\tan x$ is

(A) $-\frac{4}{3}$ (B) $-\frac{3}{4}$ (C) $\frac{3}{4}$ (D) $\frac{4}{3}$

(E) not completely determined by the given information

16. In a room containing N people, $N > 3$, at least one person has not shaken hands with everyone else in the room. What is the maximum number of people in the room that could have shaken hands with everyone else?

(A) 0 (B) 1 (C) $N - 1$ (D) N (E) none of these

17. If k is a positive number and f is a function such that, for every positive number x,

$$[f(x^2+1)]^{\sqrt{x}} = k;$$

then, for every positive number y,

$$\left[f\left(\frac{9+y^2}{y^2}\right)\right]^{\sqrt{\frac{12}{y}}}$$

is equal to

(A) $\sqrt{k}$ (B) $2k$ (C) $k\sqrt{k}$ (D) k^2 (E) $y\sqrt{k}$

18. What is the smallest positive integer n such that $\sqrt{n} - \sqrt{n-1} < .01$?

(A) 2499 (B) 2500 (C) 2501 (D) 10,000 (E) There is no such integer.

19. A positive integer n not exceeding 100 is chosen in such a way that if $n \leqslant 50$, then the probability of choosing n is p, and if $n > 50$, then the probability of choosing n is $3p$. The probability that a perfect square is chosen is

(A) .05 (B) .065 (C) .08 (D) .09 (E) .1

20. If a, b, c are non-zero real numbers such that

$$\frac{a+b-c}{c} = \frac{a-b+c}{b} = \frac{-a+b+c}{a},$$

and

$$x = \frac{(a+b)(b+c)(c+a)}{abc},$$

and $x < 0$, then x equals

(A) -1 (B) -2 (C) -4 (D) -6 (E) -8

21. For all positive numbers x distinct from 1,

$$\frac{1}{\log_3 x} + \frac{1}{\log_4 x} + \frac{1}{\log_5 x}$$

equals

(A) $\frac{1}{\log_{60} x}$ (B) $\frac{1}{\log_x 60}$

(C) $\frac{1}{(\log_3 x)(\log_4 x)(\log_5 x)}$

(D) $\frac{12}{(\log_3 x)+(\log_4 x)+(\log_5 x)}$

(E) $\frac{\log_2 x}{(\log_3 x)(\log_5 x)} + \frac{\log_3 x}{(\log_2 x)(\log_5 x)} + \frac{\log_5 x}{(\log_2 x)(\log_3 x)}$

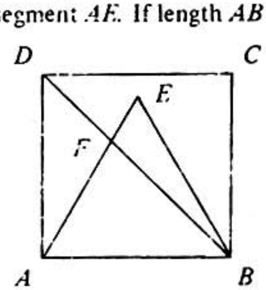
22. The following four statements, and only these, are found on a card:

> On this card exactly one statement is false.
> On this card exactly two statements are false.
> On this card exactly three statements are false.
> On this card exactly four statements are false.

(Assume each statement on the card is either true or false.) Among them the number of false statements is exactly

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

23. Vertex E of equilateral triangle ABE is in the interior of square $ABCD$, and F is the point of intersection of diagonal BD and line segment AE. If length AB is $\sqrt{1+\sqrt{3}}$ then the area of ΔABF is



(A) 1 (B) $\frac{\sqrt{2}}{2}$ (C) $\frac{\sqrt{3}}{2}$

(D) $4-2\sqrt{3}$ (E) $\frac{1}{2}+\frac{\sqrt{3}}{4}$

24. If the distinct non-zero numbers $x(y-z)$, $y(z-x)$, $z(x-y)$ form a geometric progression with common ratio r, then r satisfies the equation

(A) $r^2+r+1=0$ (B) $r^2-r+1=0$ (C) $r^4+r^2-1=0$
(D) $(r+1)^4+r=0$ (E) $(r-1)^4+r=0$

25. Let a be a positive number. Consider the set S of all points whose rectangular coordinates (x, y) satisfy all of the following conditions:

(i) $\frac{a}{2} \leqslant x \leqslant 2a$ (iii) $x+y \geqslant a$

(ii) $\frac{a}{2} \leqslant y \leqslant 2a$ (iv) $x+a \geqslant y$

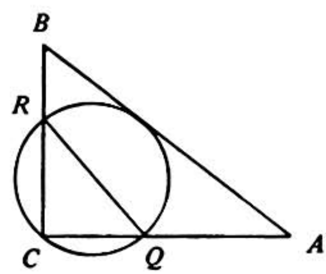
(v) $y+a \geqslant x$

The boundary of set S is a polygon with

(A) 3 sides (B) 4 sides (C) 5 sides (D) 6 sides (E) 7 sides

26. In ΔABC, $AB = 10$, $AC = 8$ and $BC = 6$. Circle P is the circle with smallest radius which passes through C and is tangent to AB. Let Q and R be the points of intersection, distinct from C, of circle P with sides AC and BC, respectively. The length of segment QR is

(A) 4.75 (B) 4.8 (C) 5
(D) $4\sqrt{2}$ (E) $3\sqrt{3}$

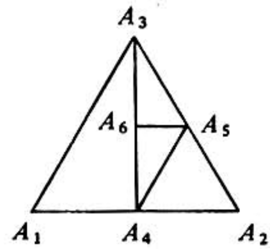


27. There is more than one integer greater than 1 which, when divided by any integer k such that $2 \leqslant k \leqslant 11$, has a remainder of 1. What is the difference between the two smallest such integers?

(A) 2310 (B) 2311 (C) 27,720 (D) 27,721 (E) none of these

28. If $\Delta A_1A_2A_3$ is equilateral and A_{n+3} is the midpoint of line segment A_nA_{n+1} for all positive integers n, then the measure of $\measuredangle A_{44}A_{45}A_{43}$ equals

(A) 30° (B) 45° (C) 60°
(D) 90° (E) 120°



29. Sides AB, BC, CD and DA, respectively, of convex quadrilateral $ABCD$ are extended past B, C, D and A to points B', C', D' and A'. Also, $AB = BB' = 6$, $BC = CC' = 7$, $CD = DD' = 8$ and $DA = AA' = 9$; and the area of $ABCD$ is 10. The area of $A'B'C'D'$ is

(A) 20 (B) 40 (C) 45 (D) 50 (E) 60

30. In a tennis tournament, n women and $2n$ men play, and each player plays exactly one match with every other player. If there are no ties and the ratio of the number of matches won by women to the number of matches won by men is $7/5$, then n equals

(A) 2 (B) 4 (C) 6 (D) 7 (E) none of these

THE DATE FOR THE SEVENTH U.S.A. MATHEMATICAL OLYMPIAD HAS NOT BEEN SET.
PARTICIPATION IS BY INVITATION, BASED ON PERFORMANCE ON THE 1978 ANNUAL HIGH SCHOOL MATHEMATICS EXAMINATION.

THE DATE AND PLACE OF THE XX INTERNATIONAL OLYMPIAD WERE NOT SET AT THE TIME OF THIS PRINTING.

THIRTIETH ANNUAL MATHEMATICS EXAMINATION

1979

30

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SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
and
CASUALTY ACTUARIAL SOCIETY

INSTRUCTIONS

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2. *The answer sheet on which you are to indicate the correct answer to each question is on the back of this cover.
3. This is a multiple choice test. Each question is followed by answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box to the right of the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box immediately to the right of No. 3. Each question has only one correct answer. Fill in the answers as you find them.
4. *There is a penalty for wrong answers.* On the average random guessing will neither increase nor decrease your score.
5. Use pencil. Scratch paper, graph paper, ruler, compasses, and eraser are permitted. *Slide rules and calculators are not permitted.*
6. *When your teacher gives the signal tear off this cover.
7. Keep the questions covered with the answer sheet while you write the information required in the first six lines.
8. When your teacher gives the signal begin working the problems. You have **90 MINUTES** working time for the test.

*This instruction may be modified if machine-scoring is used.

TUESDAY, MARCH 6, 1979

COMMITTEE ON HIGH SCHOOL CONTESTS

Chairman: Committee on High School Contests
Metropolitan Life Insurance Company, 1 Madison Avenue, New York, N.Y. 10010.
and
Mathematics Department, The City College of New York,
138th St. at Convent Ave., New York, N.Y. 10031.
Executive Director: Mathematics Department, Univ. of Nebraska
917 Oldfather Hall, Lincoln, Nebr. 68588.
Olympiad Subcommittee Chairman: Hill Mathematical Center,
Rutgers University, New Brunswick, N.J. 08903.

THIRTIETH ANNUAL MATHEMATICS EXAMINATION 1979

To be filled in by the student

PRINT ____

last name   first name   middle initial

____

(home address) no.   street

____

city   state   zip

____

school (full name)   street address

____

city   state   zip

At what date if any do you anticipate entering college full time? ____________

month   year

1		16	
2		17	
3		18	
4		19	
5		20	
6		21	
7		22	
8		23	
9		24	
10		25	
11		26	
12		27	
13		28	
14		29	
15		30	

Not to be filled in by the student.

Each question has a value of *four points. One* point is to be deducted for each wrong answer (do *not* deduct points for questions left unanswered). The total score is computed by taking four times the number right, subtracting the number wrong *and* adding 30.

Number Correct [ ] × 4 = Points Correct [ ]

− Number Wrong [ ]

+ 30

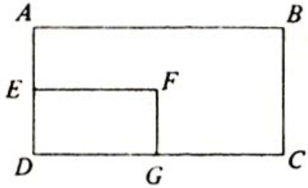
______

*Total Score [ ]

*Note: Blank papers should receive a score of 30.

1. If rectangle $ABCD$ has area 72 square meters and E and G are the midpoints of sides AD and CD, respectively, then the area of rectangle $DEFG$ in square meters is

(A) 8 (B) 9 (C) 12
(D) 18 (E) 24

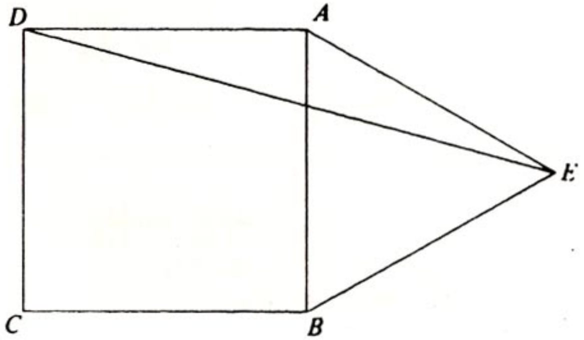


2. For all non-zero real numbers x and y such that $x - y = xy$, $\frac{1}{x} - \frac{1}{y}$ equals

(A) $\frac{1}{xy}$ (B) $\frac{1}{x-y}$ (C) 0 (D) -1 (E) $y - x$

3. In the adjoining figure, $ABCD$ is a square, ABE is an equilateral triangle and point E is outside square $ABCD$. What is the measure of $\measuredangle AED$ in degrees?

(A) 10 (B) 12.5 (C) 15 (D) 20 (E) 25



4. For all real numbers x, $x[x\{x(2-x)-4\}+10]+1=$

(A) $-x^4+2x^3+4x^2+10x+1$ (B) $-x^4-2x^3+4x^2+10x+1$
(C) $-x^4-2x^3-4x^2+10x+1$ (D) $-x^4-2x^3-4x^2-10x+1$
(E) $-x^4+2x^3-4x^2+10x+1$

5. Find the sum of the digits of the largest even three digit number (in base ten representation) which is not changed when its unit's and hundred's digits are interchanged.

(A) 22 (B) 23 (C) 24 (D) 25 (E) 26

6. $\frac{3}{2}+\frac{5}{4}+\frac{9}{8}+\frac{17}{16}+\frac{33}{32}+\frac{65}{64}-7=$

(A) $-\frac{1}{64}$ (B) $-\frac{1}{16}$ (C) 0 (D) $\frac{1}{16}$ (E) $\frac{1}{64}$

7. The square of an integer is called a *perfect square.* If x is a perfect square, the next larger perfect square is

(A) $x+1$ (B) x^2+1 (C) x^2+2x+1 (D) x^2+x
(E) $x+2\sqrt{x}+1$

8. Find the area of the smallest region bounded by the graphs of $y=|x|$ and $x^2+y^2=4$.

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$ (C) π (D) $\frac{3\pi}{2}$ (E) 2π

9. The product of $\sqrt[3]{4}$ and $\sqrt[4]{8}$ equals

(A) $\sqrt[7]{12}$ (B) $2\sqrt[7]{12}$ (C) $\sqrt[7]{32}$ (D) $\sqrt[12]{32}$ (E) $2\sqrt[12]{32}$

10. If $P_1P_2P_3P_4P_5P_6$ is a regular hexagon whose apothem (distance from the center to the midpoint of a side) is 2, and Q_i is the midpoint of side P_iP_{i+1} for $i=1, 2, 3, 4$, then the area of quadrilateral $Q_1Q_2Q_3Q_4$ is

(A) 6 (B) $2\sqrt{6}$ (C) $\frac{8\sqrt{3}}{3}$ (D) $3\sqrt{3}$ (E) $4\sqrt{3}$

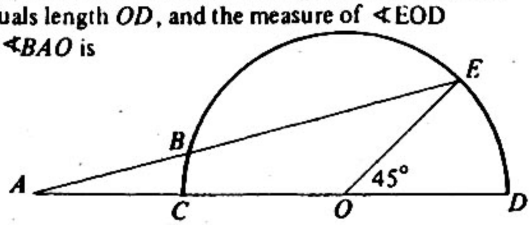
11. Find a positive integral solution to the equation

$$\frac{1+3+5+\cdots+(2n-1)}{2+4+6+\cdots+2n}=\frac{115}{116}.$$

(A) 110 (B) 115 (C) 116 (D) 231
(E) The equation has no positive integral solutions.

12. In the adjoining figure, CD is the diameter of a semi-circle with center O. Point A lies on the extension of DC past C; point E lies on the semi-circle, and B is the point of intersection (distinct from E) of line segment AE with the semi-circle. If length AB equals length OD, and the measure of $\measuredangle EOD$ is 45°, then the measure of $\measuredangle BAO$ is

(A) 10° (B) 15°
(C) 20° (D) 25°
(E) 30°



13. The inequality $y - x < \sqrt{x^2}$ is satisfied if and only if

(A) $y < 0$ or $y < 2x$ (or both inequalities hold)
(B) $y > 0$ or $y < 2x$ (or both inequalities hold)
(C) $y^2 < 2xy$
(D) $y < 0$
(E) $x > 0$ and $y < 2x$

14. In a certain sequence of numbers, the first number in the sequence is 1, and, for all $n \geq 2$, the product of the first n numbers in the sequence is n^2. The sum of the third and the fifth numbers in the sequence is

(A) $\frac{25}{9}$ (B) $\frac{31}{15}$ (C) $\frac{61}{16}$ (D) $\frac{576}{225}$ (E) 34

15. Two identical jars are filled with alcohol solutions, the ratio of the volume of alcohol to the volume of water being $p{:}1$ in one jar and $q{:}1$ in the other jar. If the entire contents of the two jars are mixed together, the ratio of the volume of alcohol to the volume of water in the mixture is

(A) $\frac{p+q}{2}$ (B) $\frac{p^2+q^2}{p+q}$ (C) $\frac{2pq}{p+q}$ (D) $\frac{2(p^2+pq+q^2)}{3(p+q)}$
(E) $\frac{p+q+2pq}{p+q+2}$

16. A circle with area A_1 is contained in the interior of a larger circle with area $A_1 + A_2$. If the radius of the larger circle is 3, and if $A_1, A_2, A_1 + A_2$ is an arithmetic progression, then the radius of the smaller circle is

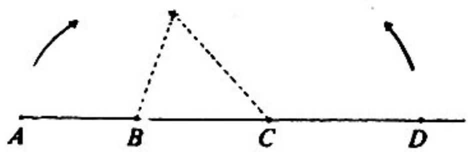
(A) $\frac{\sqrt{3}}{2}$ (B) 1 (C) $\frac{2}{\sqrt{3}}$ (D) $\frac{3}{2}$ (E) $\sqrt{3}$

17. Points A, B, C and D are distinct and lie, in the given order, on a straight line. Line segments AB, AC and AD have lengths x, y and z, respectively. If line segments AB and CD may be rotated about points B and C, respectively, so that points A and D coincide, to form a triangle with positive area, then which of the following three inequalities must be satisfied?

I $x<\frac{z}{2}$

II $y<x+\frac{z}{2}$

III $y<\frac{z}{2}$



(A) I only (B) II only (C) I and II only (D) II and III only
(E) I, II and III

18. To the nearest thousandth, $\log_{10}2$ is .301 and $\log_{10}3$ is .477. Which of the following is the best approximation of $\log_5 10$?

(A) $\frac{8}{7}$ (B) $\frac{9}{7}$ (C) $\frac{10}{7}$ (D) $\frac{11}{7}$ (E) $\frac{12}{7}$

19. Find the sum of the squares of all real numbers satisfying the equation

$$x^{256}-256^{32}=0.$$

(A) 8 (B) 128 (C) 512 (D) 65,536 (E) $2(256^{32})$

20. If $a=\frac{1}{2}$ and $(a+1)(b+1)=2$, then the radian measure of Arctan a + Arctan b equals

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{5}$ (E) $\frac{\pi}{6}$

21. The length of the hypotenuse of a right triangle is h, and the radius of the inscribed circle is r. The ratio of the area of the circle to the area of the triangle is

(A) $\frac{\pi r}{h+2r}$ (B) $\frac{\pi r}{h+r}$ (C) $\frac{\pi r}{2h+r}$ (D) $\frac{\pi r^2}{h^2+r^2}$

(E) none of these

22. Find the number of pairs (m, n) of integers which satisfy the equation $m^3 + 6m^2 + 5m = 27n^3 + 9n^2 + 9n + 1$.

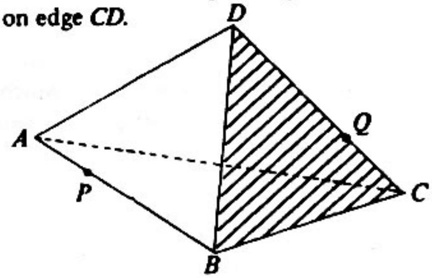
(A) 0 (B) 1 (C) 3 (D) 9 (E) infinitely many

23. The edges of a regular tetrahedron with vertices A, B, C and D each have length one. Find the least possible distance between a pair of points P and Q, where P is on edge AB and Q is on edge CD.

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$

(C) $\frac{\sqrt{2}}{2}$ (D) $\frac{\sqrt{3}}{2}$

(E) $\frac{\sqrt{3}}{3}$



24. Sides AB, BC and CD of (simple) quadrilateral $ABCD$ have lengths 4, 5 and 20, respectively. If vertex angles B and C are obtuse and $\sin C = -\cos B = \frac{3}{5}$, then side AD has length

(A) 24 (B) 24.5 (C) 24.6 (D) 24.8 (E) 25

25. If $q_1(x)$ and r_1 are the quotient and remainder, respectively, when the polynomial x^8 is divided by $x + \frac{1}{2}$, and if $q_2(x)$ and r_2 are the quotient and remainder, respectively, when $q_1(x)$ is divided by $x + \frac{1}{2}$, then r_2 equals

(A) $\frac{1}{256}$ (B) $-\frac{1}{16}$ (C) 1 (D) -16 (E) 256

26. The function f satisfies the functional equation

$$f(x) + f(y) = f(x + y) - xy - 1$$

for every pair x, y of real numbers. If $f(1) = 1$, then the number of integers $n \neq 1$ for which $f(n) = n$ is

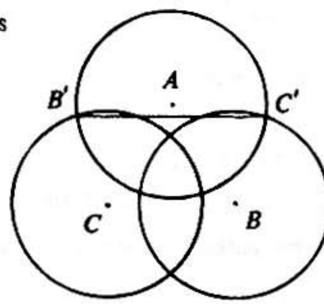
(A) 0 (B) 1 (C) 2 (D) 3 (E) infinite

27. An ordered pair (b, c) of integers, each of which has absolute value less than or equal to five, is chosen at random, with each such ordered pair having an equal likelihood of being chosen. What is the probability that the equation $x^2 + bx + c = 0$ will *not* have distinct positive real roots?

(A) $\frac{106}{121}$ (B) $\frac{108}{121}$ (C) $\frac{110}{121}$ (D) $\frac{112}{121}$ (E) none of these

28. Circles with centers A, B and C each have radius r, where $1 < r < 2$. The distance between each pair of centers is 2. If B' is the point of intersection of circle A and circle C which is outside circle B, and if C' is the point of intersection of circle A and circle B which is outside circle C, then length $B'C'$ equals

(A) $3r - 2$
(B) r^2
(C) $r + \sqrt{3(r-1)}$
(D) $1 + \sqrt{3(r^2-1)}$
(E) none of these



29. For each positive number x, let

$$f(x) = \frac{\left(x + \frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6}\right) - 2}{\left(x + \frac{1}{x}\right)^3 + \left(x^3 + \frac{1}{x^3}\right)}$$

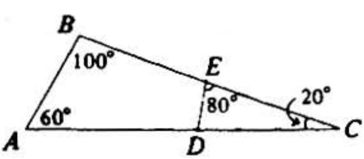
The minimum value of $f(x)$ is

(A) 1 (B) 2 (C) 3 (D) 4 (E) 6

30. In ΔABC, E is the midpoint of side BC and D is on side AC. If the length of AC is 1 and $\measuredangle BAC = 60°$, $\measuredangle ABC = 100°$, $\measuredangle ACB = 20°$ and $\measuredangle DEC = 80°$, then the area of ΔABC plus twice the area of ΔCDE equals

(A) $\frac{1}{4}\cos 10°$ (B) $\frac{\sqrt{3}}{8}$ (C) $\frac{1}{4}\cos 40°$ (D) $\frac{1}{4}\cos 50°$

(E) $\frac{1}{8}$



THIRTY FIRST ANNUAL
HIGH SCHOOL
MATHEMATICS EXAMINATION
1980
31

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The Committee on High School Contests may reexamine students before granting awards.

TUESDAY, MARCH 4, 1980

COMMITTEE ON HIGH SCHOOL CONTESTS

Chairman: Committee on High School Contests
Mathematics Department, The City College of New York,
138th St. at Convent Ave., New York, N.Y. 10031.
Executive Director: Mathematics Department, University of Nebraska
917 Oldfather Hall, Lincoln, Nebr. 68588.
Olympiad Subcommitee Chairman: Hill Mathematical Center,
Rutgers University, New Brunswick, N.J. 08903.

THIRTY FIRST ANNUAL MATHEMATICS EXAMINATION 1980

To be filled in by the student

PRINT ______________________________

last name | first name | middle initial

(home address) no. | street

city | state | zip

school (full name) | street address

city | state | zip

At what date if any do you anticipate entering college full time? ______ month ______ year

1		16	
2		17	
3		18	
4		19	
5		20	
6		21	
7		22	
8		23	
9		24	
10		25	
11		26	
12		27	
13		28	
14		29	
15		30	

Not to be filled in by the student.

Each question has a value of *four points. One* point is to be deducted for each wrong answer (do *not* deduct points for questions left unanswered). The total score is computed by taking four times the number right, subtracting the number wrong *and* adding 30.

Number Correct [ ] X 4 = [ ]

Number Wrong − [ ]

Difference [ ]

\+ 30

*Total Score [ ]

*Note: Blank papers should receive a score of 30.

1. The largest whole number such that seven times the number is less than 100 is

(A) 12 (B) 13 (C) 14 (D) 15 (E) 16

2. The degree of $(x^2+1)^4(x^3+1)^3$ as a polynomial in x is

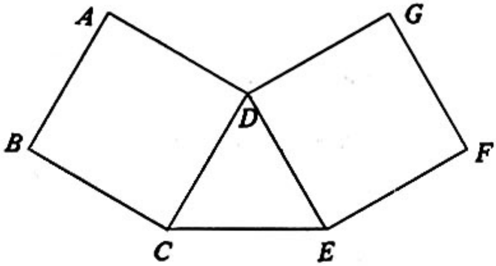
(A) 5 (B) 7 (C) 12 (D) 17 (E) 72

3. If the ratio of $2x-y$ to $x+y$ is $\frac{2}{3}$, what is the ratio of x to y?

(A) $\frac{1}{5}$ (B) $\frac{4}{5}$ (C) 1 (D) $\frac{6}{5}$ (E) $\frac{5}{4}$

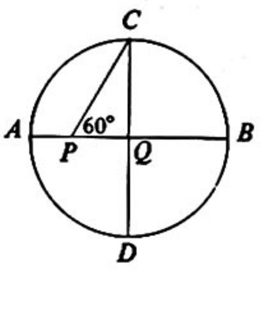
4. In the adjoining figure, CDE is an equilateral triangle and $ABCD$ and $DEFG$ are squares. The measure of $\measuredangle GDA$ is

(A) 90° (B) 105°
(C) 120° (D) 135°
(E) 150°



5. If AB and CD are perpendicular diameters of circle Q, and $\measuredangle QPC = 60°$, then the length of PQ divided by the length of AQ is

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{\sqrt{3}}{3}$
(C) $\frac{\sqrt{2}}{2}$ (D) $\frac{1}{2}$
(E) $\frac{2}{3}$

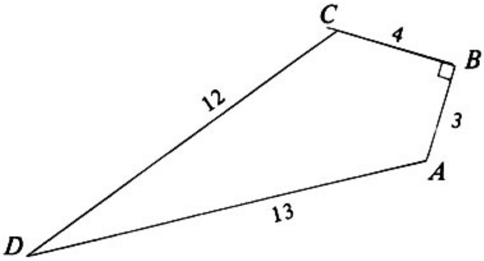


6. A positive number x satisfies the inequality $\sqrt{x} < 2x$ if and only if

(A) $x > \frac{1}{4}$ (B) $x > 2$ (C) $x > 4$ (D) $x < \frac{1}{4}$ (E) $x < 4$

7. Sides AB, BC, CD and DA of convex quadrilateral $ABCD$ have lengths 3, 4, 12 and 13, respectively; and $\measuredangle CBA$ is a right angle. The area of the quadrilateral is

(A) 32 (B) 36
(C) 39 (D) 42
(E) 48



8. How many pairs (a, b) of non-zero real numbers satisfy the equation

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}\ ?$$

(A) none (B) 1 (C) 2 (D) one pair for each $b \neq 0$
(E) two pairs for each $b \neq 0$

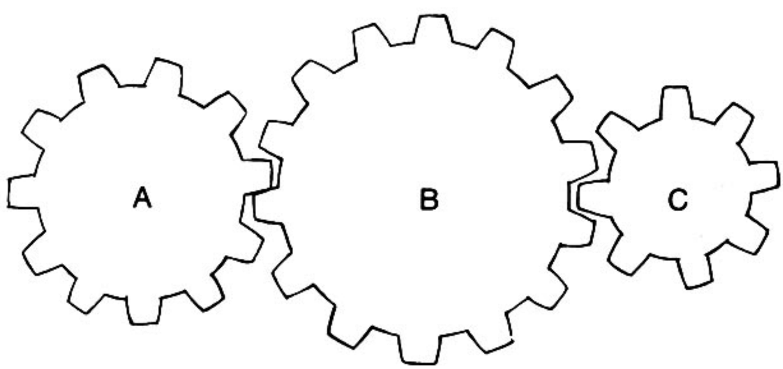
9. A man walks x miles due west, turns 150° to his left and walks 3 miles in the new direction. If he finishes at a point $\sqrt{3}$ miles from his starting point, then x is

(A) $\sqrt{3}$ (B) $2\sqrt{3}$ (C) $\frac{3}{2}$ (D) 3

(E) not uniquely determined by the given information

10. The number of teeth in three meshed circular gears A, B, C are x, y, z, respectively. (The teeth on all gears are the same size and regularly spaced as in the figure.) The angular speeds, in revolutions per minute, of A, B, C are in the proportion

(A) $x{:}y{:}z$ (B) $z{:}y{:}x$ (C) $y{:}z{:}x$ (D) $yz{:}xz{:}xy$ (E) $xz{:}yx{:}zy$



11. If the sum of the first 10 terms and the sum of the first 100 terms of a given arithmetic progression are 100 and 10, respectively, then the sum of the first 110 terms is

(A) 90 (B) -90 (C) 110 (D) -110 (E) -100

12. The equations of L_1 and L_2 are $y = mx$ and $y = nx$, respectively. Suppose L_1 makes twice as large an angle with the horizontal (measured counterclockwise from the positive x-axis) as does L_2, and that L_1 has 4 times the slope of L_2. If L_1 is not horizontal, then mn is

(A) $\frac{\sqrt{2}}{2}$ (B) $-\frac{\sqrt{2}}{2}$ (C) 2 (D) -2
(E) not uniquely determined by the given information

13. A bug (of negligible size) starts at the origin on the co-ordinate plane. First it moves 1 unit right to $(1,0)$. Then it makes a 90° turn counterclockwise and travels $\frac{1}{2}$ a unit to $\left(1,\frac{1}{2}\right)$. If it continues in this fashion, each time making a 90° turn counterclockwise and traveling half as far as in the previous move, to which of the following points will it come closest?

(A) $\left(\frac{2}{3},\frac{2}{3}\right)$ (B) $\left(\frac{4}{5},\frac{2}{5}\right)$ (C) $\left(\frac{2}{3},\frac{4}{5}\right)$ (D) $\left(\frac{2}{3},\frac{1}{3}\right)$ (E) $\left(\frac{2}{5},\frac{4}{5}\right)$

14. If the function f defined by

$$f(x) = \frac{cx}{2x+3}, \quad x \neq -\frac{3}{2},$$

satisfies $f(f(x)) = x$ for all real numbers x except $-\frac{3}{2}$, then c is

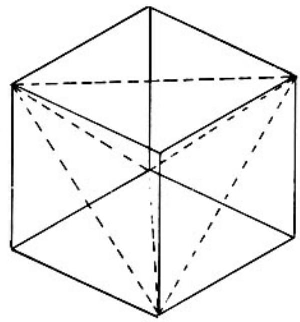
(A) -3 (B) $-\frac{3}{2}$ (C) $\frac{3}{2}$ (D) 3
(E) not uniquely determined by the given information

15. A store prices an item in dollars and cents so that when 4% sales tax is added no rounding is necessary because the result is exactly n dollars where n is a positive integer. The smallest value of n is

(A) 1 (B) 13 (C) 25 (D) 26 (E) 100

16. Four of the eight vertices of a cube are vertices of a regular tetrahedron. Find the ratio of the surface area of the cube to the surface area of the tetrahedron.

(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) $\sqrt{\frac{3}{2}}$ (D) $\frac{2}{\sqrt{3}}$ (E) 2



17. Given that $i^2 = -1$, for how many integers n is $(n+i)^4$ an integer?

(A) none (B) 1 (C) 2 (D) 3 (E) 4

18. If $b > 1$, $\sin x > 0$, $\cos x > 0$ and $\log_b \sin x = a$, then $\log_b \cos x$ equals

(A) $2\log_b(1 - b^{a/2})$ (B) $\sqrt{1-a^2}$ (C) b^{a^2} (D) $\frac{1}{2}\log_b(1 - b^{2a})$

(E) none of these

19. Let C_1, C_2 and C_3 be three parallel chords of a circle on the same side of the center. The distance between C_1 and C_2 is the same as the distance between C_2 and C_3. The lengths of the chords are 20, 16 and 8. The radius of the circle is

(A) 12 (B) $4\sqrt{7}$ (C) $\frac{5\sqrt{65}}{3}$ (D) $\frac{5\sqrt{22}}{2}$

(E) not uniquely determined by the given information

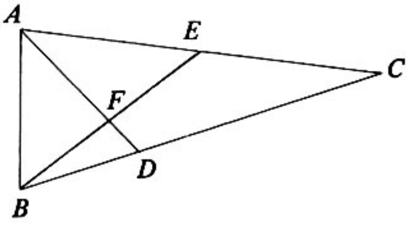
20. A box contains 2 pennies, 4 nickels and 6 dimes. Six coins are drawn without replacement, with each coin having an equal probability of being chosen. What is the probability that the value of the coins drawn is at least 50 cents?

(A) $\frac{37}{924}$ (B) $\frac{91}{924}$ (C) $\frac{127}{924}$ (D) $\frac{132}{924}$ (E) none of these

21. In triangle ABC, $\measuredangle CBA = 72°$, E is the midpoint of side AC, and D is a point on side BC such that $2BD = DC$; AD and BE intersect at F. The ratio of the area of ΔBDF to the area of quadrilateral $FDCE$ is

(A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$

(D) $\frac{2}{5}$ (E) none of these



22. For each real number x, let $f(x)$ be the minimum of the numbers $4x + 1$, $x + 2$, and $-2x + 4$. Then the maximum value of $f(x)$ is

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{5}{2}$ (E) $\frac{8}{3}$

23. Line segments drawn from the vertex opposite the hypotenuse of a right triangle to the points trisecting the hypotenuse have lengths $\sin x$ and $\cos x$, where x is a real number such that $0 < x < \frac{\pi}{2}$. The length of the hypotenuse is

(A) $\frac{4}{3}$ (B) $\frac{3}{2}$ (C) $\frac{3\sqrt{5}}{5}$ (D) $\frac{2\sqrt{5}}{3}$

(E) not uniquely determined by the given information

24. For some real number r, the polynomial $8x^3 - 4x^2 - 42x + 45$ is divisible by $(x - r)^2$. Which of the following numbers is closest to r?

(A) 1.22 (B) 1.32 (C) 1.42 (D) 1.52 (E) 1.62

25. In the non-decreasing sequence of odd integers $\{a_1, a_2, a_3, \ldots\} = \{1, 3, 3, 3, 5, 5, 5, 5, 5, \ldots\}$ each positive odd integer k appears k times. It is a fact that there are integers b, c and d such that, for all positive integers n,

$$a_n = b[\sqrt{n + c}] + d,$$

where $[x]$ denotes the largest integer not exceeding x. The sum $b + c + d$ equals

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

26. Four balls of radius 1 are mutually tangent, three resting on the floor and the fourth resting on the others. A tetrahedron, each of whose edges has length s, is circumscribed around the balls. Then s equals

(A) $4\sqrt{2}$ (B) $4\sqrt{3}$ (C) $2\sqrt{6}$ (D) $1+2\sqrt{6}$ (E) $2+2\sqrt{6}$

27. The sum $\sqrt[3]{5+2\sqrt{13}}+\sqrt[3]{5-2\sqrt{13}}$ equals

(A) $\frac{3}{2}$ (B) $\frac{\sqrt[3]{65}}{4}$ (C) $\frac{1+\sqrt[6]{13}}{2}$ (D) $\sqrt[3]{2}$ (E) none of these

28. The polynomial $x^{2n}+1+(x+1)^{2n}$ is not divisible by x^2+x+1 if n equals

(A) 17 (B) 20 (C) 21 (D) 64 (E) 65

29. How many ordered triples (x, y, z) of integers satisfy the system of equations below?

$$\begin{aligned} x^2 - 3xy + 2y^2 \qquad\qquad - z^2 &= 31 \\ -x^2 \qquad\qquad + 6yz + 2z^2 &= 44 \\ x^2 + xy \qquad\qquad\qquad + 8z^2 &= 100 \end{aligned}$$

(A) 0 (B) 1 (C) 2 (D) a finite number greater than two
(E) infinitely many

30. A six digit number (base 10) is *squarish* if it satisfies the following conditions:

(i) none of its digits is zero;
(ii) it is a perfect square; and
(iii) the first of two digits, the middle two digits and the last two digits of the number are all perfect squares when considered as two digit numbers.

How many squarish numbers are there?

(A) 0 (B) 2 (C) 3 (D) 8 (E) 9

THIRTY-SECOND ANNUAL HIGH SCHOOL MATHEMATICS EXAMINATION 1981

32

Sponsored Jointly by
MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
and CASUALTY ACTUARIAL SOCIETY

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. *The answer sheet on which you are to indicate the correct answer to each question is on the back of this cover.
3. This is a multiple choice test. Each question is followed by answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box to the right of the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box immediately to the right of No. 3. Each question has only one correct answer. Fill in the answers as you find them.
4. *There is a penalty for wrong answers.* On the average random guessing will neither increase nor decrease your score.
5. Use pencil. Scratch paper, graph paper, ruler, compasses, and eraser are permitted. *Slide rules and calculators are not permitted.*
6. *When your teacher gives the signal tear off this cover.
7. Keep the questions covered with the answer sheet while you write the information required in the first six lines.
8. When your teacher gives the signal begin working the problems. You have **90 MINUTES** working time for the text.

*This instruction may be modified if machine-scoring is used.

The results of this Examination are used to identify students who possess unusual mathematical ability. To assure the integrity of the Examination, the MAA Committee on High School Contests reserves the right to reexamine students prior to granting official status to their scores.

TUESDAY, MARCH 10, 1981

COMMITTEE ON HIGH SCHOOL CONTESTS

Chairman: Committee on High School Contests
Mathematics Department, The City College of New York,
138th St. at Convent Ave., New York, N.Y. 10031.
Executive Director: Mathematics and Statistics Department, Univ. of Nebraska
917 Oldfather Hall, Lincoln, Nebr. 68588
Olympiad Subcommittee Chairman: Hill Mathematical Center,
Rutgers University, New Brunswick, N.J. 08903.

THIRTY-SECOND ANNUAL MATHEMATICS EXAMINATION 1981

To be filled in by the student

PRINT ____

last name | first name | middle initial

____

(home address) no. | street

____

city | state | zip

____

school (full name) | street address

____

city | state | zip

At what date if any do you anticipate entering college full time? ____________

month | year

1		16	
2		17	
3		18	
4		19	
5		20	
6		21	
7		22	
8		23	
9		24	
10		25	
11		26	
12		27	
13		28	
14		29	
15		30	

Not to be filled in by the student.

Each question has a value of *four points. One* point is to be deducted for each wrong answer (do *not* deduct points for questions left unanswered). The total score is computed by taking four times the number right, subtracting the number wrong *and* adding 30.

Number Correct $\square \times 4 = \square$

Number Wrong $- \square$

Difference $\square$

$+ \quad 30$

*Total Score $\square$

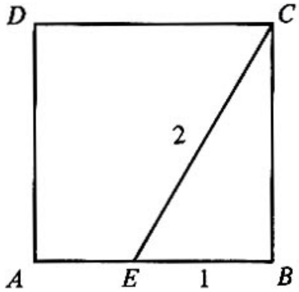
*Note: Blank papers should receive a score of 30.

1. If $\sqrt{x+2} = 2$ then $(x+2)^2$ equals

(A) $\sqrt{2}$ (B) 2 (C) 4 (D) 8 (E) 16

2. Point E is on side AB of square ABCD. If EB has length one and EC has length two, then the area of the square is

(A) $\sqrt{3}$ (B) $\sqrt{5}$
(C) 3 (D) $2\sqrt{3}$
(E) 5



3. For $x \neq 0$, $\frac{1}{x} + \frac{1}{2x} + \frac{1}{3x}$ equals

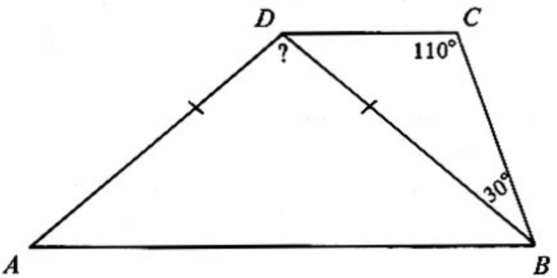
(A) $\frac{1}{2x}$ (B) $\frac{1}{6x}$ (C) $\frac{5}{6x}$ (D) $\frac{11}{6x}$ (E) $\frac{1}{6x^3}$

4. If three times the larger of two numbers is four times the smaller and the difference between the numbers is 8, then the larger of the two numbers is

(A) 16 (B) 24 (C) 32 (D) 44 (E) 52

5. In trapezoid $ABCD$, sides AB and CD are parallel, and diagonal BD and side AD have equal length. If $\measuredangle DCB = 110^\circ$ and $\measuredangle CBD = 30^\circ$, then $\measuredangle ADB =$

(A) 80° (B) 90° (C) 100°
(D) 110° (E) 120°



6. If $\frac{x}{x-1} = \frac{y^2+2y-1}{y^2+2y-2}$, then x equals

(A) y^2+2y-1 (B) y^2+2y-2 (C) y^2+2y+2
(D) y^2+2y+1 (E) $-y^2-2y+1$

7. How many of the first one hundred positive integers are divisible by all of the numbers 2, 3, 4, 5?

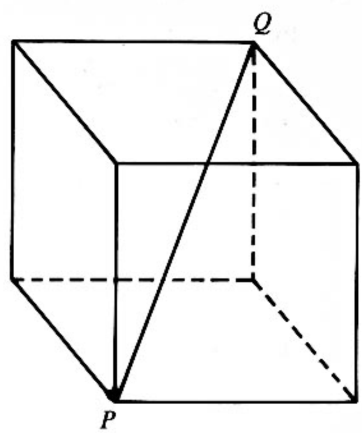
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

8. For all positive numbers x, y, z, the product
$(x+y+z)^{-1}(x^{-1}+y^{-1}+z^{-1})(xy+yz+zx)^{-1}[(xy)^{-1}+(yz)^{-1}+(zx)^{-1}]$
equals

(A) $x^{-2}y^{-2}z^{-2}$ (B) $x^{-2}+y^{-2}+z^{-2}$ (C) $(x+y+z)^{-2}$
(D) $\frac{1}{xyz}$ (E) $\frac{1}{xy+yz+zx}$

9. In the adjoining figure, PQ is a diagonal of the cube. If PQ has length a, then the surface area of the cube is

(A) $2a^2$ (B) $2\sqrt{2}\,a^2$ (C) $2\sqrt{3}a^2$
(D) $3\sqrt{3}\,a^2$ (E) $6a^2$



10. The lines L and K are symmetric to each other with respect to the line $y=x$. If the equation of line L is $y=ax+b$ with $a \neq 0$ and $b \neq 0$, then the equation of K is $y=$

(A) $\frac{1}{a}x+b$ (B) $-\frac{1}{a}x+b$ (C) $-\frac{1}{a}x-\frac{b}{a}$

(D) $\frac{1}{a}x+\frac{b}{a}$ (E) $\frac{1}{a}x-\frac{b}{a}$

11. The three sides of a right triangle have integral lengths which form an arithmetic progression. One of the sides could have length

(A) 22 (B) 58 (C) 81 (D) 91 (E) 361

12. If p, q and M are positive numbers and $q < 100$, then the number obtained by increasing M by $p\%$ and decreasing the result by $q\%$ exceeds M if and only if

(A) $p > q$ (B) $p > \frac{q}{100-q}$ (C) $p > \frac{q}{1-q}$

(D) $p > \frac{100q}{100+q}$ (E) $p > \frac{100q}{100-q}$

13. Suppose that at the end of any year, a unit of money has lost 10% of the value it had at the beginning of that year. Find the smallest integer n such that after n years the unit of money will have lost at least 90% of its value. (To the nearest thousandth $\log_{10} 3$ is .477.)

(A) 14 (B) 16 (C) 18 (D) 20 (E) 22

14. In a geometric sequence of real numbers, the sum of the first two terms is 7 and the sum of the first six terms is 91. The sum of the first four terms is

(A) 28 (B) 32 (C) 35 (D) 49 (E) 84

15. If $b > 1, x > 0$ and $(2x)^{\log_b 2} - (3x)^{\log_b 3} = 0$, then x is

(A) $\frac{1}{216}$ (B) $\frac{1}{6}$ (C) 1 (D) 6

(E) not uniquely determined

16. The base three representation of x is

$$12112211122211112222.$$

The first digit (on the left) of the base nine representation of x is

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

17. The function f is not defined for $x = 0$, but, for all non-zero real numbers x,

$f(x) + 2f\left(\frac{1}{x}\right) = 3x$. The equation $f(x) = f(-x)$ is satisfied by

(A) exactly one real number
(B) exactly two real numbers
(C) no real numbers
(D) infinitely many, but not all, non-zero real numbers
(E) all non-zero real numbers

18. The number of real solutions to the equation

$$\frac{x}{100} = \sin x$$

is

(A) 61 (B) 62 (C) 63 (D) 64 (E) 65

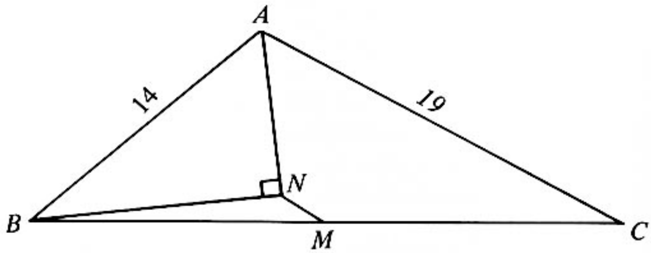
19. In $\triangle ABC$, M is the midpoint of side BC, AN bisects $\measuredangle BAC$, $BN \perp AN$ and θ is the measure of $\measuredangle BAC$. If sides AB and AC have lengths 14 and 19, respectively, then length MN equals

(A) 2 (B) $\frac{5}{2}$

(C) $\frac{5}{2} - \sin\theta$

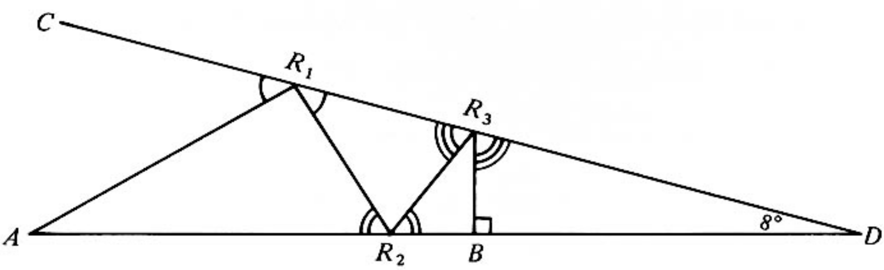
(D) $\frac{5}{2} - \frac{1}{2}\sin\theta$

(E) $\frac{5}{2} - \frac{1}{2}\sin\left(\frac{1}{2}\theta\right)$



20. A ray of light originates from point A and travels in a plane, being reflected n times between lines AD and CD, before striking a point B (which may be on AD or CD) perpendicularly and retracing its path to A. (At each point of reflection the light makes two equal angles as indicated in the adjoining figure. The figure shows the light path for $n = 3$.) If $\measuredangle CDA = 8^\circ$, what is the largest value n can have?

(A) 6
(B) 10
(C) 38
(D) 98
(E) There is no largest value.



21. In a triangle with sides of lengths a, b and c, $(a + b + c)(a + b - c) = 3ab$. The measure of the angle opposite the side of length c is

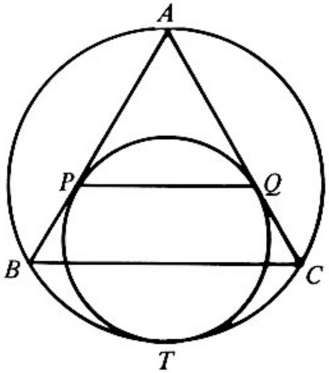
(A) 15° (B) 30° (C) 45° (D) 60° (E) 150°

22. How many lines in a three dimensional rectangular coordinate system pass through four distinct points of the form (i, j, k), where i, j and k are positive integers not exceeding four?

(A) 60 (B) 64 (C) 72 (D) 76 (E) 100

23. Equilateral $\triangle ABC$ is inscribed in a circle. A second circle is tangent internally to the circumcircle at T and tangent to sides AB and AC at points P and Q. If side BC has length 12, then segment PQ has length

(A) 6
(B) $6\sqrt{3}$
(C) 8
(D) $8\sqrt{3}$
(E) 9

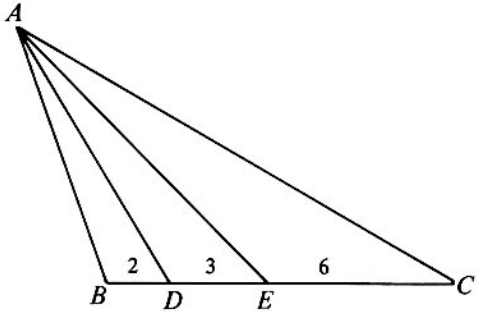


24. If θ is a constant such that $0 < \theta < \pi$ and $x + \frac{1}{x} = 2\cos\theta$, then for each positive integer n, $x^n + \frac{1}{x^n}$ equals

(A) $2\cos\theta$ (B) $2^n\cos\theta$
(C) $2\cos^n\theta$ (D) $2\cos n\theta$ (E) $2^n\cos^n\theta$

25. In triangle ABC in the adjoining figure, AD and AE trisect $\measuredangle BAC$. The lengths of BD, DE and EC are 2, 3 and 6, respectively. The length of the shortest side of $\triangle ABC$ is

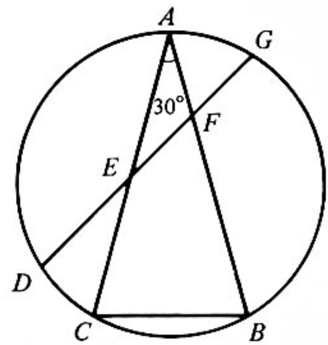
(A) $2\sqrt{10}$
(B) 11
(C) $6\sqrt{6}$
(D) 6
(E) not uniquely determined by the given information



26. Alice, Bob and Carol repeatedly take turns tossing a die. Alice begins; Bob always follows Alice; Carol always follows Bob; and Alice always follows Carol. Find the probability that Carol will be the first one to toss a six. (The probability of obtaining a six on any toss is $\frac{1}{6}$, independent of the outcome of any other toss.)

(A) $\frac{1}{3}$ (B) $\frac{2}{9}$ (C) $\frac{5}{18}$ (D) $\frac{25}{91}$ (E) $\frac{36}{91}$

27. In the adjoining figure triangle ABC is inscribed in a circle. Point D lies on $\overset{\frown}{AC}$ with $\overset{\frown}{DC} = 30°$, and point G lies on $\overset{\frown}{BA}$ with $\overset{\frown}{BG} > \overset{\frown}{GA}$. Side AB and side AC each has length equal to the length of chord DG and $\measuredangle CAB = 30°$. Chord DG intersects sides AC and AB at E and F, respectively. The ratio of the area of $\triangle AFE$ to the area of $\triangle ABC$ is



(A) $\frac{2-\sqrt{3}}{3}$ (B) $\frac{2\sqrt{3}-3}{3}$

(C) $7\sqrt{3}-12$ (D) $3\sqrt{3}-5$

(E) $\frac{9-5\sqrt{3}}{3}$

28. Consider the set of all equations $x^3 + a_2 x^2 + a_1 x + a_0 = 0$, where a_2, a_1, a_0 are real constants and $|a_i| \leqslant 2$ for $i = 0, 1, 2$. Let r be the largest positive real number which satisfies at least one of these equations. Then

(A) $1 \leqslant r < \frac{3}{2}$ (B) $\frac{3}{2} \leqslant r < 2$

(C) $2 \leqslant r < \frac{5}{2}$ (D) $\frac{5}{2} \leqslant r < 3$

(E) $3 \leqslant r < \frac{7}{2}$

29. If $a \geqslant 1$, then the sum of the real solutions of

$$\sqrt{a - \sqrt{a + x}} = x$$

is equal to

(A) $\sqrt{a} - 1$ (B) $\frac{\sqrt{a}-1}{2}$ (C) $\sqrt{a-1}$ (D) $\frac{\sqrt{a-1}}{2}$

(E) $\frac{\sqrt{4a-3}-1}{2}$

30. If a, b, c, d are the solutions of the equation $x^4 - bx - 3 = 0$, then an equation whose solutions are

$$\frac{a+b+c}{d^2}, \quad \frac{a+b+d}{c^2}, \quad \frac{a+c+d}{b^2}, \quad \frac{b+c+d}{a^2}$$

is

(A) $3x^4 + bx + 1 = 0$ (B) $3x^4 - bx + 1 = 0$
(C) $3x^4 + bx^3 - 1 = 0$ (D) $3x^4 - bx^3 - 1 = 0$
(E) none of these

THIRTY-THIRD ANNUAL
HIGH SCHOOL
MATHEMATICS EXAMINATION
1982
33
TUESDAY, MARCH 9, 1982

Sponsored Jointly by
MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
CASUALTY ACTUARIAL SOCIETY

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   In either case keep the examination questions covered while writing this information.
5. *There is a penalty for wrong answers.* On the average random guessing will neither increase nor decrease your score.
6. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted. *Slide rules and calculators are not permitted.*
7. When your teacher gives the signal begin working the problems. You have 90 MINUTES working time for the test.

The results of this Examination are used to identify students who possess unusual mathematical ability. To assure the integrity of the Examination, the MAA Committee on High School Contests reserves the right to reexamine students before deciding whether to grant official status to individual or team scores. Official status will not be granted if a school does not agree to reexamination when requested.

THIRTY-THIRD ANNUAL MATHEMATICS EXAMINATION 1982

To be filled in by the student

PRINT ____
last name first name middle initial

____
(home address) no. street

____
city state zip

____
school (full name) street address

____
city state zip

Grade Level (9, 10, 11, 12 or other number) [ ] .

1		16	
2		17	
3		18	
4		19	
5		20	
6		21	
7		22	
8		23	
9		24	
10		25	
11		26	
12		27	
13		28	
14		29	
15		30	

Not to be filled in by the student.

Each question has a value of *four points. One* point is to be deducted for each wrong answer (do *not* deduct points for questions left unanswered). The total score is computed by taking four times the number right, subtracting the number wrong *and* adding 30.

Number Correct [ ] × 4 = [ ]

Number Wrong − [ ]

Difference [ ]

+ 30

*Total Score [ ]

*Note: Blank papers should receive a score of 30.

1. When the polynomial $x^3 - 2$ is divided by the polynomial $x^2 - 2$, the remainder is

(A) 2 (B) -2 (C) $-2x - 2$ (D) $2x + 2$ (E) $2x - 2$

2. If a number eight times as large as x is increased by two, then one fourth of the result equals

(A) $2x + \frac{1}{2}$ (B) $x + \frac{1}{2}$ (C) $2x + 2$ (D) $2x + 4$ (E) $2x + 16$

3. Evaluate $(x^x)^{(x^x)}$ at $x = 2$.

(A) 16 (B) 64 (C) 256 (D) 1024 (E) 65,536

4. The perimeter of a semicircular region, measured in centimeters, is numerically equal to its area, measured in square centimeters. The radius of the semicircle, measured in centimeters, is

(A) π (B) $\frac{2}{\pi}$ (C) 1 (D) $\frac{1}{2}$ (E) $\frac{4}{\pi} + 2$

5. Two positive numbers x and y are in the ratio $a:b$, where $0<a<b$. If $x + y = c$, then the smaller of x and y is

(A) $\frac{ac}{b}$ (B) $\frac{bc - ac}{b}$ (C) $\frac{ac}{a + b}$ (D) $\frac{bc}{a + b}$ (E) $\frac{ac}{b - a}$

6. The sum of all but one of the interior angles of a convex polygon equals 2570°. The remaining angle is

(A) 90° (B) 105° (C) 120° (D) 130° (E) 144°

7. If the operation $x * y$ is defined by $x * y = (x + 1)(y + 1) - 1$, then which one of the following is **false?**

(A) $x * y = y * x$ for all real x and y.
(B) $x * (y + z) = (x * y) + (x * z)$ for all real x, y, and z.
(C) $(x - 1) * (x + 1) = (x * x) - 1$ for all real x.
(D) $x * 0 = x$ for all real x.
(E) $x * (y * z) = (x * y) * z$ for all real x, y, and z.

8. By definition $r! = r(r - 1)\cdots 1$ and $\binom{j}{k} = \frac{j!}{k!(j - k)!}$,
where r, j, k are positive integers and $k<j$. If $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}$ form an arithmetic progression with $n > 3$, then n equals

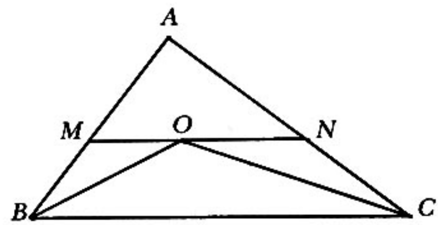
(A) 5 (B) 7 (C) 9 (D) 11 (E) 12

9. A vertical line divides the triangle with vertices (0,0), (1,1) and (9,1) in the xy-plane into two regions of equal area. The equation of the line is $x =$

(A) 2.5 (B) 3.0 (C) 3.5 (D) 4.0 (E) 4.5

10. In the adjoining diagram, BO bisects $\measuredangle CBA$, CO bisects $\measuredangle ACB$, and MN is parallel to BC. If $AB = 12$, $BC = 24$, and $AC = 18$, then the perimeter of $\triangle AMN$ is

(A) 30 (B) 33
(C) 36 (D) 39
(E) 42



11. How many integers with four different digits are there between 1,000 and 9,999 such that the absolute value of the difference between the first digit and the last digit is 2?

(A) 672 (B) 784 (C) 840 (D) 896 (E) 1,008

12. Let $f(x) = ax^7 + bx^3 + cx - 5$, where a, b and c are constants. If $f(-7) = 7$, then $f(7)$ equals

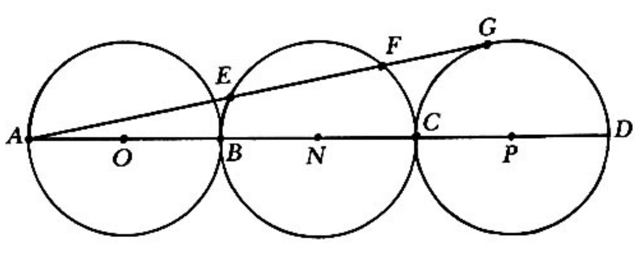
(A) -17 (B) -7 (C) 14 (D) 21 (E) not uniquely determined

13. If $a > 1$, $b > 1$ and $p = \dfrac{\log_b(\log_b a)}{\log_b a}$, then a^p equals

(A) 1 (B) b (C) $\log_a b$ (D) $\log_b a$ (E) $a^{\log_b a}$

14. In the adjoining figure, points B and C lie on line segment AD, and AB, BC and CD are diameters of circles O, N and P, respectively. Circles O, N and P all have radius 15 and the line AG is tangent to circle P at G. If AG intersects circle N at points E and F, then chord EF has length

(A) 20 (B) $15\sqrt{2}$
(C) 24 (D) 25
(E) none of these



15. Let $[z]$ denote the greatest integer not exceedng z. Let x and y satisfy the simultaneous equations

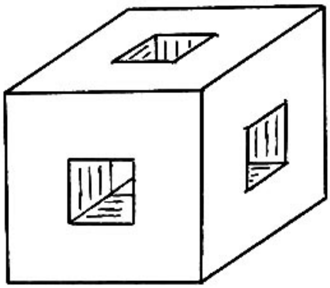
$$y = 2[x] + 3$$
$$y = 3[x - 2] + 5.$$

If x is not an integer, then $x + y$ is

(A) an integer
(B) between 4 and 5
(C) between -4 and 4
(D) between 15 and 16
(E) 16.5

16. In the adjoining figure, a wooden cube has edges of length 3 meters. Square holes, of side one meter, centered in each face are cut through to the opposite face. The edges of the holes are parallel to the edges of the cube. The entire surface area including the inside, in square meters, is

(A) 54
(B) 72
(C) 76
(D) 84
(E) 86



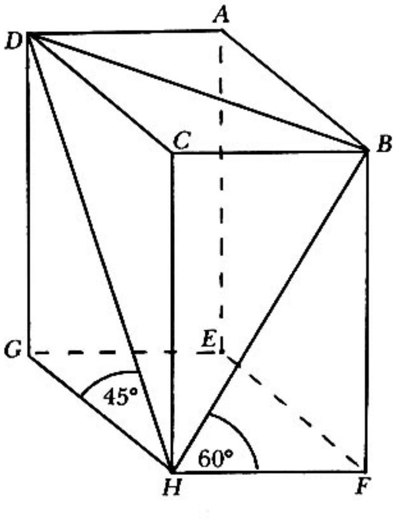
17. How many real numbers x satisfy the equation

$$3^{2x+2} - 3^{x+3} - 3^{x} + 3 = 0\ ?$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

18. In the adjoining figure of a rectangular solid, $\measuredangle DHG = 45°$ and $\measuredangle FHB = 60°$. Find the cosine of $\measuredangle BHD$.

(A) $\frac{\sqrt{3}}{6}$
(B) $\frac{\sqrt{2}}{6}$
(C) $\frac{\sqrt{6}}{3}$
(D) $\frac{\sqrt{6}}{4}$
(E) $\frac{\sqrt{6}-\sqrt{2}}{4}$



19. Let $f(x) = |x - 2| + |x - 4| - |2x - 6|$, for $2 \leqslant x \leqslant 8$. The sum of the largest and smallest values of $f(x)$ is

(A) 1 (B) 2 (C) 4 (D) 6 (E) none of these

20. The number of pairs of positive integers (x,y) which satisfy the equation $x^2 + y^2 = x^3$ is

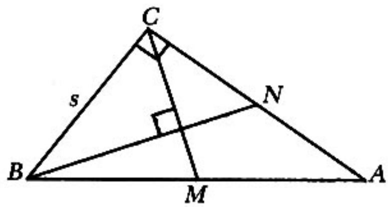
(A) 0 (B) 1 (C) 2 (D) not finite (E) none of these

21. In the adjoining figure, the triangle ABC is a right triangle with $\measuredangle BCA = 90°$. Median CM is perpendicular to median BN, and side $BC = s$. The length of BN is

(A) $s\sqrt{2}$ (B) $\frac{3}{2}s\sqrt{2}$

(C) $2s\sqrt{2}$ (D) $\frac{s\sqrt{5}}{2}$

(E) $\frac{s\sqrt{6}}{2}$

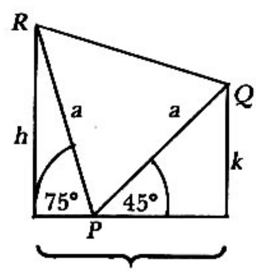


22. In a narrow alley of width w a ladder of length a is placed with its foot at a point P between the walls. Resting against one wall at Q, a distance k above the ground, the ladder makes a $45°$ angle with the ground. Resting against the other wall at R, a distance h above the ground, the ladder makes a $75°$ angle with the ground. The width w is equal to

(A) a (B) RQ

(C) k (D) $\frac{h + k}{2}$

(E) h

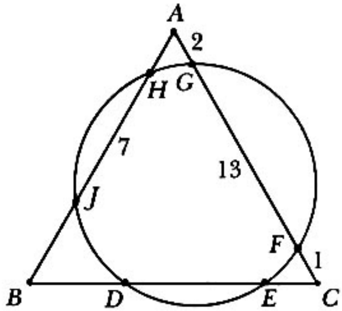


23. The lengths of the sides of a triangle are consecutive integers, and the largest angle is twice the smallest angle. The cosine of the smallest angle is

(A) $\frac{3}{4}$ (B) $\frac{7}{10}$ (C) $\frac{2}{3}$ (D) $\frac{9}{14}$ (E) none of these

24. In the adjoining figure, the circle meets the sides of an equilateral triangle at six points. If $AG = 2$, $GF = 13$, $FC = 1$ and $HJ = 7$, then DE equals

(A) $2\sqrt{22}$ (B) $7\sqrt{3}$
(C) 9 (D) 10
(E) 13



25. The adjoining figure is a map of part of a city: the small rectangles are blocks and the spaces in between are streets. Each morning a student walks from intersection A to intersection B, always walking along streets shown, always going east or south. For variety, at each intersection where he has a choice, he chooses with probability $\frac{1}{2}$ (independent of all other choices) whether to go east or south. Find the probability that, on any given morning, he walks through intersection C.

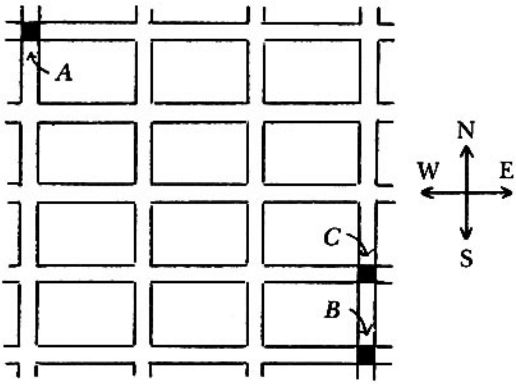
(A) $\frac{11}{32}$

(B) $\frac{1}{2}$

(C) $\frac{4}{7}$

(D) $\frac{21}{32}$

(E) $\frac{3}{4}$



26. If the base 8 representation of a perfect square is $ab3c$, where $a \neq 0$, then c is

(A) 0 (B) 1 (C) 3 (D) 4 (E) not uniquely determined

27. Suppose $z = a + bi$ is a solution of the polynomial equation

$$c_4z^4 + ic_3z^3 + c_2z^2 + ic_1z + c_0 = 0,$$

where c_0, c_1, c_2, c_3, a and b are real constants and $i^2 = -1$.
Which one of the following must also be a solution?

(A) $-a-bi$ (B) $a-bi$ (C) $-a+bi$ (D) $b+ai$ (E) none of these

28. A set of consecutive positive integers beginning with 1 is written on a blackboard. One number is erased. The average (arithmetic mean) of the remaining numbers is $35\frac{7}{17}$. What number was erased?

(A) 6 (B) 7 (C) 8 (D) 9 (E) can not be determined

29. Let x, y and z be three positive real numbers whose sum is 1. If no one of these numbers is more than twice any other, then the minimum possible value of the product xyz is

(A) $\frac{1}{32}$ (B) $\frac{1}{36}$ (C) $\frac{4}{125}$ (D) $\frac{1}{127}$ (E) none of these

30. Find the units digit in the decimal expansion of

$$(15 + \sqrt{220})^{19} + (15 + \sqrt{220})^{82}.$$

(A) 0 (B) 2 (C) 5 (D) 9 (E) none of these

34th ANNUAL
AMERICAN HIGH SCHOOL
MATHEMATICS EXAMINATION
(AHSME)

TUESDAY, MARCH 1, 1983

A Prize Examination Sponsored by:
MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
CASUALTY ACTUARIAL SOCIETY

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NEW. The first annual **American Invitational Mathematics Examination** (AIME) will be given on Tuesday, March 22, 1983. It will be a 2½-hour, 15-question, short answer examination, not multiple choice. Those students who score 95 or above on this AHSME examination are invited to take the AIME. (The student's school must also agree to administer the AIME.) Your AHSME School Manager has a sample AIME for you to look at. Invitations to take the USA Mathematical Olympiad in May will be based on AIME results.

To be filled in by the student

PRINT ____
last name first name middle initial

____
(home address) no. street

____
city state or province zip or postcode

____
school (full name) street address

____
city state or province zip or postcode

Grade Level (9, 10, 11, 12 or other number) ☐ Sex: Male ☐ Female ☐

1		16	
2		17	
3		18	
4		19	
5		20	
6		21	
7		22	
8		23	
9		24	
10		25	
11		26	
12		27	
13		28	
14		29	
15		30	

Not to be filled in by the student.

Each question has a value of *four points. One* point is to be deducted for each wrong answer (do *not* deduct points for questions left unanswered). The total score is computed by taking four times the number right, subtracting the number wrong *and* adding 30.

Number Correct ☐ × 4 = ☐

Number Wrong − ☐

Difference ☐

\+ 30

*Total Score ☐

*Note: Blank papers should receive a score of 30.

1. If $x \neq 0$, $\frac{x}{2} = y^2$ and $\frac{x}{4} = 4y$, then x equals

(A) 8 (B) 16 (C) 32 (D) 64 (E) 128

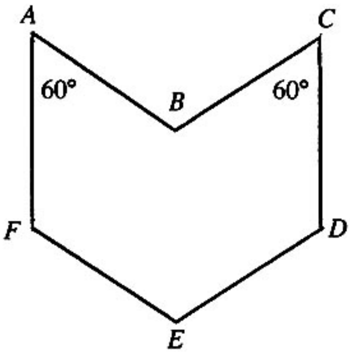
2. Point P is outside circle C on the plane. At most how many points on C are 3 cm from P?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 8

3. Three primes, p, q and r, satisfy $p+q=r$ and $1 < p < q$. Then p equals

(A) 2 (B) 3 (C) 7 (D) 13 (E) 17

4. In the adjoining plane figure, sides AF and CD are parallel, as are sides AB and FE, and sides BC and ED. Each side has length 1. Also $\measuredangle FAB = \measuredangle BCD = 60^\circ$. The area of the figure is



(A) $\frac{\sqrt{3}}{2}$ (B) 1 (C) $\frac{3}{2}$

(D) $\sqrt{3}$ (E) 2

5. Triangle ABC has a right angle at C. If $\sin A = \frac{2}{3}$, then $\tan B$ is

(A) $\frac{3}{5}$ (B) $\frac{\sqrt{5}}{3}$ (C) $\frac{2}{\sqrt{5}}$ (D) $\frac{\sqrt{5}}{2}$ (E) $\frac{5}{3}$

6. When x^5, $x+\frac{1}{x}$ and $1+\frac{2}{x}+\frac{3}{x^2}$ are multiplied, the product is a polynomial of degree

(A) 2 (B) 3 (C) 6 (D) 7 (E) 8

7. Alice sells an item at \$10 less than the list price and receives 10% of her selling price as her commission. Bob sells the same item at \$20 less than the list price and receives 20% of his selling price as his commission. If they both get the same commission, then the list price is

(A) \$20 (B) \$30 (C) \$50 (D) \$70 (E) \$100

8. Let $f(x) = \frac{x+1}{x-1}$. Then for $x^2 \neq 1$, $f(-x)$ is

(A) $\frac{1}{f(x)}$ (B) $-f(x)$ (C) $\frac{1}{f(-x)}$ (D) $-f(-x)$ (E) $f(x)$

9. In a certain population the ratio of the number of women to the number of men is 11 to 10. If the average (arithmetic mean) age of the women is 34 and the average age of the men is 32, then the average age of the population is

(A) $32\frac{9}{10}$ (B) $32\frac{20}{21}$ (C) 33 (D) $33\frac{1}{21}$ (E) $33\frac{1}{10}$

10. Segment AB is both a diameter of a circle of radius 1 and a side of an equilateral triangle ABC. The circle also intersects AC and BC at points D and E, respectively. The length of AE is

(A) $\frac{3}{2}$ (B) $\frac{5}{3}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\sqrt{3}$ (E) $\frac{2+\sqrt{3}}{2}$

11. Simplify $\sin(x-y)\cos y + \cos(x-y)\sin y$.

(A) 1 (B) $\sin x$ (C) $\cos x$ (D) $\sin x \cos 2y$ (E) $\cos x \cos 2y$

12. If $\log_7\left(\log_3(\log_2 x)\right) = 0$, then $x^{-1/2}$ equals

(A) $\frac{1}{3}$ (B) $\frac{1}{2\sqrt{3}}$ (C) $\frac{1}{3\sqrt{3}}$ (D) $\frac{1}{\sqrt{42}}$ (E) none of these

13. If $xy = a$, $xz = b$, and $yz = c$, and none of these quantities is 0, then $x^2 + y^2 + z^2$ equals

(A) $\frac{ab+ac+bc}{abc}$ (B) $\frac{a^2+b^2+c^2}{abc}$ (C) $\frac{(a+b+c)^2}{abc}$

(D) $\frac{(ab+ac+bc)^2}{abc}$ (E) $\frac{(ab)^2+(ac)^2+(bc)^2}{abc}$

14. The units digit of $3^{1001}\,7^{1002}\,13^{1003}$ is

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

15. Three balls marked 1, 2 and 3 are placed in an urn. One ball is drawn, its number is recorded, and then the ball is returned to the urn. This process is repeated and then repeated once more, and each ball is equally likely to be drawn on each occasion. If the sum of the numbers recorded is 6, what is the probability that the ball numbered 2 was drawn all three times?

(A) $\frac{1}{27}$ (B) $\frac{1}{8}$ (C) $\frac{1}{7}$ (D) $\frac{1}{6}$ (E) $\frac{1}{3}$

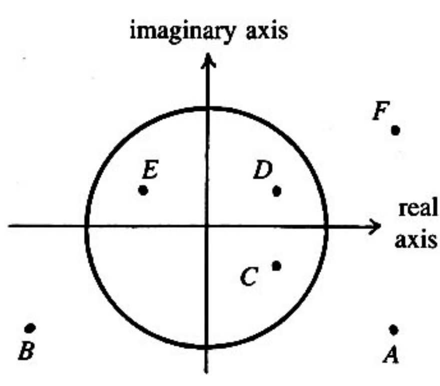
16. Let $x = .123456789101112...998999$, where the digits are obtained by writing the integers 1 through 999 in order. The 1983^{rd} digit to the right of the decimal point is

(A) 2 (B) 3 (C) 5 (D) 7 (E) 8

17. The diagram to the right shows several numbers in the complex plane. The circle is the unit circle centered at the origin. One of these numbers is the reciprocal of F. Which one?

(A) A (B) B (C) C

(D) D (E) E



18. Let f be a polynomial function such that, for all real x,

$$f(x^2 + 1) = x^4 + 5x^2 + 3.$$

For all real x, $f(x^2 - 1)$ is

(A) $x^4 + 5x^2 + 1$ (B) $x^4 + x^2 - 3$ (C) $x^4 - 5x^2 + 1$

(D) $x^4 + x^2 + 3$ (E) none of these

19. Point D is on side CB of triangle ABC. If $\measuredangle CAD = \measuredangle DAB = 60°$, $AC = 3$ and $AB = 6$, then the length of AD is

(A) 2 (B) 2.5 (C) 3 (D) 3.5 (E) 4

20. If $\tan\alpha$ and $\tan\beta$ are the roots of $x^2 - px + q = 0$, and $\cot\alpha$ and $\cot\beta$ are the roots of $x^2 - rx + s = 0$, then rs is necessarily

(A) pq (B) $\frac{1}{pq}$ (C) $\frac{p}{q^2}$ (D) $\frac{q}{p^2}$ (E) $\frac{p}{q}$

21. Find the smallest positive number from the numbers below.

(A) $10 - 3\sqrt{11}$ (B) $3\sqrt{11} - 10$ (C) $18 - 5\sqrt{13}$

(D) $51 - 10\sqrt{26}$ (E) $10\sqrt{26} - 51$

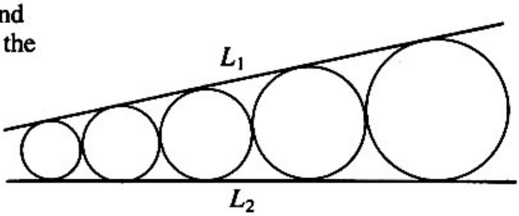
22. Consider the two functions $f(x) = x^2 + 2bx + 1$ and $g(x) = 2a(x + b)$, where the variable x and the constants a and b are real numbers. Each such pair of constants a and b may be considered as a point (a, b) in an ab-plane. Let S be the set of such points (a, b) for which the graphs of $y = f(x)$ and $y = g(x)$ do **not** intersect (in the xy-plane). The area of S is

(A) 1 (B) π (C) 4 (D) 4π (E) infinite

23. In the adjoining figure the five circles are tangent to one another consecutively and to the lines L_1 and L_2. If the radius of the largest circle is 18 and that of the smallest one is 8, then the radius of the middle circle is

(A) 12 (B) 12.5 (C) 13

(D) 13.5 (E) 14



24. How many non-congruent right triangles are there such that the perimeter in cm and area in cm^2 are numerically equal?

(A) none (B) 1 (C) 2 (D) 4 (E) infinitely many

25. If $60^a = 3$ and $60^b = 5$, then $12^{[(1-a-b)/2(1-b)]}$ is

(A) $\sqrt{3}$ (B) 2 (C) $\sqrt{5}$ (D) 3 (E) $\sqrt{12}$

26. The probability that event A occurs is $\frac{3}{4}$; the probability that event B occurs is $\frac{2}{3}$. Let p be the probability that both A and B occur. The smallest interval necessarily containing p is the interval

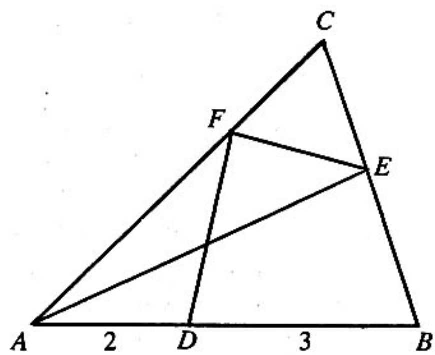
(A) $\left[\frac{1}{12},\frac{1}{2}\right]$ (B) $\left[\frac{5}{12},\frac{1}{2}\right]$ (C) $\left[\frac{1}{2},\frac{2}{3}\right]$ (D) $\left[\frac{5}{12},\frac{2}{3}\right]$
(E) $\left[\frac{1}{12},\frac{2}{3}\right]$

27. A large sphere is on a horizontal field on a sunny day. At a certain time the shadow of the sphere reaches out a distance of 10 m from the point where the sphere touches the ground. At the same instant a meter stick (held vertically with one end on the ground) casts a shadow of length 2 m. What is the radius of the sphere in meters? (Assume the sun's rays are parallel and the meter stick is a line segment.)

(A) $\frac{5}{2}$ (B) $9 - 4\sqrt{5}$ (C) $8\sqrt{10} - 23$ (D) $6 - \sqrt{15}$ (E) $10\sqrt{5} - 20$

28. Triangle ABC in the figure has area 10. Points D, E and F, all distinct from A, B and C, are on sides AB, BC and CA respectively, and $AD = 2$, $DB = 3$. If triangle ABE and quadrilateral $DBEF$ have equal areas, then that area is

(A) 4 (B) 5 (C) 6
(D) $\frac{5}{3}\sqrt{10}$
(E) not uniquely determined

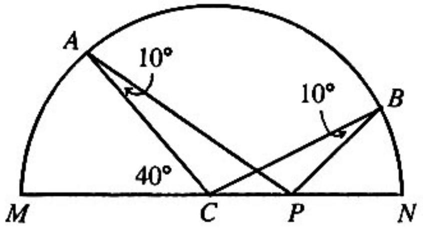


29. A point P lies in the same plane as a given square of side 1. Let the vertices of the square, taken counterclockwise, be A, B, C and D. Also, let the distances from P to A, B and C, respectively, be u, v and w. What is the greatest distance that P can be from D if $u^2 + v^2 = w^2$?

(A) $1 + \sqrt{2}$ (B) $2\sqrt{2}$ (C) $2 + \sqrt{2}$ (D) $3\sqrt{2}$ (E) $3 + \sqrt{2}$

30. Distinct points A and B are on a semicircle with diameter MN and center C. The point P is on CN and $\measuredangle CAP = \measuredangle CBP = 10^\circ$. If $\widehat{MA} = 40^\circ$, then $\widehat{BN}$ equals

(A) 10° (B) 15° (C) 20°
(D) 25° (E) 30°



Students and teachers with questions or comments about this Examination may write to:

Professor Stephen B. Maurer, Chairman
MAA Committee on High School Contests
Swarthmore College, Swarthmore, PA 19081
current address: Alfred P. Sloan Foundation
630 Fifth Avenue
New York, NY 10111-0242

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Professor Walter E. Mientka, Executive Director
MAA Committee on High School Contests
Department of Mathematics and Statistics
University of Nebraska
Lincoln, NE 68588-0322

35th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)

TUESDAY, FEBRUARY 28, 1984

A Prize Examination Sponsored by:
MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
CASUALTY ACTUARIAL SOCIETY

INSTRUCTIONS

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2. This is a thirty question multiple choice test. Each question is followed by answers marked A, B, C, D, E. Only one of these is correct.
3. *There is a penalty for wrong answers*. On the average, random guessing will neither increase nor decrease your score. Hard to read or ambiguous answers will be marked *wrong*.
4. Use pencil. Scratch paper, graph paper, ruler, compass and eraser are permitted. *Calculators and slide rules are not permitted.*
5. Students should note that figures are not necessarily drawn to scale.
6. The back of this cover is a form on which you are asked to write certain information and record your answers. You must use this form — unless your region has centralized computer scoring (for which your teacher will give you a different form and instructions).
7. When your teacher indicates, tear off this cover and fill in the information requested on the back of it. *Keep the examination questions covered while writing this information.*
8. When your teacher gives the signal, begin working the problems. You will have **90 MINUTES** working time for the test.

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The second annual **American Invitational Mathematics Examination** (AIME) will be given on Tuesday, March 20, 1984. It will be a 2½-hour, 15-question, short answer examination, not multiple choice. Those students who score 95 or above on this AHSME examination are invited to take the AIME. (The student's school must also agree to administer the AIME.) Invitations to take the USA Mathematical Olympiad in May will be based on AIME results.

To be filled in by the student

PRINT ____
last name first name middle initial

____
(home address) no. street

____
city state or province zip or postcode

____
school (full name) street address

____
city state or province zip or postcode

Grade Level (9, 10, 11, 12 or other number) ☐ Date: ____________

1		16	
2		17	
3		18	
4		19	
5		20	
6		21	
7		22	
8		23	
9		24	
10		25	
11		26	
12		27	
13		28	
14		29	
15		30	

Not to be filled in by the student.

Each question has a value of *four points. One* point is to be deducted for each wrong answer (do *not* deduct points for questions left unanswered). The total score is computed by taking four times the number right, subtracting the number wrong *and* adding 30.

Number Correct ☐ × 4 = ☐

Number Wrong − ☐

Difference ☐

+ 30

*Total Score ☐

*Note: Blank papers should receive a score of 30.

1. $\dfrac{1000^2}{252^2 - 248^2}$ equals

(A) 62,500 (B) 1000 (C) 500 (D) 250 (E) $\frac{1}{2}$

2. If x, y and $y - \frac{1}{x}$ are not 0, then

$$\frac{x - \frac{1}{y}}{y - \frac{1}{x}}$$ equals

(A) 1 (B) $\frac{x}{y}$ (C) $\frac{y}{x}$ (D) $\frac{x}{y} - \frac{y}{x}$ (E) $xy - \frac{1}{xy}$

3. Let n be the smallest nonprime integer greater than 1 with no prime factor less than 10. Then

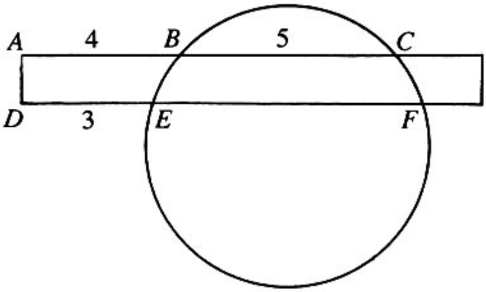
(A) $100 < n \le 110$ (B) $110 < n \le 120$ (C) $120 < n \le 130$

(D) $130 < n \le 140$ (E) $140 < n \le 150$.

4. A rectangle intersects a circle as shown: $AB = 4$, $BC = 5$ and $DE = 3$. Then EF equals

(A) 6 (B) 7 (C) $\frac{20}{3}$

(D) 8 (E) 9



5. The largest integer n for which $n^{200} < 5^{300}$ is

(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

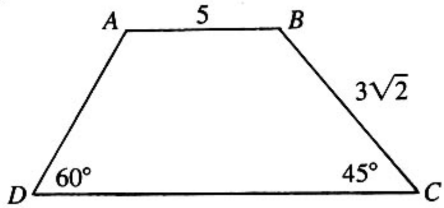
6. In a certain school, there are three times as many boys as girls and nine times as many girls as teachers. Using the letters b, g, t to represent the number of boys, girls and teachers, respectively, then the total number of boys, girls and teachers can be represented by the expression

(A) $31b$ (B) $\frac{37}{27}b$ (C) $13g$ (D) $\frac{37}{27}g$ (E) $\frac{37}{27}t$

7. When Dave walks to school, he averages 90 steps per minute, each of his steps 75 cm long. It takes him 16 minutes to get to school. His brother, Jack, going to the same school by the same route, averages 100 steps per minute, but his steps are only 60 cm long. How long does it take Jack to get to school?

(A) $14\frac{2}{9}$ min. (B) 15 min. (C) 18 min. (D) 20 min. (E) $22\frac{2}{9}$ min.

8. Figure $ABCD$ is a trapezoid with $AB \parallel DC$, $AB = 5$, $BC = 3\sqrt{2}$, $\measuredangle BCD = 45°$ and $\measuredangle CDA = 60°$. The length of DC is



(A) $7 + \frac{2}{3}\sqrt{3}$ (B) 8 (C) $9\frac{1}{2}$

(D) $8 + \sqrt{3}$ (E) $8 + 3\sqrt{3}$

9. The number of digits in $4^{16}5^{25}$ (when written in the usual base 10 form) is

(A) 31 (B) 30 (C) 29 (D) 28 (E) 27

10. Four complex numbers lie at the vertices of a square in the complex plane. Three of the numbers are $1 + 2i$, $-2 + i$ and $-1 - 2i$. The fourth number is

(A) $2 + i$ (B) $2 - i$ (C) $1 - 2i$ (D) $-1 + 2i$ (E) $-2 - i$

11. A calculator has a key which replaces the displayed entry with its square, and another key which replaces the displayed entry with its reciprocal. Let y be the final result if one starts with an entry $x \neq 0$ and alternately squares and reciprocates n times each. Assuming the calculator is completely accurate (e.g., no roundoff or overflow), then y equals

(A) $x^{((-2)^n)}$ (B) x^{2n} (C) x^{-2n} (D) $x^{-(2^n)}$ (E) $x^{((-1)^n 2n)}$

12. If the sequence $\{a_n\}$ is defined by

$$a_1 = 2,$$

$$a_{n+1} = a_n + 2n \qquad (n \geq 1),$$

then a_{100} equals

(A) 9900 (B) 9902 (C) 9904 (D) 10100 (E) 10102

13. $\dfrac{2\sqrt{6}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$ equals

(A) $\sqrt{2}+\sqrt{3}-\sqrt{5}$ (B) $4-\sqrt{2}-\sqrt{3}$ (C) $\sqrt{2}+\sqrt{3}+\sqrt{6}-5$

(D) $\frac{1}{2}(\sqrt{2}+\sqrt{5}-\sqrt{3})$ (E) $\frac{1}{3}(\sqrt{3}+\sqrt{5}-\sqrt{2})$

14. The product of all real roots of the equation $x^{\log_{10} x} = 10$ is

(A) 1 (B) -1 (C) 10 (D) 10^{-1} (E) none of these

15. If $\sin 2x \sin 3x = \cos 2x \cos 3x$, then one value for x is

(A) 18° (B) 30° (C) 36° (D) 45° (E) 60°

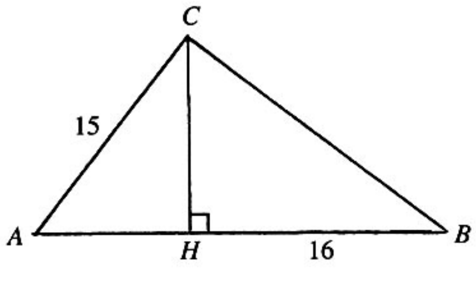
16. The function $f(x)$ satisfies $f(2+x) = f(2-x)$ for all real numbers x. If the equation $f(x) = 0$ has exactly four distinct real roots, then the sum of these roots is

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

17. A right triangle ABC with hypotenuse AB has side $AC = 15$. Altitude CH divides AB into segments AH and HB, with $HB = 16$.
The area of ΔABC is

(A) 120 (B) 144 (C) 150

(D) 216 (E) $144\sqrt{5}$



18. A point (x,y) is to be chosen in the coordinate plane so that it is equally distant from the x-axis, the y-axis, and the line $x + y = 2$. Then x is

(A) $\sqrt{2}-1$ (B) $\frac{1}{2}$ (C) $2-\sqrt{2}$ (D) 1 (E) not uniquely determined

19. A box contains 11 balls, numbered 1, 2, 3, …, 11. If 6 balls are drawn simultaneously at random, what is the probability that the sum of the numbers on the balls drawn is odd?

(A) $\frac{100}{231}$ (B) $\frac{115}{231}$ (C) $\frac{1}{2}$ (D) $\frac{118}{231}$ (E) $\frac{6}{11}$.

20. The number of distinct solutions of the equation

$$\left|x - |2x+1|\right| = 3 \quad \text{is}$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

21. The number of triples (a,b,c) of positive integers which satisfy the simultaneous equations

$$ab + bc = 44,$$

$$ac + bc = 23,$$

is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

22. Let a and c be fixed positive numbers. For each real number t let (x_t, y_t) be the vertex of the parabola $y = ax^2 + tx + c$. If the set of vertices (x_t, y_t) for all real values of t is graphed in the plane, the graph is

(A) a straight line

(B) a parabola

(C) part, but not all, of a parabola

(D) one branch of a hyperbola

(E) none of these

23. $\dfrac{\sin 10^\circ + \sin 20^\circ}{\cos 10^\circ + \cos 20^\circ}$ equals

(A) $\tan 10^\circ + \tan 20^\circ$ (B) $\tan 30^\circ$ (C) $\frac{1}{2}(\tan 10^\circ + \tan 20^\circ)$

(D) $\tan 15^\circ$ (E) $\frac{1}{4}\tan 60^\circ$

24. If a and b are positive real numbers and each of the equations $x^2 + ax + 2b = 0$ and $x^2 + 2bx + a = 0$ has real roots, then the smallest possible value of $a + b$ is

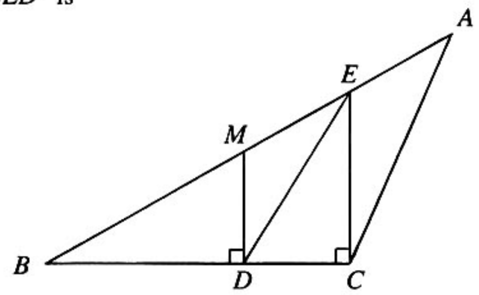
(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

25. The total area of all the faces of a rectangular solid is $22\ \text{cm}^2$, and the total length of all its edges is 24 cm. Then the length in cm of any one of its internal diagonals is

(A) $\sqrt{11}$ (B) $\sqrt{12}$ (C) $\sqrt{13}$ (D) $\sqrt{14}$

(E) not uniquely determined

26. In the obtuse triangle ABC, $AM = MB$, $MD \perp BC$, $EC \perp BC$. If the area of ΔABC is 24, then the area of ΔBED is

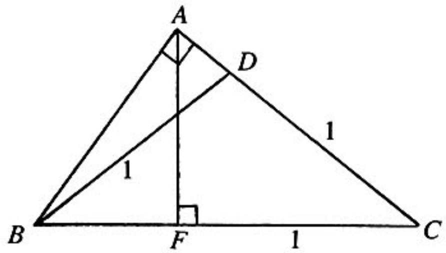


(A) 9 (B) 12

(C) 15 (D) 18

(E) not uniquely determined

27. In ΔABC, D is on AC and F is on BC. Also, $AB \perp AC$, $AF \perp BC$, and $BD = DC = FC = 1$.
Find AC.



(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) $\sqrt[3]{2}$

(D) $\sqrt[3]{3}$ (E) $\sqrt[4]{3}$.

28. The number of distinct pairs of integers (x,y) such that $0 < x < y$ and $\sqrt{1984} = \sqrt{x} + \sqrt{y}$ is

(A) 0 (B) 1 (C) 3 (D) 4 (E) 7

29. Find the largest value of $\frac{y}{x}$ for pairs of real numbers (x, y) which satisfy $(x-3)^2 + (y-3)^2 = 6$.

(A) $3 + 2\sqrt{2}$ (B) $2 + \sqrt{3}$ (C) $3\sqrt{3}$ (D) 6 (E) $6 + 2\sqrt{3}$

30. For any complex number $w = a + bi$, $|w|$ is defined to be the real number $\sqrt{a^2 + b^2}$. If $w = \cos 40° + i \sin 40°$, then $\left|w + 2w^2 + 3w^3 + \cdots + 9w^9\right|^{-1}$ equals

(A) $\frac{1}{9}\sin 40°$ (B) $\frac{2}{9}\sin 20°$ (C) $\frac{1}{9}\cos 40°$

(D) $\frac{1}{18}\cos 20°$ (E) none of these

Answers and detailed solutions to the 1984 AHSME are available in a Solutions Pamphlet. Your AHSME School Manager has a copy.

Students and teachers with questions or comments about this Examination may write to:

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Lincoln, NE 68588-0322

36th ANNUAL AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION (AHSME)

TUESDAY, FEBRUARY 26, 1985

A Prize Examination Sponsored by:
MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
CASUALTY ACTUARIAL SOCIETY
AMERICAN STATISTICAL ASSOCIATION

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9. When your proctor gives the signal, begin working the problems. You will have **90 MINUTES** working time for the test.
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The results of this AHSME are used to identify students with unusual mathematical ability. To assure that this purpose is served, the MAA Committee on High School Contests reserves the right to reexamine students before deciding whether to grant official status to individual or team scores. Reexamination will be requested when there is a reasonable basis to believe that scores have been obtained by extremely lucky guessing or dishonesty. Official status will not be granted if a student or school does not agree to a requested reexamination. The Committee also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

36th AHSME 1985

To be filled in by the student

PRINT ______________________________ [ ]

last name first name middle initial age

____

(home address) no. street

____

city state or province zip or postcode

____

school (full name) street address

____

city state or province zip or postcode

Grade Level (9, 10, 11, 12 or other number) [ ] Date:__________

1		16	
2		17	
3		18	
4		19	
5		20	
6		21	
7		22	
8		23	
9		24	
10		25	
11		26	
12		27	
13		28	
14		29	
15		30	

Not to be filled in by the student.

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Number Correct [ ] × 4 = [ ]

Number Wrong − [ ]

Difference [ ]

\+ 30

*Total Score [ ]

*Note: Blank papers should receive a score of 30.

1. If $2x + 1 = 8$, then $4x + 1 =$

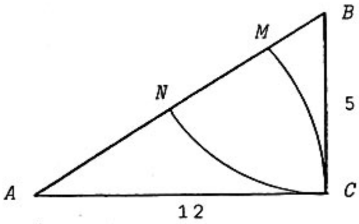
   (A) 15 (B) 16 (C) 17 (D) 18 (E) 19

2. In an arcade game, the "monster" is the shaded sector of a circle of radius 1 cm, as shown in the figure. The missing piece (the mouth) has central angle $60°$. What is the perimeter of the monster in cm?

   (A) $\pi + 2$ (B) 2π (C) $\frac{5}{3}\pi$

   (D) $\frac{5}{6}\pi + 2$ (E) $\frac{5}{3}\pi + 2$

3. In right ΔABC with legs 5 and 12, arcs of circles are drawn, one with center A and radius 12, the other with center B and radius 5. They intersect the hypotenuse in M and N. Then MN has length



   (A) 2 (B) $\frac{13}{5}$ (C) 3

   (D) 4 (E) $\frac{24}{5}$

4. A large bag of coins contains pennies, dimes and quarters. There are twice as many dimes as pennies and three times as many quarters as dimes. An amount of money which could be in the bag is

   (A) \$306 (B) \$333 (C) \$342 (D) \$348 (E) \$360

5. Which terms must be removed from the sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12}$$

   if the sum of the remaining terms is to equal 1?

   (A) $\frac{1}{4}$ and $\frac{1}{8}$ (B) $\frac{1}{4}$ and $\frac{1}{12}$ (C) $\frac{1}{8}$ and $\frac{1}{12}$

   (D) $\frac{1}{6}$ and $\frac{1}{10}$ (E) $\frac{1}{8}$ and $\frac{1}{10}$

6. One student in a class of boys and girls is to be chosen to represent the class. Each student is equally likely to be chosen and the probability that a boy is chosen is 2/3 of the probability that a girl is chosen. The ratio of the number of boys to the total number of boys and girls is

   (A) 1/3 (B) 2/5 (C) 1/2 (D) 3/5 (E) 2/3

7. In some computer languages (such as APL), when there are no parentheses in an algebraic expression, the operations are grouped from right to left. Thus, $a \times b - c$ in such languages means the same as $a(b-c)$ in ordinary algebraic notation. If $a \div b - c + d$ is evaluated in such a language, the result in ordinary algebraic notation would be

(A) $\frac{a}{b} - c + d$ (B) $\frac{a}{b} - c - d$ (C) $\frac{d+c-b}{a}$

(D) $\frac{a}{b-c+d}$ (E) $\frac{a}{b-c-d}$

8. Let a, a', b, b' be real numbers with a and a' nonzero. The solution to $ax + b = 0$ is less than the solution to $a'x + b' = 0$ if and only if

(A) $a'b < ab'$ (B) $ab' < a'b$ (C) $ab < a'b'$ (D) $\frac{b}{a} < \frac{b'}{a'}$ (E) $\frac{b'}{a'} < \frac{b}{a}$

9. The odd positive integers, 1, 3, 5, 7, ... , are arranged in five columns continuing with the pattern shown on the right. Counting from the left, the column in which 1985 appears is the

	1	3	5	7
15	13	11	9	
	17	19	21	23
31	29	27	25	
	33	35	37	39
47	45	43	41	
	49	51	53	55
.	.	.	.	
	.	.	.	.
.	.	.	.	
	.	.	.	.

(A) first (B) second

(C) third (D) fourth

(E) fifth

10. An arbitrary circle can intersect the graph of $y = \sin x$ in

(A) at most 2 points (B) at most 4 points (C) at most 6 points

(D) at most 8 points (E) more than 16 points

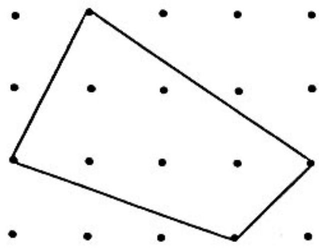
11. How many distinguishable rearrangements of the letters in CONTEST have both the vowels first? (For instance, OETCNST is one such arrangement, but OTETSNC is not.)

(A) 60 (B) 120 (C) 240 (D) 720 (E) 2520

12. Let p, q and r be distinct prime numbers, where 1 is not considered a prime. Which of the following is the smallest positive perfect cube having $n = pq^2r^4$ as a divisor?

(A) $p^8q^8r^8$ (B) $(pq^2r^2)^3$ (C) $(p^2q^2r^2)^3$ (D) $(pqr^2)^3$ (E) $4p^3q^3r^3$

13. Pegs are put in a board 1 unit apart both horizontally and vertically. A rubber band is stretched over 4 pegs as shown in the figure, forming a quadrilateral. Its area in square units is



(A) 4 (B) 4.5 (C) 5

(D) 5.5 (E) 6

14. Exactly three of the interior angles of a convex polygon are obtuse. What is the maximum number of sides of such a polygon?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

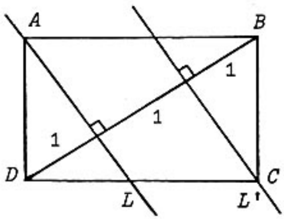
15. If a and b are positive numbers such that $a^b = b^a$ and $b = 9a$, then the value of a is

(A) 9 (B) 1/9 (C) $\sqrt[9]{9}$ (D) $\sqrt[3]{9}$ (E) $\sqrt[4]{3}$

16. If $A = 20°$ and $B = 25°$, then the value of $(1 + \tan A)(1 + \tan B)$ is

(A) $\sqrt{3}$ (B) 2 (C) $1 + \sqrt{2}$ (D) $2(\tan A + \tan B)$ (E) none of these

17. Diagonal DB of rectangle $ABCD$ is divided into three segments of length 1 by parallel lines L and L' that pass through A and C and are perpendicular to DB. The area of $ABCD$, rounded to one decimal place, is



(A) 4.1 (B) 4.2 (C) 4.3

(D) 4.4 (E) 4.5

18. Six bags of marbles contain 18, 19, 21, 23, 25 and 34 marbles, respectively. One bag contains chipped marbles only. The other 5 bags contain no chipped marbles. Jane takes three of the bags and George takes two of the others. Only the bag of chipped marbles remains. If Jane gets twice as many marbles as George, how many chipped marbles are there?

(A) 18 (B) 19 (C) 21 (D) 23 (E) 25

19. Consider the graphs of $y = Ax^2$ and $y^2 + 3 = x^2 + 4y$, where A is a **positive** constant and x and y are real variables. In how many points do the two graphs intersect?

(A) exactly 4 (B) exactly 2

(C) at least 1, but the number varies for different positive values of A

(D) 0 for at least one positive value of A

(E) none of these

20. A wooden cube with edge length n units (where n is an integer > 2) is painted black all over. By slices parallel to its faces, the cube is cut into n^3 smaller cubes each of unit edge length. If the number of smaller cubes with just one face painted black is equal to the number of smaller cubes completely free of paint, what is n?

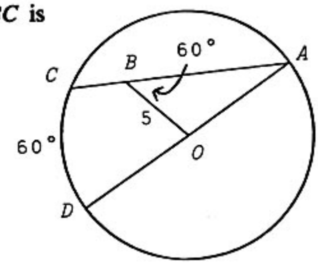
(A) 5 (B) 6 (C) 7 (D) 8 (E) none of these

21. How many integers x satisfy the equation

$$(x^2 - x - 1)^{x+2} = 1 \quad ?$$

(A) 2 (B) 3 (C) 4 (D) 5 (E) none of these

22. In a circle with center O, AD is a diameter, ABC is a chord, $BO = 5$ and $\angle ABO = \widehat{CD} = 60°$. Then the length of BC is



(A) 3 (B) $3 + \sqrt{3}$

(C) $5 - \frac{\sqrt{3}}{2}$ (D) 5

(E) none of the above

23. If $x = \frac{-1 + i\sqrt{3}}{2}$ and $y = \frac{-1 - i\sqrt{3}}{2}$, where $i^2 = -1$, then which of the following is **not** correct?

(A) $x^5 + y^5 = -1$ (B) $x^7 + y^7 = -1$ (C) $x^9 + y^9 = -1$

(D) $x^{11} + y^{11} = -1$ (E) $x^{13} + y^{13} = -1$

24. A non-zero digit is chosen in such a way that the probability of choosing digit d is $\log_{10}(d+1) - \log_{10} d$. The probability that the digit 2 is chosen is exactly 1/2 the probability that the digit chosen is in the set

(A) {2,3} (B) {3,4} (C) {4,5,6,7,8} (D) {5,6,7,8,9} (E) {4,5,6,7,8,9}

25. The volume of a certain rectangular solid is $8\,\text{cm}^3$, its total surface area is $32\,\text{cm}^2$, and its three dimensions are in geometric progression. The sum of the lengths in cm of all the edges of this solid is

(A) 28 (B) 32 (C) 36 (D) 40 (E) 44

26. Find the least positive integer n for which $\frac{n-13}{5n+6}$ is a non-zero reducible fraction.

(A) 45 (B) 68 (C) 155 (D) 226 (E) none of these

27. Consider a sequence $x_1, x_2, x_3, \ldots$, defined by:

$$x_1 = \sqrt[3]{3}, \quad x_2 = (\sqrt[3]{3})^{\sqrt[3]{3}}, \quad \text{and in general,}$$
$$x_n = (x_{n-1})^{\sqrt[3]{3}} \text{ for } n > 1.$$

What is the smallest value of n for which x_n is an integer?

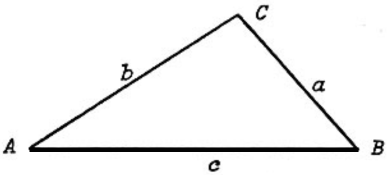
(A) 2 (B) 3 (C) 4 (D) 9 (E) 27

28. In $\triangle ABC$, we have $\angle C = 3\angle A$, $a = 27$ and $c = 48$. What is b ?

(A) 33 (B) 35

(C) 37 (D) 39

(E) not uniquely determined



29. In their base 10 representations, the integer a consists of a sequence of 1985 eights and the integer b consists of a sequence of 1985 fives. What is the sum of the digits of the base 10 representation of the integer $9ab$?

(A) 15880 (B) 17856 (C) 17865 (D) 17874 (E) 19851

30. Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x. Then the number of real solutions to $4x^2 - 40\lfloor x \rfloor + 51 = 0$ is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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AMERICAN MATHEMATICS COMPETITIONS

37th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION (AHSME)

TUESDAY, FEBRUARY 25, 1986

Sponsors:
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INSTRUCTIONS AND INFORMATION

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a thirty question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. **NEW SCORING RULES.** You will receive 5 points for each correct answer, 0 points for each incorrect answer, and 2 points for each problem left *unanswered*. You will no longer receive 30 points to start with.
4. Guessing. Be careful! You get a sure 2 points if you don't answer a question; a wrong guess will gain you 0. If you feel you have only 1 chance in 3 of being right, don't guess; with the new system such guessing will, on average, *lower* your score. However, if you feel you have 1 chance in 2 (for instance, by eliminating 3 of the 5 answers), go ahead and guess. In any event, try to *solve* the problem first.
5. Use pencil. Scratch paper, graph paper, ruler, compass and eraser are permitted. *Calculators and slide rules are not permitted.*
6. Figures are not necessarily drawn to scale.
7. The back of this cover is a form on which you are asked to write certain information, sign a Certification and record your answers. You must use this form—unless your region has centralized computer scoring (for which your proctor will give you a different form and instructions).
8. When the proctor indicates, tear off this cover and provide the requested information and signature on the back. *Keep the examination questions covered.*
9. When your proctor gives the signal, begin working the problems. You will have **90 MINUTES** working time for the test.
10. Students who score 100 or above on this AHSME will be invited to take the 4th annual **American Invitational Mathematics Examination** (AIME) on March 18, 1986. For more information about the AIME, see the back cover of this AHSME.

The results of this AHSME are used to identify students with unusual mathematical ability. To assure that this purpose is served, the MAA Committee on the American Mathematics Competitions reserves the right to reexamine students before deciding whether to grant official status to individual or team scores. Reexamination will be requested when, after an inquiry, there is reasonable basis to believe that scores have been obtained by extremely lucky guessing or dishonesty. Official status will not be granted if a student or school does not agree to a requested reexamination. The Committee also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

PRINT ______________________________ [ ]

last name | first name | middle initial | age

(home address) no. | street

city | state or province | zip or postcode

school (full name) | street address

city | state or province | zip or postcode

Grade Level (9, 10, 11, 12 or other number) [ ] Date: ____________

1		16	
2		17	
3		18	
4		19	
5		20	
6		21	
7		22	
8		23	
9		24	
10		25	
11		26	
12		27	
13		28	
14		29	
15		30	

CERTIFICATION

I accept the procedures on reexamination and disqualification stated on the bottom of the front cover.

____

Student's Signature

Your score is **not official** without your signature above.

SCORING

(Not to be filled in by the student)
See Instruction 3 on the front cover.

number correct [ ] × 5 = [ ]
+
number unanswered [ ] × 2 = [ ]

Total Score = sum = [ ]

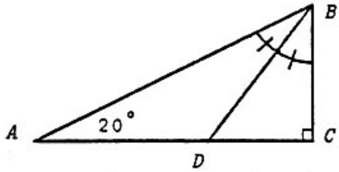
1. $[x-(y-z)]-[(x-y)-z]=$

(A) $2y$ (B) $2z$ (C) $-2y$ (D) $-2z$ (E) 0

2. If the line L in the xy-plane has half the slope and twice the y-intercept of the line $y=\frac{2}{3}x+4$, then an equation for L is

(A) $y=\frac{1}{3}x+8$ (B) $y=\frac{4}{3}x+2$ (C) $y=\frac{1}{3}x+4$
(D) $y=\frac{4}{3}x+4$ (E) $y=\frac{1}{3}x+2$

3. In the figure, ΔABC has a right angle at C and $\angle A=20^\circ$. If BD is the bisector of $\angle ABC$, then $\angle BDC=$



(A) 40° (B) 45° (C) 50°
(D) 55° (E) 60°

4. Let S be the statement

"If the sum of the digits of the whole number n is divisible by 6, then n is divisible by 6."

A value of n which shows S to be false is

(A) 30 (B) 33 (C) 40 (D) 42 (E) none of these

5. Simplify $\left(\sqrt[6]{27}-\sqrt{6\frac{3}{4}}\right)^2$.

(A) $\frac{3}{4}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{3\sqrt{3}}{4}$ (D) $\frac{3}{2}$ (E) $\frac{3\sqrt{3}}{2}$

6. Using a table of a certain height, two identical blocks of wood are placed as shown in Figure 1. Length r is found to be 32 inches. After rearranging the blocks as in Figure 2, length s is found to be 28 inches. How high is the table?

(A) 28 inches
(B) 29 inches
(C) 30 inches
(D) 31 inches
(E) 32 inches

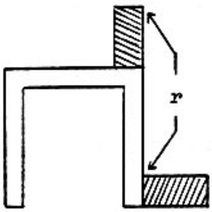


Figure 1

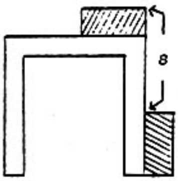


Figure 2

7. The sum of the greatest integer less than or equal to x and the least integer greater than or equal to x is 5. The solution set for x is

(A) $\{5/2\}$ (B) $\{x \mid 2 \le x \le 3\}$ (C) $\{x \mid 2 \le x < 3\}$
(D) $\{x \mid 2 < x \le 3\}$ (E) $\{x \mid 2 < x < 3\}$

8. The population of the United States in 1980 was 226,504,825. The area of the country is 3,615,122 square miles. There are $(5280)^2$ square feet in one square mile. Which number below best approximates the average number of square feet per person?

(A) 5,000 (B) 10,000 (C) 50,000 (D) 100,000 (E) 500,000

9. The product $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{9^2}\right)\left(1-\frac{1}{10^2}\right)$ equals

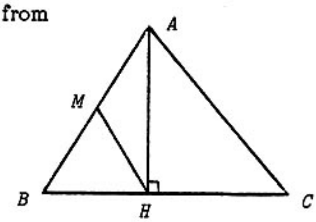
(A) $\frac{5}{12}$ (B) $\frac{1}{2}$ (C) $\frac{11}{20}$ (D) $\frac{2}{3}$ (E) $\frac{7}{10}$

10. The 120 permutations of AHSME are arranged in dictionary order, as if each were an ordinary five-letter word. The last letter of the 86^{th} word in this list is

(A) A (B) H (C) S (D) M (E) E

11. In ΔABC, $AB=13$, $BC=14$ and $CA=15$. Also, M is the midpoint of side AB and H is the foot of the altitude from A to BC. The length of HM is

(A) 6 (B) 6.5 (C) 7
(D) 7.5 (E) 8



12. John scores 93 on this year's AHSME. Had the old scoring system still been in effect, he would score only 84 for the same answers. How many questions does he leave unanswered? (The new scoring system is explained on the cover. In the old system, one started with 30 points, received 4 more for each correct answer, lost one point for each wrong answer, and neither gained nor lost points for unanswered questions.)

(A) 6 (B) 9 (C) 11 (D) 14 (E) not uniquely determined

13. A parabola $y = ax^2+bx+c$ has vertex (4,2). If (2,0) is on the parabola, then abc equals

(A) −12 (B) −6 (C) 0 (D) 6 (E) 12

14. Suppose hops, skips and jumps are specific units of length. If b hops equals c skips, d jumps equals e hops, and f jumps equals g meters, then one meter equals how many skips?

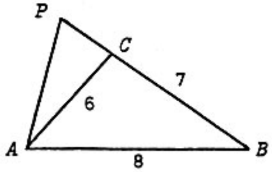
(A) $\frac{bdg}{cef}$ (B) $\frac{cdf}{beg}$ (C) $\frac{cdg}{bef}$ (D) $\frac{cef}{bdg}$ (E) $\frac{ceg}{bdf}$

15. A student attempted to compute the average, A, of x, y and z by computing the average of x and y, and then computing the average of the result and z. Whenever $x < y < z$, the student's final result is

(A) correct
(B) always less than A
(C) always greater than A
(D) sometimes less than A and sometimes equal to A
(E) sometimes greater than A and sometimes equal to A

16. In $\triangle ABC$, $AB = 8$, $BC = 7$, $CA = 6$ and side BC is extended, as shown in the figure, to a point P so that $\triangle PAB$ is similar to $\triangle PCA$. The length of PC is

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11



17. A drawer in a darkened room contains 100 red socks, 80 green socks, 60 blue socks and 40 black socks. A youngster selects socks one at a time from the drawer but is unable to see the color of the socks drawn. What is the smallest number of socks that must be selected to guarantee that the selection contains at least 10 pairs? (A pair of socks is two socks of the same color. No sock may be counted in more than one pair.)

(A) 21 (B) 23 (C) 24 (D) 30 (E) 50

18. A plane intersects a right circular cylinder of radius 1 forming an ellipse. If the major axis of the ellipse is 50% longer than the minor axis, the length of the major axis is

(A) 1 (B) $\frac{3}{2}$ (C) 2 (D) $\frac{9}{4}$ (E) 3

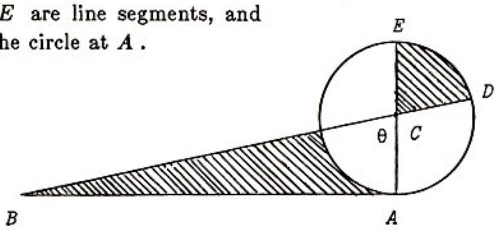
19. A park is in the shape of a regular hexagon 2 km on a side. Starting at a corner, Alice walks along the perimeter of the park for a distance of 5 km. How many kilometers is she from her starting point?

(A) $\sqrt{13}$ (B) $\sqrt{14}$ (C) $\sqrt{15}$ (D) $\sqrt{16}$ (E) $\sqrt{17}$

20. Suppose x and y are inversely proportional and positive. If x increases by $p\,\%$, then y decreases by

(A) $p\,\%$ (B) $\frac{p}{1+p}\,\%$ (C) $\frac{100}{p}\,\%$ (D) $\frac{p}{100+p}\,\%$ (E) $\frac{100p}{100+p}\,\%$

21. In the configuration below, θ is measured in radians, C is the center of the circle, BCD and ACE are line segments, and AB is tangent to the circle at A.



A necessary and sufficient condition for the equality of the two shaded areas, given $0<\theta<\pi/2$, is

(A) $\tan\theta=\theta$ (B) $\tan\theta=2\theta$ (C) $\tan\theta=4\theta$

(D) $\tan 2\theta=\theta$ (E) $\tan\frac{\theta}{2}=\theta$

22. Six distinct integers are picked at random from $\{1,2,3,...,10\}$. What is the probability that, among those selected, the second smallest is 3?

(A) 1/60 (B) 1/6 (C) 1/3 (D) 1/2 (E) none of these

23. Let $N = 69^5+5\cdot 69^4+10\cdot 69^3+10\cdot 69^2+5\cdot 69+1$.
How many positive integers are factors of N?

(A) 3 (B) 5 (C) 69 (D) 125 (E) 216

24. Let $p(x)=x^2+bx+c$, where b and c are integers. If $p(x)$ is a factor of both x^4+6x^2+25 and $3x^4+4x^2+28x+5$, what is $p(1)$?

(A) 0 (B) 1 (C) 2 (D) 4 (E) 8

25. If $\lfloor x\rfloor$ is the greatest integer less than or equal to x, then

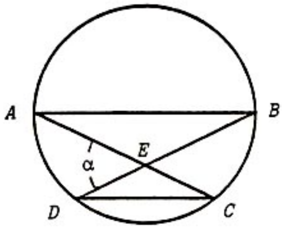
$$\sum_{N=1}^{1024} \lfloor \log_2 N \rfloor =$$

(A) 8192 (B) 8204 (C) 9218 (D) $\lfloor \log_2(1024!) \rfloor$ (E) none of these

26. It is desired to construct a right triangle in the coordinate plane so that its legs are parallel to the x and y axes and so that the medians to the midpoints of the legs lie on the lines $y = 3x + 1$ and $y = mx + 2$. The number of different constants m for which such a triangle exists is

(A) 0 (B) 1 (C) 2 (D) 3 (E) more than 3

27. In the adjoining figure, AB is a diameter of the circle, CD is a chord parallel to AB, and AC intersects BD at E, with $\angle AED = \alpha$. The ratio of the area of ΔCDE to that of ΔABE is

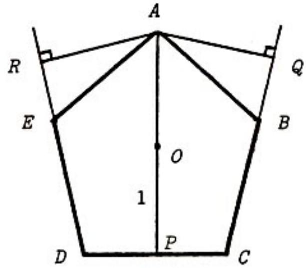


(A) $\cos\alpha$ (B) $\sin\alpha$

(C) $\cos^2\alpha$ (D) $\sin^2\alpha$

(E) $1 - \sin\alpha$

28. $ABCDE$ is a regular pentagon. AP, AQ and AR are the perpendiculars dropped from A onto CD, CB extended and DE extended, respectively. Let O be the center of the pentagon. If $OP = 1$, then $AO + AQ + AR$ equals



(A) 3 (B) $1 + \sqrt{5}$ (C) 4

(D) $2 + \sqrt{5}$ (E) 5

29. Two of the altitudes of the scalene triangle ABC have length 4 and 12. If the length of the third altitude is also an integer, what is the biggest it can be?

(A) 4 (B) 5 (C) 6 (D) 7 (E) none of these

30. The number of real solutions (x, y, z, w) of the simultaneous equations

$$2y = x + \frac{17}{x}, \quad 2z = y + \frac{17}{y}, \quad 2w = z + \frac{17}{z}, \quad 2x = w + \frac{17}{w}$$

is

(A) 1 (B) 2 (C) 4 (D) 8 (E) 16

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(AHSME)

TUESDAY, MARCH 3, 1987

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NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
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AMERICAN STATISTICAL ASSOCIATION
AMERICAN MATHEMATICAL ASSOCIATION OF TWO-YEAR COLLEGES

INSTRUCTIONS AND INFORMATION

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a thirty question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. SCORING RULES. You will receive 5 points for each correct answer, 0 points for each incorrect answer, and 2 points for each problem left *unanswered.*
4. Guessing. Be careful! You get a sure 2 points if you don't answer a question; a wrong guess will gain you 0. If you feel you have only 1 chance in 3 of being right, don't guess; such guessing will, on average, *lower* your score. However, if you feel you have 1 chance in 2 (for instance, by eliminating 3 of the 5 answers), go ahead and guess. In any event, try to *solve* the problem first.
5. Use pencil. Scratch paper, graph paper, ruler, compass, protractor and eraser are permitted. *Calculators and slide rules are not permitted.*
6. Figures are not necessarily drawn to scale.
7. The back of this cover is a form on which you are asked to write certain information, sign a Certification and record your answers. You must use this form—unless your region has centralized computer scoring (for which your proctor will give you a different form and instructions.)
8. When the proctor indicates, tear off this cover and provide the requested information and signature on the back. *Keep the examination questions covered.*
9. When your proctor gives the signal, begin working the problems. You will have **90 MINUTES** working time for the test.
10. Students who score 100 or above on this AHSME will be invited to take the 5th annual **American Invitational Mathematics Examination** (AIME) on March 24, 1987. For more information about the AIME, see the back cover of this AHSME.

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PRINT ____ [ ]

last name | first name | middle initial | age

(home address) no. | street

city | state or province | zip or postcode

school (full name) | street address

city | state or province | zip or postcode

Grade Level (9, 10, 11, 12 or other number) [ ] Date: ______________

1		16	
2		17	
3		18	
4		19	
5		20	
6		21	
7		22	
8		23	
9		24	
10		25	
11		26	
12		27	
13		28	
14		29	
15		30	

CERTIFICATION

I accept the procedures on reexamination and disqualification stated on the bottom of the front cover.

____

Student's Signature

Your score is **not official** without your signature above.

SCORING

(Not to be filled in by the student)
See Instruction 3 on the front cover.

number correct [ ] × 5 = [ ]

+

number unanswered [ ] × 2 = [ ]

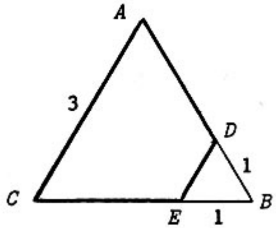
Total Score = sum = [ ]

1. $(1+x^2)(1-x^3)$ equals

(A) $1-x^5$ (B) $1-x^6$ (C) $1+x^2-x^3$ (D) $1+x^2-x^3-x^5$ (E) $1+x^2-x^3-x^6$

2. As shown in the figure, a triangular corner with side lengths $DB = EB = 1$ is cut from equilateral triangle ABC of side length 3. The perimeter of the remaining quadrilateral $ADEC$ is

(A) 6 (B) $6\frac{1}{2}$ (C) 7 (D) $7\frac{1}{2}$ (E) 8



3. How many primes less than 100 have 7 as the ones digit? (Assume the usual base 10 representation.)

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

4. $$\frac{2^1+2^0+2^{-1}}{2^{-2}+2^{-3}+2^{-4}}$$ equals

(A) 6 (B) 8 (C) $\frac{31}{2}$ (D) 24 (E) 512

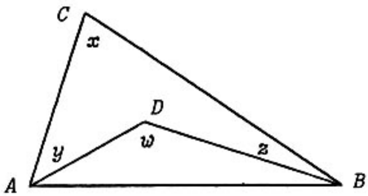
5. A student recorded the exact percentage frequency distribution for a set of measurements, as shown to the right. However, the student neglected to indicate N, the total number of measurements. What is the smallest possible value of N?

(A) 5 (B) 8 (C) 16 (D) 25 (E) 50

measured value	percent frequency
0	12.5
1	0
2	50
3	25
4	12.5
	100

6. In the ΔABC shown, D is some interior point, and x, y, z, w are the measures of angles in degrees. Solve for x in terms of y, z and w.

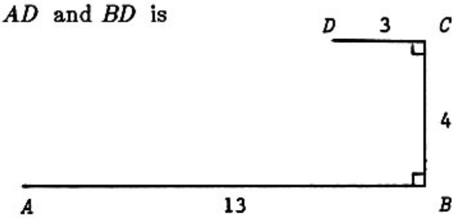
(A) $w-y-z$ (B) $w-2y-2z$ (C) $180-w-y-z$ (D) $2w-y-z$ (E) $180-w+y+z$



7. If $a-1=b+2=c-3=d+4$, which of the four quantities a, b, c, d is the largest?

(A) a (B) b (C) c (D) d (E) no one is always largest

8. In the figure the sum of the distances AD and BD is



(A) between 10 and 11
(B) 12
(C) between 15 and 16
(D) between 16 and 17
(E) 17

9. The first four terms of an arithmetic sequence are $a, x, b, 2x$. The ratio of a to b is

(A) 1/4 (B) 1/3 (C) 1/2 (D) 2/3 (E) 2

10. How many ordered triples (a, b, c) of non-zero real numbers have the property that each number is the product of the other two?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

11. Let c be a constant. The simultaneous equations

$$x - y = 2,$$
$$cx + y = 3$$

have a solution (x, y) inside Quadrant I if and only if

(A) $c=-1$ (B) $c>-1$ (C) $c<3/2$ (D) $0<c<3/2$ (E) $-1<c<3/2$

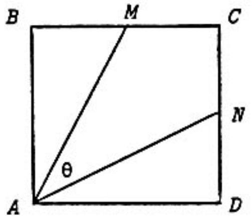
12. In an office, at various times during the day the boss gives the secretary a letter to type, each time putting the letter on top of the pile in the secretary's in-box. When there is time, the secretary takes the top letter off the pile and types it. If there are five letters in all, and the boss delivers them in the order 1 2 3 4 5, which of the following could **not** be the order in which the secretary types them?

(A) 1 2 3 4 5 (B) 2 4 3 5 1 (C) 3 2 4 1 5 (D) 4 5 2 3 1 (E) 5 4 3 2 1

13. A long piece of paper 5 cm wide is made into a roll for cash registers by wrapping it 600 times around a cardboard tube of diameter 2 cm, forming a roll 10 cm in diameter. Approximate the length of the paper in meters. (Pretend the paper forms 600 concentric circles with diameters evenly spaced from 2 cm to 10 cm.)

(A) 36π (B) 45π (C) 60π (D) 72π (E) 90π

14. $ABCD$ is a square and M and N are the midpoints of BC and CD respectively. Then $\sin\theta =$



(A) $\frac{\sqrt{5}}{5}$ (B) $\frac{3}{5}$ (C) $\frac{\sqrt{10}}{5}$

(D) $\frac{4}{5}$ (E) none of these

15. If (x, y) is a solution to the system

$$xy = 6 \quad \text{and} \quad x^2y + xy^2 + x + y = 63,$$

find $x^2 + y^2$.

(A) 13 (B) $\frac{1173}{32}$ (C) 55 (D) 69 (E) 81

16. A cryptographer devises the following method for encoding positive integers. First, the integer is expressed in base 5. Second, a 1-to-1 correspondence is established between the digits that appear in the expressions in base 5 and the elements of the set $\{V, W, X, Y, Z\}$. Using this correspondence, the cryptographer finds that three consecutive integers in increasing order are coded as VYZ, VYX, VVW, respectively. What is the base-10 expression for the integer coded as XYZ?

(A) 48 (B) 71 (C) 82 (D) 108 (E) 113

17. In a mathematics competition, the sum of the scores of Bill and Dick equalled the sum of the scores of Ann and Carol. If the scores of Bill and Carol had been interchanged, then the sum of the scores of Ann and Carol would have exceeded the sum of the scores of the other two. Also, Dick's score exceeded the sum of the scores of Bill and Carol. Determine the order in which the four contestants finished, from highest to lowest. Assume all scores were nonnegative.

(A) Dick, Ann, Carol, Bill
(B) Dick, Ann, Bill, Carol
(C) Dick, Carol, Bill, Ann
(D) Ann, Dick, Carol, Bill
(E) Ann, Dick, Bill, Carol

18. It takes A algebra books (all the same thickness) and H geometry books (all the same thickness, which is greater than that of an algebra book) to completely fill a certain shelf. Also, S of the algebra books and M of the geometry books would fill the same shelf. Finally, E of the algebra books alone would fill this shelf. Given that A, H, S, M, E are distinct positive integers, it follows that E is

(A) $\frac{AM+SH}{M+H}$ (B) $\frac{AM^2+SH^2}{M^2+H^2}$ (C) $\frac{AH-SM}{M-H}$ (D) $\frac{AM-SH}{M-H}$ (E) $\frac{AM^2-SH^2}{M^2-H^2}$

19. Which of the following is closest to $\sqrt{65}-\sqrt{63}$?

(A) .12 (B) .13 (C) .14 (D) .15 (E) .16

20. Evaluate

$$\log_{10}(\tan 1^\circ) + \log_{10}(\tan 2^\circ) + \log_{10}(\tan 3^\circ) + \cdots + \log_{10}(\tan 88^\circ) + \log_{10}(\tan 89^\circ).$$

(A) 0 (B) $\frac{1}{2}\log_{10}(\frac{1}{2}\sqrt{3})$ (C) $\frac{1}{2}\log_{10}2$ (D) 1 (E) none of these

21. There are two natural ways to inscribe a square in a given isosceles right triangle. If it is done as in Figure 1 below, then one finds that the area of the square is $441\,\text{cm}^2$. What is the area (in cm^2) of the square inscribed in the same $\triangle ABC$ as shown in Figure 2 below?

(A) 378
(B) 392
(C) 400
(D) 441
(E) 484

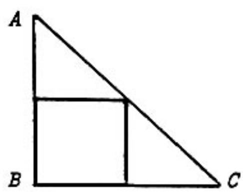


Figure 1

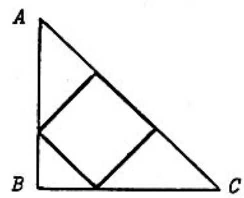


Figure 2

22. A ball was floating in a lake when the lake froze. The ball was removed (without breaking the ice), leaving a hole 24 cm across at the top and 8 cm deep. What was the radius of the ball (in centimeters)?

(A) 8 (B) 12 (C) 13 (D) $8\sqrt{3}$ (E) $6\sqrt{6}$

23. If p is a prime and both roots of $x^2+px-444p=0$ are integers, then

(A) $1<p\le 11$ (B) $11<p\le 21$ (C) $21<p\le 31$ (D) $31<p\le 41$
(E) $41<p\le 51$

24. How many polynomial functions f of degree ≥ 1 satisfy

$$f(x^2) = [f(x)]^2 = f(f(x)) \ ?$$

(A) 0 (B) 1 (C) 2 (D) finitely many but more than 2 (E) infinitely many

25. ABC is a triangle: $A=(0,0)$, $B=(36,15)$ and both the coordinates of C are integers. What is the minimum area $\triangle ABC$ can have?

(A) 1/2 (B) 1 (C) 3/2 (D) 13/2 (E) there is no minimum

26. The amount 2.5 is split into two nonnegative real numbers uniformly at random, for instance, into 2.143 and .357, or into $\sqrt{3}$ and $2.5-\sqrt{3}$. Then each number is rounded to its nearest integer, for instance, 2 and 0 in the first case above, 2 and 1 in the second. What is the probability that the two integers sum to 3?

(A) 1/4 (B) 2/5 (C) 1/2 (D) 3/5 (E) 3/4

27. A cube of cheese $C = \{(x, y, z) \mid 0 \le x, y, z \le 1\}$ is cut along the planes $x = y$, $y = z$ and $z = x$. How many pieces are there? (No cheese is moved until all three cuts are made.)

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

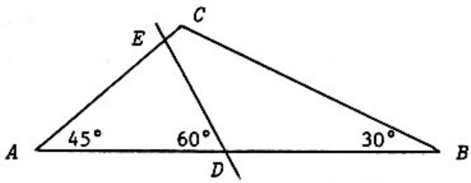
28. Let a, b, c, d be real numbers. Suppose that all the roots of $z^4 + az^3 + bz^2 + cz + d = 0$ are complex numbers lying on a circle in the complex plane centered at $0+0i$ and having radius 1. The sum of the reciprocals of the roots is necessarily

(A) a (B) b (C) c (D) $-a$ (E) $-b$

29. Consider the sequence of numbers defined recursively by $t_1 = 1$ and for $n > 1$ by $t_n = 1 + t_{(n/2)}$ when n is even and by $t_n = 1/t_{(n-1)}$ when n is odd. Given that $t_n = 19/87$, the sum of the digits of n is

(A) 15 (B) 17 (C) 19 (D) 21 (E) 23

30. In the figure, ΔABC has $\angle A = 45°$ and $\angle B = 30°$. A line DE, with D on AB and $\angle ADE = 60°$, divides ΔABC into two pieces of equal area. (Note: the figure may not be accurate; perhaps E is on CB instead of AC.) The ratio AD/AB is



(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{2}{2+\sqrt{2}}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{1}{\sqrt[3]{6}}$ (E) $\frac{1}{\sqrt[4]{12}}$

Important Information on Back Cover

SOLUTIONS

Your School Examination Manager has at least one copy of the 1987 AHSME Solutions Pamphlet. It is meant to be loaned or given to students (but not duplicated).

WRITE TO US!

Questions and comments about the problems and solutions for this AHSME (but **not** requests for the Solutions Pamphlet) should be addressed to:

Prof. Stephen B. Maurer, CAMC Chairman
Department of Mathematics
Swarthmore College, Swarthmore, PA 19081

Comments about administrative arrangements, and orders for any of the publications listed below, should be addressed to:

Prof. Walter E. Mientka, CAMC Executive Director
Department of Mathematics & Statistics
University of Nebraska, Lincoln, NE 68588-0322

1987 AIME

The AIME is a 15-question, 3-hour, short-answer examination. You are invited to participate only if you receive a score of 100 or above on this AHSME. (Your school must also agree to administer the AIME.) The top scoring students on the AIME will be invited to take the **USA Mathematical Olympiad** on April 28, 1987. The best way to prepare for the AIME and USAMO is to study previous years of these exams.

PUBLICATIONS

MINIMUM ORDER: $5 (before handling fee), US FUNDS ONLY. Individual orders from US and Canada must be prepaid. Orders are mailed 4th Class, unless you wish 1st Class, in which case add a 20% handling fee. Make checks payable to MAA/CAMC.

FOREIGN ORDERS: do **NOT** prepay; an invoice will be sent.

Examinations: **Each price is for one copy of an exam and its solutions for one year.** Specify the years you want and how many copies of each. All prices effective to September 1, 1987.

AJHSME (Junior High Exam), 1985-86, 50¢ per copy per year.
AHSME 1972-1987, 50¢ per copy per year.
AIME, 1983-86, $1 per copy per year.
USA and International Mathematical Olympiads (together), 1976-86, $1 per copy per year
National Summary of Results and Awards, 1976-86, $3 per copy per year.

Books

Contest Problem Book I ($7.50), AHSME exams and solutions, 1950-60.
Contest Problem Book II ($7.50), AHSMEs 1961-65.
Contest Problem Book III ($8.50), AHSMEs 1966-72.
Contest Problem Book IV ($10.00), AHSMEs 1973-82.
International Mathematical Olympiads ($8.50), exams and solutions, 1959-77.

Journal

The ARBELOS contains short articles and challenging problems; recommended especially for AIME and USAMO qualifiers. $5.50 for 1986-87 subscription (5 issues). Back volumes from 1982-86 available at $4.50 each. Canada and APO/FPO—add $2 per volume.

AMERICAN MATHEMATICS COMPETITIONS

39th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)

TUESDAY, MARCH 1, 1988

Sponsors:

MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES   MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
CASUALTY ACTUARIAL SOCIETY
AMERICAN STATISTICAL ASSOCIATION
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PRINT ______________________________________ [ ]

last name | first name | middle initial | age

____

(home address) no. | street

____

city | state or province | zip or postcode

____

school (full name) | street address

____

city | state or province | zip or postcode

Grade Level (9, 10, 11, 12 or other number) [ ] Date: ____________

1		16	
2		17	
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9		24	
10		25	
11		26	
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13		28	
14		29	
15		30	

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(Not to be filled in by the student)
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number correct [ ] × 5 = [ ]

+

number unanswered [ ] × 2 = [ ]

Total Score = sum = [ ]

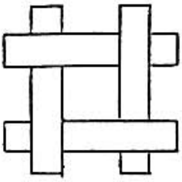
1. $\sqrt{8} + \sqrt{18} =$

(A) $\sqrt{26}$ (B) $2(\sqrt{2}+\sqrt{3})$ (C) 7 (D) $5\sqrt{2}$ (E) $2\sqrt{13}$

2. Triangles ABC and XYZ are similar, with A corresponding to X and B to Y. If $AB = 3$, $BC = 4$ and $XY = 5$, then YZ is

(A) $3\frac{3}{4}$ (B) 6 (C) $6\frac{1}{4}$ (D) $6\frac{2}{3}$ (E) 8

3. Four rectangular paper strips of length 10 and width 1 are put flat on a table and overlap perpendicularly as shown. How much area of the table is covered?



(A) 36 (B) 40 (C) 44

(D) 96 (E) 100

4. The slope of the line $\frac{x}{3} + \frac{y}{2} = 1$ is

(A) $-\frac{3}{2}$ (B) $-\frac{2}{3}$ (C) $\frac{1}{3}$ (D) $\frac{2}{3}$ (E) $\frac{3}{2}$

5. If b and c are constants and

$$(x+2)(x+b) = x^2 + cx + 6,$$

then c is

(A) -5 (B) -3 (C) -1 (D) 3 (E) 5

6. A figure is an equiangular parallelogram if and only if it is a

(A) rectangle (B) regular polygon (C) rhombus (D) square (E) trapezoid

7. Estimate the time it takes to send 60 blocks of data over a communications channel if each block consists of 512 "chunks" and the channel can transmit 120 chunks per second.

(A) .04 seconds (B) .4 seconds (C) 4 seconds (D) 4 minutes (E) 4 hours

8. If $\frac{b}{a} = 2$ and $\frac{c}{b} = 3$, what is the ratio of $a + b$ to $b + c$?

(A) $\frac{1}{3}$ (B) $\frac{3}{8}$ (C) $\frac{3}{5}$ (D) $\frac{2}{3}$ (E) $\frac{3}{4}$

9. An $8' \times 10'$ table sits in the corner of a square room, as in Figure 1 below. The owners desire to move the table to the position shown in Figure 2. The side of the room is S feet. What is the smallest integer value of S for which the table can be moved as desired without tilting it or taking it apart?

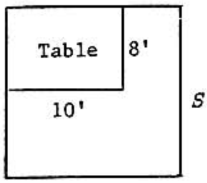


Figure 1

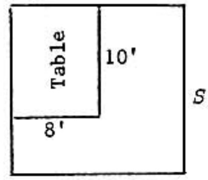


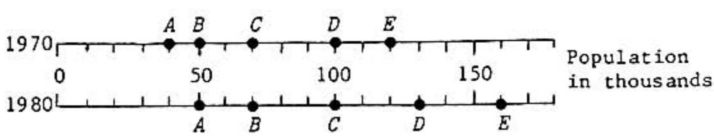
Figure 2

(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

10. In an experiment, a scientific constant C is determined to be 2.43865 with an error of at most ± 0.00312. The experimenter wishes to announce a value for C in which every digit is significant. That is, whatever C is, the announced value must be the correct result when C is rounded to that number of digits. The most accurate value the experimenter can announce for C is

(A) 2. (B) 2.4 (C) 2.43 (D) 2.44 (E) 2.439

11. On each horizontal line in the figure below, the five large dots indicate the populations of cities A, B, C, D and E in the year indicated. Which city had the greatest percentage increase in population from 1970 to 1980?



(A) A (B) B (C) C (D) D (E) E

12. Each integer 1 through 9 is written on a separate slip of paper and all nine slips are put into a hat. Jack picks one of these slips at random and puts it back. Then Jill picks a slip at random. Which digit is most likely to be the units digit of the **sum** of Jack's integer and Jill's integer?

(A) 0 (B) 1 (C) 8 (D) 9 (E) each digit is equally likely

13. If $\sin x = 3\cos x$ then what is $\sin x \cos x$?

(A) 1/6 (B) 1/5 (C) 2/9 (D) 1/4 (E) 3/10

14. For any real number a and positive integer k, define

$$\binom{a}{k} = \frac{a(a-1)(a-2)\cdots(a-(k-1))}{k(k-1)(k-2)\cdots(2)(1)}$$

What is

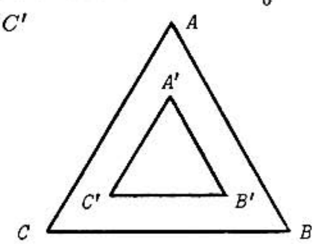
$$\binom{-\frac{1}{2}}{100} \div \binom{\frac{1}{2}}{100} \ ?$$

(A) -199 (B) -197 (C) -1 (D) 197 (E) 199

15. If a and b are integers such that x^2-x-1 is a factor of ax^3+bx^2+1, then b is

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

16. ABC and $A'B'C'$ are equilateral triangles with parallel sides and the same center, as in the figure. The distance between side BC and side $B'C'$ is $\frac{1}{6}$ the altitude of $\triangle ABC$. The ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$ is



(A) $\frac{1}{36}$ (B) $\frac{1}{6}$ (C) $\frac{1}{4}$

(D) $\frac{\sqrt{3}}{4}$ (E) $\frac{9+8\sqrt{3}}{36}$

17. If $|x|+x+y=10$ and $x+|y|-y=12$, find $x+y$.

(A) -2 (B) 2 (C) $\frac{18}{5}$ (D) $\frac{22}{3}$ (E) 22

18. At the end of a professional bowling tournament, the top 5 bowlers have a play-off. First #5 bowls #4. The loser receives 5th prize and the winner bowls #3 in another game. The loser of this game receives 4th prize and the winner bowls #2. The loser of this game receives 3rd prize and the winner bowls #1. The winner of this game gets 1st prize and the loser gets 2nd prize. In how many orders can bowlers #1 through #5 receive the prizes?

(A) 10 (B) 16 (C) 24 (D) 120 (E) none of these

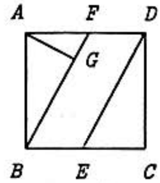
19. Simplify

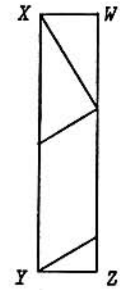
$$\frac{bx(a^2x^2+2a^2y^2+b^2y^2)+ay(a^2x^2+2b^2x^2+b^2y^2)}{bx+ay}$$

(A) $a^2x^2+b^2y^2$ (B) $(ax+by)^2$ (C) $(ax+by)(bx+ay)$

(D) $2(a^2x^2+b^2y^2)$ (E) $(bx+ay)^2$

20. In one of the adjoining figures a square of side 2 is dissected into four pieces so that E and F are the midpoints of opposite sides and AG is perpendicular to BF. These four pieces can then be reassembled into a rectangle as shown in the second figure. The ratio of height to base, XY/YZ, in this rectangle is





(A) 4 (B) $1+2\sqrt{3}$ (C) $2\sqrt{5}$
(D) $(8+4\sqrt{3})/3$ (E) 5

21. The complex number z satisfies $z+|z|=2+8i$. What is $|z|^2$? Note: if $z=a+bi$, then $|z|=\sqrt{a^2+b^2}$.

(A) 68 (B) 100 (C) 169 (D) 208 (E) 289

22. For how many integers x does a triangle with side lengths 10, 24 and x have all its angles acute?

(A) 4 (B) 5 (C) 6 (D) 7 (E) more than 7

23. The six edges of tetrahedron $ABCD$ measure 7, 13, 18, 27, 36 and 41 units. If the length of edge AB is 41, then the length of edge CD is

(A) 7 (B) 13 (C) 18 (D) 27 (E) 36

24. An isosceles trapezoid is circumscribed around a circle. The longer base of the trapezoid is 16, and one of the base angles is arcsin(.8). Find the area of the trapezoid.

(A) 72 (B) 75 (C) 80 (D) 90 (E) not uniquely determined

25. X, Y, and Z are pairwise disjoint sets of people. The average ages of people in the sets X, Y, Z, $X\cup Y$, $X\cup Z$ and $Y\cup Z$ are given in the table below.

Set	X	Y	Z	$X\cup Y$	$X\cup Z$	$Y\cup Z$
Average age of people in the set	37	23	41	29	39.5	33

Find the average age of the people in the set $X\cup Y\cup Z$.

(A) 33 (B) 33.5 (C) $33.6\overline{6}$ (D) $33.83\overline{3}$ (E) 34

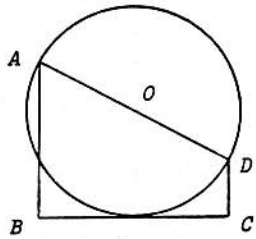
26. Suppose that p and q are positive numbers for which

$$\log_9(p) = \log_{12}(q) = \log_{16}(p+q).$$

What is the value of q/p?

(A) 4/3 (B) $\frac{1}{2}(1+\sqrt{3})$ (C) 8/5 (D) $\frac{1}{2}(1+\sqrt{5})$ (E) 16/9

27. In the figure, $AB \perp BC$, $BC \perp CD$ and BC is tangent to the circle with center O and diameter AD. In which one of the following cases is the area of $ABCD$ an integer?



(A) $AB=3$, $CD=1$ (B) $AB=5$, $CD=2$

(C) $AB=7$, $CD=3$ (D) $AB=9$, $CD=4$

(E) $AB=11$, $CD=5$

28. An unfair coin has probability p of coming up heads on a single toss. Let w be the probability that, in 5 independent tosses of this coin, heads come up exactly 3 times. If $w=144/625$, then

(A) p must be 2/5 (B) p must be 3/5

(C) p must be greater than 3/5 (D) p is not uniquely determined

(E) there is no value of p for which $w=144/625$

29. You plot weight (y) against height (x) for three of your friends and obtain the points (x_1,y_1), (x_2,y_2), (x_3,y_3). If $x_1<x_2<x_3$ and $x_3-x_2=x_2-x_1$, which of the following is necessarily the slope of the line which best fits the data? "Best fits" means that the sum of the squares of the vertical distances from the data points to the line is smaller than for any other line.

(A) $\dfrac{y_3-y_1}{x_3-x_1}$ (B) $\dfrac{(y_2-y_1)-(y_3-y_2)}{x_3-x_1}$ (C) $\dfrac{2y_3-y_1-y_2}{2x_3-x_1-x_2}$

(D) $\dfrac{y_2-y_1}{x_2-x_1}+\dfrac{y_3-y_2}{x_3-x_2}$ (E) none of these

30. Let $f(x)=4x-x^2$. Given x_0, consider the sequence defined by $x_n=f(x_{n-1})$ for all $n\geq 1$. For how many real numbers x_0 will the sequence $x_0, x_1, x_2, \ldots$ take on only a finite number of different values?

(A) 0 (B) 1 or 2 (C) 3, 4, 5 or 6

(D) more than 6 but finitely many (E) infinitely many

Important Information on Back Cover

AMERICAN MATHEMATICS COMPETITIONS

40th ANNUAL AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION (AHSME)

TUESDAY, FEBRUARY 28, 1989

Sponsored by

Mathematical Association of America
Society of Actuaries Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society American Statistical Association
American Mathematical Association of Two-Year Colleges
American Mathematical Society

1. **DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.**
2. **This is a thirty question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.**
3. **SCORING RULES: You will receive 5 points for each correct answer, 2 points for each problem left unanswered, and 0 points for each incorrect answer.**
4. **Solve the problem carefully. Note the scoring rules. To guess before eliminating 3 of the 5 choices will, on the average, lower your score.**
5. **Use pencil. Scratch paper, graph paper, ruler, compass, protractor and eraser are permitted.** ***Calculators and slide rules are not permitted.***
6. **Figures are not necessarily drawn to scale.**
7. **The back of this page is a form on which you are asked to write certain information, sign a Certification and record your answers. You must use this form unless your region has centralized computer scoring (for which your proctor will give you a different form and instructions).**
8. **Difficult to read or ambiguous answers will be marked wrong. Printed capital letters are recommended.**
9. **Carefully follow the instructions of your proctor in order to keep the examination questions concealed while providing the requested information and signature on page 2.**
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PRINT ____ [ ]

last name | first name | middle initial | age

____

(home address) no. | street

____

city | state or province | zip or postcode

____

school (full name) | street address

____

city | state or province | zip or postcode

Grade Level (9, 10, 11, 12 or other number) [ ] Date: ____________

1		16	
2		17	
3		18	
4		19	
5		20	
6		21	
7		22	
8		23	
9		24	
10		25	
11		26	
12		27	
13		28	
14		29	
15		30	

CERTIFICATION

I accept the procedures on reexamination and disqualification stated on the bottom of the front cover.

____

Student's Signature

Your score is **not official** without your signature above.

SCORING

(Not to be filled in by the student)
See Instruction 3 on the front cover.

number correct [ ] × 5 = [ ]
+
number unanswered [ ] × 2 = [ ]

Total Score = sum = [ ]

1. $(-1)^{5^2} + 1^{2^5} =$

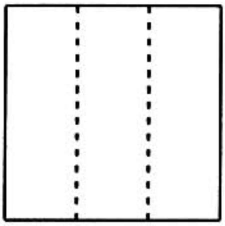
(A) -7 (B) -2 (C) 0 (D) 1 (E) 57

2. $\sqrt{\frac{1}{9} + \frac{1}{16}} =$

(A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $\frac{2}{7}$ (D) $\frac{5}{12}$ (E) $\frac{7}{12}$

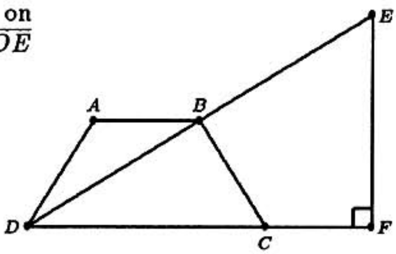
3. A square is cut into three rectangles along two lines parallel to a side, as shown. If the perimeter of each of the three rectangles is 24, then the area of the original square is

(A) 24 (B) 36 (C) 64 (D) 81 (E) 96



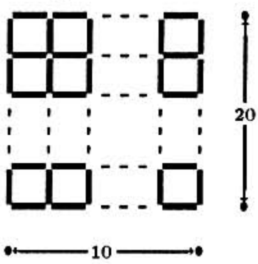
4. In the figure, $ABCD$ is an isosceles trapezoid with side lengths $AD = BC = 5$, $AB = 4$, and $DC = 10$. The point C is on $\overline{DF}$ and B is the midpoint of hypotenuse $\overline{DE}$ in the right triangle DEF. Then $CF =$

(A) 3.25 (B) 3.5 (C) 3.75
(D) 4.0 (E) 4.25



5. Toothpicks of equal length are used to build a rectangular grid as shown. If the grid is 20 toothpicks high and 10 toothpicks wide, then the number of toothpicks used is

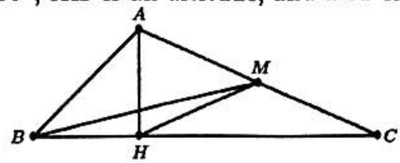
(A) 30 (B) 200
(C) 410 (D) 420
(E) 430



6. If $a, b > 0$ and the triangle in the first quadrant bounded by the coordinate axes and the graph of $ax + by = 6$ has area 6, then $ab =$

(A) 3 (B) 6 (C) 12 (D) 108 (E) 432

7. In $\triangle ABC$, $\angle A = 100°$, $\angle B = 50°$, $\angle C = 30°$, $\overline{AH}$ is an altitude, and $\overline{BM}$ is a median. Then $\angle MHC =$



(A) 15° **(B)** 22.5° **(C)** 30° **(D)** 40° **(E)** 45°

8. For how many integers n between 1 and 100 does $x^2 + x - n$ factor into the product of two linear factors with integer coefficients?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 9 **(E)** 10

9. Mr. and Mrs. Zeta want to name baby Zeta so that its monogram (first, middle, and last initials) will be in alphabetical order with no letters repeated. How many such monograms are possible?

(A) 276 **(B)** 300 **(C)** 552 **(D)** 600 **(E)** 15600

10. Consider the sequence defined recursively by $u_1 = a$ (any positive number), and $u_{n+1} = -1/(u_n + 1)$, $n = 1, 2, 3, \ldots$. For which of the following values of n must $u_n = a$?

(A) 14 **(B)** 15 **(C)** 16 **(D)** 17 **(E)** 18

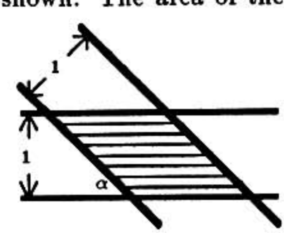
11. Let a, b, c and d be integers with $a < 2b$, $b < 3c$, and $c < 4d$. If $d < 100$, the largest possible value for a is

(A) 2367 **(B)** 2375 **(C)** 2391 **(D)** 2399 **(E)** 2400

12. The traffic on a certain east-west highway moves at a constant speed of 60 miles per hour in both directions. An eastbound driver passes 20 westbound vehicles in a five-minute interval. Assume vehicles in the westbound lane are equally spaced. Which of the following is closest to the number of westbound vehicles present in a 100-mile section of highway?

(A) 100 **(B)** 120 **(C)** 200 **(D)** 240 **(E)** 400

13. Two strips of width 1 overlap at an angle of α as shown. The area of the overlap (shown shaded) is



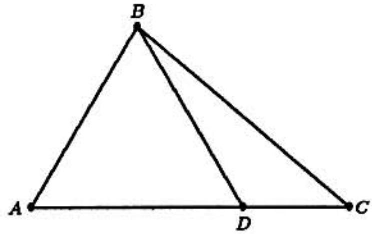
(A) $\sin\alpha$ **(B)** $\dfrac{1}{\sin\alpha}$ **(C)** $\dfrac{1}{1-\cos\alpha}$ **(D)** $\dfrac{1}{\sin^2\alpha}$ **(E)** $\dfrac{1}{(1-\cos\alpha)^2}$

14. $\cot 10 + \tan 5 =$

(A) csc 5 **(B)** csc 10 **(C)** sec 5 **(D)** sec 10 **(E)** sin 15

15. In $\triangle ABC$, $AB = 5$, $BC = 7$, $AC = 9$ and D is on $\overline{AC}$ with $BD = 5$. Find the ratio $AD : DC$.

(A) $4:3$ **(B)** $7:5$ **(C)** $11:6$
(D) $13:5$ **(E)** $19:8$



16. A *lattice point* is a point in the plane with integer coordinates. How many lattice points are on the line segment whose endpoints are $(3, 17)$ and $(48, 281)$? (Include both endpoints of the segment in your count.)

(A) 2 **(B)** 4 **(C)** 6 **(D)** 16 **(E)** 46

17. The perimeter of an equilateral triangle exceeds the perimeter of a square by 1989 cm. The length of each side of the triangle exceeds the length of each side of the square by d cm. The square has perimeter greater than 0. How many positive integers are not possible values for d?

(A) 0 **(B)** 9 **(C)** 221 **(D)** 663 **(E)** infinitely many

18. The set of all real numbers x for which

$$x + \sqrt{x^2+1} - \frac{1}{x + \sqrt{x^2+1}}$$

is a rational number is the set of all

(A) integers x **(B)** rational x **(C)** real x
(D) x for which $\sqrt{x^2+1}$ is rational
(E) x for which $x + \sqrt{x^2+1}$ is rational

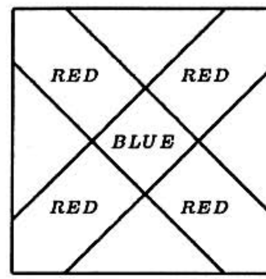
19. A triangle is inscribed in a circle. The vertices of the triangle divide the circle into three arcs of lengths 3, 4, and 5. What is the area of the triangle?

(A) 6 **(B)** $\frac{18}{\pi^2}$ **(C)** $\frac{9}{\pi^2}(\sqrt{3}-1)$ **(D)** $\frac{9}{\pi^2}(\sqrt{3}+1)$ **(E)** $\frac{9}{\pi^2}(\sqrt{3}+3)$

20. Let x be a real number selected uniformly at random between 100 and 200. If $\lfloor\sqrt{x}\rfloor = 12$, find the probability that $\lfloor\sqrt{100x}\rfloor = 120$. ($\lfloor v\rfloor$ *means the greatest integer less than or equal to* v.)

(A) $\frac{2}{25}$ (B) $\frac{241}{2500}$ (C) $\frac{1}{10}$ (D) $\frac{96}{625}$ (E) 1

21. A square flag has a red cross of uniform width with a blue square in the center on a white background as shown. (The cross is symmetric with respect to each of the diagonals of the square.) If the entire cross (both the red arms and the blue center) takes up 36% of the area of the flag, what percent of the area of the flag is blue?

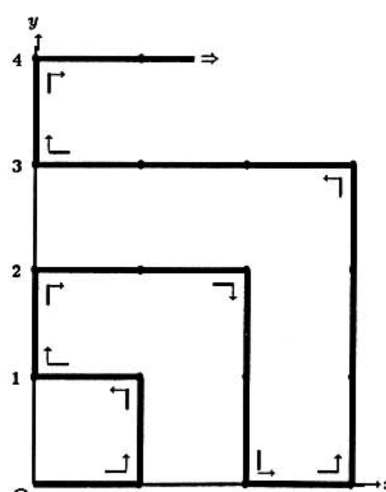


(A) .5 (B) 1 (C) 2 (D) 3 (E) 6

22. A child has a set of 96 distinct blocks. Each block is one of 2 materials (*plastic, wood*), 3 sizes (*small, medium, large*), 4 colors (*blue, green, red, yellow*), and 4 shapes (*circle, hexagon, square, triangle*). How many blocks in the set are different from the *"plastic medium red circle"* in exactly two ways? (The *"wood medium red square"* is such a block.)

(A) 29 (B) 39 (C) 48 (D) 56 (E) 62

23. A particle moves through the first quadrant as follows. During the first minute it moves from the origin to $(1, 0)$. Thereafter, it continues to follow the directions indicated in the figure, going back and forth between the positive x and y axes, moving one unit of distance parallel to an axis in each minute. At which point will the particle be after exactly 1989 minutes?



(A) $(35, 44)$ (B) $(36, 45)$
(C) $(37, 45)$ (D) $(44, 35)$
(E) $(45, 36)$

24. Five people are sitting at a round table. Let $f \geq 0$ be the number of people sitting next to at least one female and $m \geq 0$ be the number of people sitting next to at least one male. The number of possible ordered pairs (f, m) is

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

25. In a certain cross-country meet between two teams of five runners each, a runner who finishes in the n^{th} position contributes n to his team's score. The team with the lower score wins. If there are no ties among the runners, how many different winning scores are possible?

(A) 10 **(B)** 13 **(C)** 27 **(D)** 120 **(E)** 126

26. A regular octahedron is formed by joining the centers of adjoining faces of a cube. The ratio of the volume of the octahedron to the volume of the cube is

(A) $\frac{\sqrt{3}}{12}$ **(B)** $\frac{\sqrt{6}}{16}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{\sqrt{2}}{8}$ **(E)** $\frac{1}{4}$

27. Let n be a positive integer. If the equation $2x + 2y + z = n$ has 28 solutions in positive integers x, y and z, then n must be either

(A) 14 or 15 **(B)** 15 or 16 **(C)** 16 or 17 **(D)** 17 or 18
(E) 18 or 19

28. Find the sum of the roots of $\tan^2 x - 9\tan x + 1 = 0$ that are between $x = 0$ and $x = 2\pi$ radians.

(A) $\frac{\pi}{2}$ **(B)** π **(C)** $\frac{3\pi}{2}$ **(D)** 3π **(E)** 4π

29. Find $\sum_{k=0}^{49}(-1)^k\binom{99}{2k}$, where $\binom{n}{j} = \frac{n!}{j!(n-j)!}$.

(A) -2^{50} **(B)** -2^{49} **(C)** 0 **(D)** 2^{49} **(E)** 2^{50}

30. Suppose that 7 boys and 13 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row $GBBGGGBGBGGGBGBGGBGG$ we have $S = 12$. The average value of S (if all possible orders of these 20 people are considered) is closest to

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

SOLUTIONS

Your School Examination Manager has at least one copy of the 1989 AHSME Solutions Pamphlet. It is meant to be lent or given to students (but not duplicated).

WRITE TO US!

Questions and comments about the problems and solutions for this AHSME (but **not** requests for the Solutions Pamphlet) should be addressed to:

Prof Leo J Schneider, CAMC Chairman
Department of Mathematics and Computer Science
John Carroll University, University Heights, OH 44118-4581 USA

Comments about administrative arrangements and orders for any publications listed below should be addressed to:

Prof Walter E Mientka, CAMC Executive Director
Department of Mathematics and Statistics
University of Nebraska, Lincoln, NE 68588-0322 USA

1989 AIME

The AIME is a 15-question, 3-hour, short-answer examination. You are invited to participate only if you receive a score of 100 or above on this AHSME. (Your school must also agree to administer the AIME.) The top-scoring students on the AHSME/AIME will be invited to take the **USA Mathematical Olympiad** on April 25, 1989; see the AHSME or AIME Teacher's Manual for more details. The best way to prepare for the AIME and USAMO is to study previous years of these exams. Copies may be ordered as indicated below.

PUBLICATIONS

MINIMUM ORDER: $5 (before handling fee), US FUNDS ONLY. Canada and US orders must be prepaid. Orders are mailed 4th class, unless you specify 1st class, in which case add $3.00 or 20% of total order, whichever is larger. Make checks payable to the American Mathematics Competitions (AMC).

FOREIGN ORDERS: do **NOT** prepay; an invoice will be sent.

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Examinations: Each price is for one copy of an exam and its solutions for one year. Specify the years you want and how many copies of each. All prices effective to July 1, 1989.

- **AJHSME** (Junior High Exam), 1985-1988, 50 cents per copy per year.
- **AHSME** 1972-89, 50 cents per copy per year.
- **AIME** 1983-88, $1 per copy per year.
- USA and International Mathematical **Olympiads** (together), 1976-88, $1 per copy per year.
- National **Summary** of Results and Awards, 1976-88, $4 per copy per year.

Books (Exams and solutions):

- Contest Problem Book I, AHSMEs 1950-60, $8.50.
- Contest Problem Book II, AHSMEs 1961-65, $8.50.
- Contest Problem Book III, AHSMEs 1966-72, $9.50.
- Contest Problem Book IV, AHSMEs 1973-82, $10.50.
- International Mathematical Olympiads, 1959-77, $9.50.
- International Mathematical Olympiads, 1978-85, $11.00.

Journal

- The ARBELOS (short articles and challenging problems); recommended especially for AIME and USAMO qualifiers. Six volumes, 1982-1987, $6.00 each. Canada and APO/FPO—add $3 per volume.

AMERICAN MATHEMATICS COMPETITIONS

41st ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)
TUESDAY, FEBRUARY 27, 1990

Sponsored by

Mathematical Association of America
Society of Actuaries   Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society   American Statistical Association
American Mathematical Association of Two-Year Colleges
American Mathematical Society

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PRINT ____ [ ]

last name first name middle initial age

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(home address) no. street

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city state or province zip or postcode

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school (full name) street address

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city state or province zip or postcode

Grade Level (9, 10, 11, 12 or other number) [ ] Date: ______________

1		16	
2		17	
3		18	
4		19	
5		20	
6		21	
7		22	
8		23	
9		24	
10		25	
11		26	
12		27	
13		28	
14		29	
15		30	

CERTIFICATION

I accept the procedures on reexamination and disqualification stated on the bottom of the front cover.

____

Student's Signature

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SCORING

(Not to be filled in by the student)
See Instruction 3 on the front cover.

number correct [ ] $\times$ 5 = [ ]

+

number unanswered [ ] $\times$ 2 = [ ]

Total Score = sum = [ ]

1. If $\frac{x/4}{2} = \frac{4}{x/2}$ then $x =$

(A) $\pm 1/2$ (B) ± 1 (C) ± 2 (D) ± 4 (E) ± 8

2. $\left(\frac{1}{4}\right)^{-\frac{1}{4}} =$

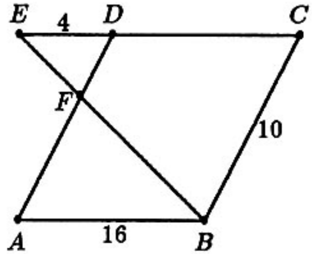
(A) -16 (B) $-\sqrt{2}$ (C) $-\frac{1}{16}$ (D) $\frac{1}{256}$ (E) $\sqrt{2}$

3. The consecutive angles of a trapezoid form an arithmetic sequence. If the smallest angle is $75°$, then the largest angle is

(A) $95°$ (B) $100°$ (C) $105°$ (D) $110°$ (E) $115°$

4. Let $ABCD$ be a parallelogram with $\angle ABC = 120°$, $AB = 16$ and $BC = 10$. Extend $\overline{CD}$ through D to E so that $DE = 4$. If $\overline{BE}$ intersects $\overline{AD}$ at F, then FD is closest to

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5



5. Which of these numbers is largest?

(A) $\sqrt{\sqrt[3]{5 \cdot 6}}$ (B) $\sqrt{6\sqrt[3]{5}}$ (C) $\sqrt{5\sqrt[3]{6}}$ (D) $\sqrt[3]{5\sqrt{6}}$ (E) $\sqrt[3]{6\sqrt{5}}$

6. Points A and B are 5 units apart. How many lines in a given plane containing A and B are 2 units from A and 3 units from B?

(A) 0 (B) 1 (C) 2 (D) 3 (E) more than 3

7. A triangle with integral sides has perimeter 8. The area of the triangle is

(A) $2\sqrt{2}$ (B) $\frac{16}{9}\sqrt{3}$ (C) $2\sqrt{3}$ (D) 4 (E) $4\sqrt{2}$

8. The number of real solutions of the equation $|x-2| + |x-3| = 1$ is

(A) 0 (B) 1 (C) 2 (D) 3 (E) more than 3

9. Each edge of a cube is colored either red or black. Every face of the cube has at least one black edge. The smallest possible number of black edges is

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

10. An $11\times11\times11$ wooden cube is formed by gluing together 11^3 unit cubes. What is the greatest number of unit cubes that can be seen from a single point?

(A) 328 **(B)** 329 **(C)** 330 **(D)** 331 **(E)** 332

11. How many positive integers less than 50 have an odd number of positive integer divisors?

(A) 3 **(B)** 5 **(C)** 7 **(D)** 9 **(E)** 11

12. Let f be the function defined by $f(x) = ax^2 - \sqrt{2}$ for some positive a. If $f\big(f(\sqrt{2})\big) = -\sqrt{2}$ then $a =$

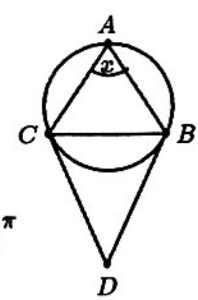
(A) $\frac{2-\sqrt{2}}{2}$ **(B)** $\frac{1}{2}$ **(C)** $2-\sqrt{2}$ **(D)** $\frac{\sqrt{2}}{2}$ **(E)** $\frac{2+\sqrt{2}}{2}$

13. If the following instructions are carried out by a computer, which value of X will be printed because of instruction 5?

1. START X AT 3 AND S AT 0.
2. INCREASE THE VALUE OF X BY 2.
3. INCREASE THE VALUE OF S BY THE VALUE OF X.
4. IF S IS AT LEAST 10000, THEN GO TO INSTRUCTION 5; OTHERWISE, GO TO INSTRUCTION 2 AND PROCEED FROM THERE.
5. PRINT THE VALUE OF X.
6. STOP.

(A) 19 **(B)** 21 **(C)** 23 **(D)** 199 **(E)** 201

14. An acute isosceles triangle, ABC, is inscribed in a circle. Through B and C, tangents to the circle are drawn, meeting at point D. If $\angle ABC = \angle ACB = 2\angle D$ and x is the radian measure of $\angle A$, then $x =$



(A) $\frac{3}{7}\pi$ **(B)** $\frac{4}{9}\pi$ **(C)** $\frac{5}{11}\pi$ **(D)** $\frac{6}{13}\pi$ **(E)** $\frac{7}{15}\pi$

15. Four whole numbers, when added three at a time, give the sums 180, 197, 208 and 222. What is the largest of the four numbers?

(A) 77 (B) 83 (C) 89 (D) 95

(E) cannot be determined from the given information

16. At one of George Washington's parties, each man shook hands with everyone except his spouse, and no handshakes took place between women. If 13 married couples attended, how many handshakes were there among these 26 people?

(A) 78 (B) 185 (C) 234 (D) 312 (E) 325

17. How many of the numbers, $100, 101, \ldots, 999$, have three different digits in increasing order or in decreasing order?

(A) 120 (B) 168 (C) 204 (D) 216 (E) 240

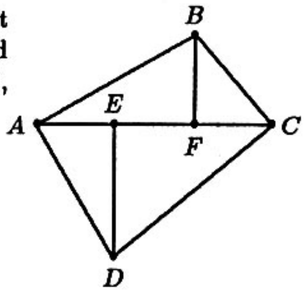
18. First a is chosen at random from the set $\{1, 2, 3, \ldots, 99, 100\}$, and then b is chosen at random from the same set. The probability that the integer $3^a + 7^b$ has units digit 8 is

(A) $\frac{1}{16}$ (B) $\frac{1}{8}$ (C) $\frac{3}{16}$ (D) $\frac{1}{5}$ (E) $\frac{1}{4}$

19. For how many integers N between 1 and 1990 is the improper fraction $\dfrac{N^2+7}{N+4}$ not in lowest terms?

(A) 0 (B) 86 (C) 90 (D) 104 (E) 105

20. In the figure, $ABCD$ is a quadrilateral with right angles at A and C. Points E and F are on $\overline{AC}$, and $\overline{DE}$ and $\overline{BF}$ are perpendicular to $\overline{AC}$. If $AE = 3$, $DE = 5$ and $CE = 7$, then $BF =$



(A) 3.6 (B) 4 (C) 4.2

(D) 4.5 (E) 5

21. Consider a pyramid $P\text{-}ABCD$ whose base $ABCD$ is square and whose vertex P is equidistant from A, B, C and D. If $AB = 1$ and $\angle APB = 2\theta$ then the volume of the pyramid is

(A) $\frac{\sin\theta}{6}$ **(B)** $\frac{\cot\theta}{6}$ **(C)** $\frac{1}{6\sin\theta}$ **(D)** $\frac{1-\sin 2\theta}{6}$ **(E)** $\frac{\sqrt{\cos 2\theta}}{6\sin\theta}$

22. If the six solutions of $x^6 = -64$ are written in the form $a + bi$, where a and b are real, then the product of those solutions with $a > 0$ is

(A) -2 **(B)** 0 **(C)** $2i$ **(D)** 4 **(E)** 16

23. If $x, y > 0$, $\log_y x + \log_x y = \frac{10}{3}$ and $xy = 144$, then $\frac{x+y}{2} =$

(A) $12\sqrt{2}$ **(B)** $13\sqrt{3}$ **(C)** 24 **(D)** 30 **(E)** 36

24. All students at Adams High School and at Baker High School take a certain exam. The average scores for boys, for girls, and for boys and girls combined, at Adams HS and Baker HS are shown in the table, as is the average for boys at the two schools combined. What is the average score for the girls at the two schools combined?

	Adams	Baker	Adams & Baker
Boys :	71	81	79
Girls :	76	90	?
Boys & Girls :	74	84	

(A) 81 **(B)** 82 **(C)** 83 **(D)** 84 **(E)** 85

25. Nine congruent spheres are packed inside a unit cube in such a way that one of them has its center at the center of the cube and each of the others is tangent to the center sphere and to three faces of the cube. What is the radius of each sphere?

(A) $1 - \frac{\sqrt{3}}{2}$ **(B)** $\frac{2\sqrt{3}-3}{2}$ **(C)** $\frac{\sqrt{2}}{6}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{\sqrt{3}(2-\sqrt{2})}{4}$

26. Ten people form a circle. Each picks a number and tells it to the two neighbors adjacent to him in the circle. Then each person computes and announces the average of the numbers of his two neighbors. The figure shows the average announced by each person (not the original number the person picked). The number picked by the person who announced the average 6 was

"1" "2" "3" "4" "5" "6" "7" "8" "9" "10"

(A) 1 **(B)** 5 **(C)** 6 **(D)** 10
(E) not uniquely determined from the given information

27. Which of these triples could not be the lengths of the three altitudes of a triangle?

(A) $1, \sqrt{3}, 2$ **(B)** $3, 4, 5$ **(C)** $5, 12, 13$ **(D)** $7, 8, \sqrt{113}$ **(E)** $8, 15, 17$

28. A quadrilateral that has consecutive sides of lengths 70, 90, 130 and 110 is inscribed in a circle and also has a circle inscribed in it. The point of tangency of the inscribed circle to the side of length 130 divides that side into segments of lengths x and y. Find $|x-y|$.

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

29. A subset of the integers $1, 2, \ldots, 100$ has the property that none of its members is 3 times another. What is the largest number of members such a subset can have?

(A) 50 **(B)** 66 **(C)** 67 **(D)** 76 **(E)** 78

30. If $R_n = \frac{1}{2}(a^n + b^n)$ where $a = 3 + 2\sqrt{2}$, $b = 3 - 2\sqrt{2}$, and $n = 0, 1, 2, \ldots$, then R_{12345} is an integer. Its units digit is

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

SOLUTIONS

Your School Examination Manager has at least one copy of the 1990 AHSME Solutions Pamphlet. It is meant to be lent or given to students (but not duplicated).

WRITE TO US!

Questions and comments about the problems and solutions for this AHSME (but not requests for the Solutions Pamphlet) should be addressed to:

Prof Leo J Schneider, AHSME Chairman
Department of Mathematics and Computer Science
John Carroll University, University Heights, OH 44118-4581 USA

Comments about administrative arrangements and orders for any publications listed below should be addressed to:

Prof Walter E Mientka, AMC Executive Director
Department of Mathematics and Statistics
University of Nebraska, Lincoln, NE 68588-0322 USA

1990 AIME

The AIME will be held on Tuesday, March 20, 1990. It is a 15-question, 3-hour, short-answer examination. You will be invited to participate only if you receive a score of 100 or above on this AHSME. (Your school must also agree to administer the AIME.) The top-scoring students on the AHSME/AIME will be invited to take the USA Mathematical Olympiad on April 24, 1990; see the AHSME or AIME Teacher's Manual for more details. The best way to prepare for the AIME and USAMO is to study previous years of these exams. Copies may be ordered as indicated below.

PUBLICATIONS

MINIMUM ORDER: $5 (before handling fee), **US FUNDS ONLY. Canada and US orders must be prepaid. Orders are mailed 4th class, unless you specify 1st class, in which case add $3.00 or 20% of total order, whichever is larger, with a maximum of $10.00. Make checks payable to the American Mathematics Competitions.**

FOREIGN ORDERS: do **NOT** prepay; an invoice will be sent.

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Examinations: Each price is for one copy of an exam and its solutions for one year. Specify the years you want and how many copies of each. All prices effective to July 1, 1990.

- **AJHSME** (Junior High Exam), 1985-1989, 50 cents per copy per year.
- **AHSME** 1972-90, 50 cents per copy per year.
- **AIME** 1983-89, $2 per copy per year.
- USA and International Mathematical **Olympiads** (together), 1976-89, $2 per copy per year.
- National **Summary** of Results and Awards, 1980-89, $4 per copy per year.

Books (Exams and solutions):
- Contest Problem Book I, AHSMEs 1950-60, $8.50.
- Contest Problem Book II, AHSMEs 1961-65, $8.50.
- Contest Problem Book III, AHSMEs 1966-72, $10.00.
- Contest Problem Book IV, AHSMEs 1973-82, $11.00.
- USA Mathematical Olympiads, 1972-86, $13.00.
- International Mathematical Olympiads, 1959-77, $10.00.
- International Mathematical Olympiads, 1978-85, $11.00.

Journal
- The ARBELOS (short articles and challenging problems); recommended especially for AIME and USAMO qualifiers. Six volumes, 1982-1987, $7.00 each.

AMERICAN MATHEMATICS COMPETITIONS

42nd ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)
TUESDAY, FEBRUARY 26, 1991

Sponsored by

Mathematical Association of America
Society of Actuaries Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society American Statistical Association
American Mathematical Association of Two-Year Colleges
American Mathematical Society

1. **DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.**
2. **This is a thirty question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.**
3. **SCORING RULES: You will receive 5 points for each correct answer, 2 points for each problem left unanswered, and 0 points for each incorrect answer.**
4. **Solve the problem carefully. Note the scoring rules. To guess before eliminating 3 of the 5 choices will, on the average, lower your score.**
5. **Use pencil. Scratch paper, graph paper, ruler, compass, protractor and eraser are permitted.** ***Calculators, slide rules and math tables are not permitted.***
6. **Figures are not necessarily drawn to scale.**
7. **The back of this page is a form on which you are asked to write certain information, sign a Certification and record your answers. You must use this form unless your region has centralized computer scoring (for which your proctor will give you a different form and instructions).**
8. **Difficult to read or ambiguous answers will be marked wrong. Printed capital letters are recommended.**
9. **Carefully follow the instructions of your proctor in order to keep the examination questions concealed while providing the requested information and signature on page 2.**
10. **When your proctor gives the signal, begin working the problems. You will have 90 MINUTES working time for the test.**

Students who score 100 or above on this AHSME will be invited to take the 9th annual American Invitational Mathematics Examination (AIME) on March 19, 1991. More details about the AIME and other information are on page 8.

The results of this AHSME are used to identify students with unusual mathematical ability. To assure that this purpose is served, the Committee on the American Mathematics Competitions reserves the right to re-examine students before deciding whether to grant official status to individual or team scores. Re-examination will be requested when, after an inquiry, there is reasonable basis to believe that scores have been obtained by extremely lucky guessing or dishonesty. Official status will not be granted if a student or school does not agree to a requested re-examination. The Committee also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

PRINT ____ [ ]

last name | first name | middle initial | age

____

(home address) no. | street

____

city | state or province | zip or postcode

____

school (full name) | street address

____

city | state or province | zip or postcode

Grade Level (9, 10, 11, 12 or other number) [ ] Date: ______________

1		16	
2		17	
3		18	
4		19	
5		20	
6		21	
7		22	
8		23	
9		24	
10		25	
11		26	
12		27	
13		28	
14		29	
15		30	

CERTIFICATION

I accept the procedures on reexamination and disqualification stated on the bottom of the front cover.

____

Student's Signature

Your score is **not official** without your signature above.

SCORING

(Not to be filled in by the student)
See Instruction 3 on the front cover.

number correct [ ] × 5 = [ ]

+

number unanswered [ ] × 2 = [ ]

Total Score = sum = [ ]

1. If for any three distinct numbers a, b and c we define $\boxed{a,b,c}$ by

$$\boxed{a,b,c} = \frac{c+a}{c-b},$$

then $\boxed{1,-2,-3} =$

(A) -2 (B) $-\frac{2}{5}$ (C) $-\frac{1}{4}$ (D) $\frac{2}{5}$ (E) 2

2. $|3-\pi| =$

(A) $\frac{1}{7}$ (B) $.14$ (C) $3-\pi$ (D) $3+\pi$ (E) $\pi-3$

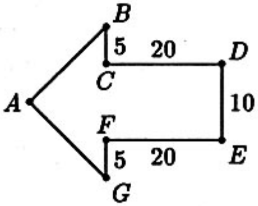
3. $\left(4^{-1}-3^{-1}\right)^{-1} =$

(A) -12 (B) -1 (C) $\frac{1}{12}$ (D) 1 (E) 12

4. Which of the following triangles cannot exist?

(A) An acute isosceles triangle (B) An isosceles right triangle
(C) An obtuse right triangle (D) A scalene right triangle
(E) A scalene obtuse triangle

5. In the arrow-shaped polygon [see figure], the angles at vertices A, C, D, E and F are right angles, $BC = FG = 5$, $CD = FE = 20$, $DE = 10$, and $AB = AG$. The area of the polygon is closest to



(A) 288 (B) 291 (C) 294
(D) 297 (E) 300

6. If $x \geq 0$, then $\sqrt{x\sqrt{x\sqrt{x}}} =$

(A) $x\sqrt{x}$ (B) $x\sqrt[4]{x}$ (C) $\sqrt[8]{x}$ (D) $\sqrt[8]{x^3}$ (E) $\sqrt[8]{x^7}$

7. If $x = \frac{a}{b}$, $a \neq b$ and $b \neq 0$, then $\frac{a+b}{a-b} =$

(A) $\frac{x}{x+1}$ (B) $\frac{x+1}{x-1}$ (C) 1 (D) $x-\frac{1}{x}$ (E) $x+\frac{1}{x}$

8. Liquid X does not mix with water. Unless obstructed, it spreads out on the surface of water to form a circular film 0.1 cm thick. A rectangular box measuring 6 cm by 3 cm by 12 cm is filled with liquid X. Its contents are poured onto a large body of water. What will be the radius, in centimeters, of the resulting circular film?

(A) $\frac{\sqrt{216}}{\pi}$ (B) $\sqrt{\frac{216}{\pi}}$ (C) $\sqrt{\frac{2160}{\pi}}$ (D) $\frac{216}{\pi}$ (E) $\frac{2160}{\pi}$

9. From time $t = 0$ to time $t = 1$ a population increased by $i\%$, and from time $t = 1$ to time $t = 2$ the population increased by $j\%$. Therefore, from time $t = 0$ to time $t = 2$ the population increased by

(A) $(i+j)\%$ (B) $ij\%$ (C) $(i+ij)\%$ (D) $\left(i+j+\frac{ij}{100}\right)\%$

(E) $\left(i+j+\frac{i+j}{100}\right)\%$

10. Point P is 9 units from the center of a circle of radius 15. How many different chords of the circle contain P and have integer lengths?

(A) 11 (B) 12 (C) 13 (D) 14 (E) 29

11. Jack and Jill run 10 kilometers. They start at the same point, run 5 kilometers up a hill, and return to the starting point by the same route. Jack has a 10-minute head start and runs at the rate of 15 km/hr uphill and 20 km/hr downhill. Jill runs 16 km/hr uphill and 22 km/hr downhill. How far from the top of the hill are they when they pass going in opposite directions?

(A) $\frac{5}{4}$ km (B) $\frac{35}{27}$ km (C) $\frac{27}{20}$ km (D) $\frac{7}{3}$ km (E) $\frac{28}{9}$ km

12. The measures (in degrees) of the interior angles of a convex hexagon form an arithmetic sequence of positive integers. Let m° be the measure of the largest interior angle of the hexagon. The largest possible value of m° is

(A) 165° (B) 167° (C) 170° (D) 175° (E) 179°

13. Horses X, Y and Z are entered in a three-horse race in which ties are not possible. If the odds against X winning are 3-to-1 and the odds against Y winning are 2-to-3, what are the odds against Z winning? (By *"odds against H winning are p-to-q"* we mean that the probability of H winning the race is $\frac{q}{p+q}$.)

(A) 3-to-20 (B) 5-to-6 (C) 8-to-5 (D) 17-to-3 (E) 20-to-3

14. If x is the cube of a positive integer and d is the number of positive integers that are divisors of x, then d could be

(A) 200 (B) 201 (C) 202 (D) 203 (E) 204

15. A circular table has exactly 60 chairs around it. There are N people seated at this table in such a way that the next person to be seated must sit next to someone. The smallest possible value of N is

(A) 15 (B) 20 (C) 30 (D) 40 (E) 58

16. One hundred students at Century High School participated in the **AHSME** last year, and their mean score was 100. The number of non-seniors taking the **AHSME** was 50% more than the number of seniors, and the mean score of the seniors was 50% higher than that of the non-seniors. What was the mean score of the seniors?

(A) 100 (B) 112.5 (C) 120 (D) 125 (E) 150

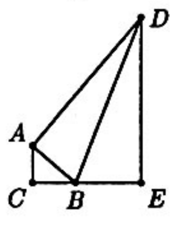
17. A positive integer N is a *palindrome* if the integer obtained by reversing the sequence of digits of N is equal to N. The year 1991 is the only year in the current century with the following two properties:
(a) It is a palindrome.
(b) It factors as a product of a 2-digit prime palindrome and a 3-digit prime palindrome.
How many years in the millenium between 1000 and 2000 (including the year 1991) have properties (a) and (b)?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

18. If S is the set of points z in the complex plane such that $(3+4i)z$ is a *real* number, then S is a

(A) right triangle (B) circle (C) hyperbola
(D) line (E) parabola

19. Triangle ABC has a right angle at C, $AC = 3$ and $BC = 4$. Triangle ABD has a right angle at A and $AD = 12$. Points C and D are on opposite sides of $\overline{AB}$. The line through D parallel to $\overline{AC}$ meets $\overline{CB}$ extended at E. If $\dfrac{DE}{DB} = \dfrac{m}{n}$, where m and n are relatively prime positive integers, then $m+n =$



(A) 25 (B) 128 (C) 153 (D) 243 (E) 256

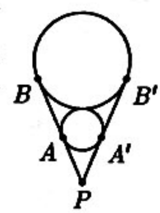
20. The sum of all real x such that $(2^x-4)^3+(4^x-2)^3=(4^x+2^x-6)^3$ is

(A) 3/2 (B) 2 (C) 5/2 (D) 3 (E) 7/2

21. If $f\left(\frac{x}{x-1}\right)=\frac{1}{x}$ for all $x\neq 0,1$ and $0<\theta<\frac{\pi}{2}$, then $f\left(\sec^2\theta\right)=$

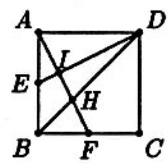
(A) $\sin^2\theta$ (B) $\cos^2\theta$ (C) $\tan^2\theta$ (D) $\cot^2\theta$ (E) $\csc^2\theta$

22. Two circles are externally tangent. Lines $\overline{PAB}$ and $\overline{PA'B'}$ are common tangents with A and A' on the smaller circle and B and B' on the larger circle. If $PA=AB=4$, then the area of the smaller circle is



(A) 1.44π (B) 2π (C) 2.56π
(D) $\sqrt{8}\pi$ (E) 4π

23. If $ABCD$ is a 2×2 square, E is the midpoint of $\overline{AB}$, F is the midpoint of $\overline{BC}$, $\overline{AF}$ and $\overline{DE}$ intersect at I, and $\overline{BD}$ and $\overline{AF}$ intersect at H, then the area of quadrilateral $BEIH$ is



(A) $\frac{1}{3}$ (B) $\frac{2}{5}$ (C) $\frac{7}{15}$ (D) $\frac{8}{15}$ (E) $\frac{3}{5}$

24. The graph, G, of $y=\log_{10}x$ is rotated 90° counter-clockwise about the origin to obtain a new graph G'. Which of the following is an equation for G'?

(A) $y=\log_{10}\left(\frac{x+90}{9}\right)$ (B) $y=\log_x 10$ (C) $y=\frac{1}{x+1}$
(D) $y=10^{-x}$ (E) $y=10^x$

25. If $T_n=1+2+3+\cdots+n$ and

$$P_n=\frac{T_2}{T_2-1}\cdot\frac{T_3}{T_3-1}\cdot\frac{T_4}{T_4-1}\cdot\;\cdots\;\cdot\frac{T_n}{T_n-1}\quad\text{for } n=2,3,4,\ldots,$$

then P_{1991} is closest to which of the following numbers?

(A) 2.0 (B) 2.3 (C) 2.6 (D) 2.9 (E) 3.2

26. An n-digit positive integer is *cute* if its n digits are an arrangement of the set $\{1, 2, \ldots, n\}$ and its first k digits form an integer that is divisible by k, for $k = 1, 2, \ldots, n$. For example, 321 is a cute 3-digit integer because 1 divides 3, 2 divides 32, and 3 divides 321. How many cute 6-digit integers are there?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

27. If $x + \sqrt{x^2 - 1} + \dfrac{1}{x - \sqrt{x^2 - 1}} = 20$ then $x^2 + \sqrt{x^4 - 1} + \dfrac{1}{x^2 + \sqrt{x^4 - 1}} =$

(A) 5.05 (B) 20 (C) 51.005 (D) 61.25 (E) 400

28. Initially an urn contains 100 black marbles and 100 white marbles. Repeatedly, three marbles are removed from the urn and replaced from a pile outside the urn as follows:

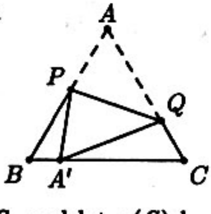
MARBLES REMOVED	REPLACED WITH
3 black	1 black
2 black, 1 white	1 black, 1 white
1 black, 2 white	2 white
3 white	1 black, 1 white.

Which of the following sets of marbles could be the contents of the urn after repeated applications of this procedure?

(A) 2 black marbles (B) 2 white marbles (C) 1 black marble
(D) 1 black and 1 white marble (E) 1 white marble

29. Equilateral triangle ABC has been creased and folded so that vertex A now rests at A' on $\overline{BC}$ as shown. If $BA' = 1$ and $A'C = 2$ then the length of crease $\overline{PQ}$ is

(A) $\frac{8}{5}$ (B) $\frac{7}{20}\sqrt{21}$ (C) $\frac{1+\sqrt{5}}{2}$ (D) $\frac{13}{8}$
(E) $\sqrt{3}$



30. For any set S, let $|S|$ denote the number of elements in S, and let $n(S)$ be the number of subsets of S, including the empty set and the set S itself. If A, B and C are sets for which

$$n(A) + n(B) + n(C) = n(A \cup B \cup C) \quad \text{and} \quad |A| = |B| = 100,$$

then what is the minimum possible value of $|A \cap B \cap C|$?

(A) 96 (B) 97 (C) 98 (D) 99 (E) 100

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University of Nebraska, Lincoln, NE 68588-0322 USA

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AMERICAN MATHEMATICS COMPETITIONS

43nd ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)
THURSDAY, FEBRUARY 27, 1992

Sponsored by

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American Mathematical Society

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1. $6^6+6^6+6^6+6^6+6^6+6^6=$

(A) 6^6 **(B)** 6^7 **(C)** 36^6 **(D)** 6^{36} **(E)** 36^{36}

2. If $3(4x+5\pi)=P$, then $6(8x+10\pi)=$

(A) $2P$ **(B)** $4P$ **(C)** $6P$ **(D)** $8P$ **(E)** $18P$

3. An urn is filled with coins and beads, all of which are either silver or gold. Twenty percent of the objects in the urn are beads. Forty percent of the coins in the urn are silver. What percent of the objects in the urn are gold coins?

(A) 40% **(B)** 48% **(C)** 52% **(D)** 60% **(E)** 80%

4. If $m>0$ and the points $(m,3)$ and $(1,m)$ lie on a line with slope m, then $m=$

(A) 1 **(B)** $\sqrt{2}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** $\sqrt{5}$

5. If a, b and c are positive integers and a and b are odd, then $3^a+(b-1)^2c$ is

(A) odd for all choices of c

(B) even for all choices of c

(C) odd, if c is even; even, if c is odd

(D) odd, if c is odd; even, if c is even

(E) odd, if c is not a multiple of 3; even, if c is a multiple of 3

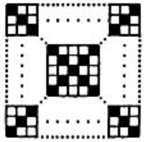
6. If $x>y>0$, then $\dfrac{x^y y^x}{y^y x^x}=$

(A) $(x-y)^{y/x}$ **(B)** $\left(\dfrac{x}{y}\right)^{x-y}$ **(C)** 1 **(D)** $\left(\dfrac{x}{y}\right)^{y-x}$ **(E)** $(x-y)^{x/y}$

7. The ratio of w to x is $4:3$, of y to z is $3:2$ and of z to x is $1:6$. What is the ratio of w to y?

(A) $1:3$ **(B)** $16:3$ **(C)** $20:3$ **(D)** $27:4$ **(E)** $12:1$

8. A square floor is tiled with congruent square tiles. The tiles on the two diagonals of the floor are black. The rest of the tiles are white. If there are 101 black tiles, then the total number of tiles is



(A) 121 (B) 625 (C) 676 (D) 2500 (E) 2601

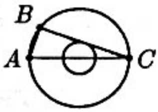
9. Five equilateral triangles, each with side $2\sqrt{3}$, are arranged so they are all on the same side of a line containing one side of each. Along this line, the midpoint of the base of one triangle is a vertex of the next. The area of the region of the plane that is covered by the union of the five triangular regions is

(A) 10 (B) 12 (C) 15 (D) $10\sqrt{3}$ (E) $12\sqrt{3}$

10. The number of positive integers k for which the equation $kx - 12 = 3k$ has an integer solution for x is

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

11. The ratio of the radii of two concentric circles is $1:3$. If $\overline{AC}$ is a diameter of the larger circle, $\overline{BC}$ is a chord of the larger circle that is tangent to the smaller circle, and $AB = 12$, then the radius of the larger circle is



(A) 13 (B) 18 (C) 21 (D) 24 (E) 26

12. Let $y = mx + b$ be the image when the line $x - 3y + 11 = 0$ is reflected across the x-axis. The value of $m + b$ is

(A) -6 (B) -5 (C) -4 (D) -3 (E) -2

13. How many pairs of positive integers (a, b) with $a+b \leq 100$ satisfy the equation

$$\frac{a + b^{-1}}{a^{-1} + b} = 13?$$

(A) 1 (B) 5 (C) 7 (D) 9 (E) 13

14. Which of the following equations have the same graph?

I. $y = x - 2$ II. $y = \dfrac{x^2 - 4}{x + 2}$ III. $(x + 2)y = x^2 - 4$

(A) I and II only **(B)** I and III only **(C)** II and III only
(D) I, II and III **(E)** None. All the equations have different graphs

15. Let $i = \sqrt{-1}$. Define a sequence of complex numbers by $z_1 = 0$, $z_{n+1} = z_n^2 + i$ for $n \geq 1$. In the complex plane, how far from the origin is z_{111}?

(A) 1 **(B)** $\sqrt{2}$ **(C)** $\sqrt{3}$ **(D)** $\sqrt{110}$ **(E)** $\sqrt{2^{55}}$

16. If $\dfrac{y}{x - z} = \dfrac{x + y}{z} = \dfrac{x}{y}$ for three positive numbers x, y and z, all different, then $\dfrac{x}{y} =$

(A) $\frac{1}{2}$ **(B)** $\frac{3}{5}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{5}{3}$ **(E)** 2

17. The two-digit integers from 19 to 92 are written consecutively to form the large integer

$$N = 19202122\ldots909192.$$

If 3^k is the highest power of 3 that is a factor of N, then $k =$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** more than 3

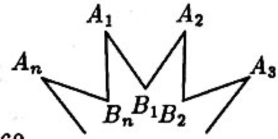
18. The increasing sequence of positive integers $a_1, a_2, a_3, \ldots$ has the property that $a_{n+2} = a_n + a_{n+1}$ for all $n \geq 1$. If $a_7 = 120$, then a_8 is

(A) 128 **(B)** 168 **(C)** 193 **(D)** 194 **(E)** 210

19. For each vertex of a solid cube, consider the tetrahedron determined by the vertex and the midpoints of the three edges that meet at that vertex. The portion of the cube that remains when these eight tetrahedra are cut away is called a *cuboctahedron*. The ratio of the volume of the cuboctahedron to the volume of the original cube is closest to which of these?

(A) 75% **(B)** 78% **(C)** 81% **(D)** 84% **(E)** 87%

20. Part of an "n-pointed regular star" is shown. It is a simple closed polygon in which all $2n$ edges are congruent, angles A_1, A_2, …, A_n are congruent and angles B_1, B_2, …, B_n are congruent. If the acute angle at A_1 is $10°$ less than the acute angle at B_1, then $n =$



(A) 12 (B) 18 (C) 24 (D) 36 (E) 60

21. For a finite sequence $A = (a_1, a_2, \ldots, a_n)$ of numbers, the *Cesàro sum* of A is defined to be

$$\frac{S_1 + S_2 + \cdots + S_n}{n},$$

where $S_k = a_1 + a_2 + \cdots + a_k$ $(1 \le k \le n)$. If the Cesàro sum of the 99-term sequence $(a_1, a_2, \ldots, a_{99})$ is 1000, what is the Cesàro sum of the 100-term sequence $(1, a_1, a_2, \ldots, a_{99})$?

(A) 991 (B) 999 (C) 1000 (D) 1001 (E) 1009

22. Ten points are selected on the positive x-axis, $\mathbf{X}^+$, and five points are selected on the positive y-axis, $\mathbf{Y}^+$. The fifty segments connecting the ten selected points on $\mathbf{X}^+$ to the five selected points on $\mathbf{Y}^+$ are drawn. What is the maximum possible number of points of intersection of these fifty segments that could lie in the interior of the first quadrant?

(A) 250 (B) 450 (C) 500 (D) 1250 (E) 2500

23. What is the size of the largest subset, S, of $\{1, 2, 3, \ldots, 50\}$ such that no pair of distinct elements of S has a sum divisible by 7?

(A) 6 (B) 7 (C) 14 (D) 22 (E) 23

24. Let $ABCD$ be a parallelogram of area 10 with $AB = 3$ and $BC = 5$. Locate E, F and G on segments $\overline{AB}$, $\overline{BC}$ and $\overline{AD}$, respectively, with $AE = BF = AG = 2$. Let the line through G parallel to $\overline{EF}$ intersect $\overline{CD}$ at H. The area of the quadrilateral $EFHG$ is

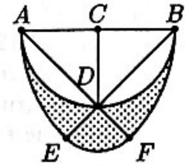
(A) 4 (B) 4.5 (C) 5 (D) 5.5 (E) 6

25. In triangle ABC, $\angle ABC = 120°$, $AB = 3$ and $BC = 4$. If perpendiculars constructed to $\overline{AB}$ at A and to $\overline{BC}$ at C meet at D, then $CD =$

(A) 3 (B) $\frac{8}{\sqrt{3}}$ (C) 5 (D) $\frac{11}{2}$ (E) $\frac{10}{\sqrt{3}}$

SEE NEXT PAGE For Problems 26 - 30

26. Semicircle $\widehat{AB}$ has center C and radius 1. Point D is on $\widehat{AB}$ and $\overline{CD} \perp \overline{AB}$. Extend $\overline{BD}$ and $\overline{AD}$ to E and F, respectively, so that circular arcs $\widehat{AE}$ and $\widehat{BF}$ have B and A as their respective centers. Circular arc $\widehat{EF}$ has center D. The area of the shaded "smile", $AEFBDA$, is



(A) $\left(2-\sqrt{2}\right)\pi$ (B) $2\pi - \pi\sqrt{2} - 1$ (C) $\left(1-\frac{\sqrt{2}}{2}\right)\pi$

(D) $\frac{5\pi}{2} - \pi\sqrt{2} - 1$ (E) $\left(3-2\sqrt{2}\right)\pi$

27. A circle of radius r has chords $\overline{AB}$ of length 10 and $\overline{CD}$ of length 7. When $\overline{AB}$ and $\overline{CD}$ are extended through B and C, respectively, they intersect at P, which is outside the circle. If $\angle APD = 60^\circ$ and $BP = 8$, then $r^2 =$

(A) 70 (B) 71 (C) 72 (D) 73 (E) 74

28. Let $i = \sqrt{-1}$. The product of the real parts of the roots of $z^2 - z = 5 - 5i$ is

(A) -25 (B) -6 (C) -5 (D) $\frac{1}{4}$ (E) 25

29. An "unfair" coin has a 2/3 probability of turning up heads. If this coin is tossed 50 times, what is the probability that the total number of heads is even?

(A) $25\left(\frac{2}{3}\right)^{50}$ (B) $\frac{1}{2}\left(1-\frac{1}{3^{50}}\right)$ (C) $\frac{1}{2}$

(D) $\frac{1}{2}\left(1+\frac{1}{3^{50}}\right)$ (E) $\frac{2}{3}$

30. Let $ABCD$ be an isosceles trapezoid with bases $AB = 92$ and $CD = 19$. Suppose $AD = BC = x$ and a circle with center on $\overline{AB}$ is tangent to segments $\overline{AD}$ and $\overline{BC}$. If m is the smallest possible value of x, then $m^2 =$

(A) 1369 (B) 1679 (C) 1748 (D) 2109 (E) 8825

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44th ANNUAL
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(AHSME)
THURSDAY, FEBRUARY 25, 1993

Sponsored by

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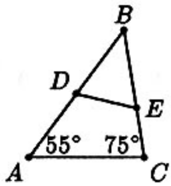
1. For integers a, b and c, define $\boxed{a,b,c}$ to mean $a^b - b^c + c^a$. Then $\boxed{1,-1,2}$ equals

(A) -4 (B) -2 (C) 0 (D) 2 (E) 4

2. In $\triangle ABC$, $\angle A = 55°$, $\angle C = 75°$, D is on side $\overline{AB}$ and E is on side $\overline{BC}$. If $DB = BE$, then $\angle BED =$



(A) 50° (B) 55° (C) 60° (D) 65° (E) 70°

3. $\dfrac{15^{30}}{45^{15}} =$

(A) $\left(\frac{1}{3}\right)^{15}$ (B) $\left(\frac{1}{3}\right)^{2}$ (C) 1 (D) 3^{15} (E) 5^{15}

4. Define the operation "$\circ$" by $x \circ y = 4x - 3y + xy$, for all real numbers x and y. For how many real numbers y does $3 \circ y = 12$?

(A) 0 (B) 1 (C) 3 (D) 4 (E) more than 4

5. Last year a bicycle cost \$160 and a cycling helmet cost \$40. This year the cost of the bicycle increased by 5%, and the cost of the helmet increased by 10%. The percent increase in the combined cost of the bicycle and the helmet is

(A) 6% (B) 7% (C) 7.5% (D) 8% (E) 15%

6. $\sqrt{\dfrac{8^{10} + 4^{10}}{8^4 + 4^{11}}} =$

(A) $\sqrt{2}$ (B) 16 (C) 32 (D) $12^{2/3}$ (E) 512.5

7. The symbol R_k stands for an integer whose base-ten representation is a sequence of k ones. For example, $R_3 = 111$, $R_5 = 11111$, etc. When R_{24} is divided by R_4, the quotient $Q = \dfrac{R_{24}}{R_4}$ is an integer whose base-ten representation is a sequence containing only ones and zeros. The number of zeros in Q is

(A) 10 (B) 11 (C) 12 (D) 13 (E) 15

8. Let C_1 and C_2 be circles of radius 1 that are in the same plane and tangent to each other. How many circles of radius 3 are in this plane and tangent to both C_1 and C_2?

(A) 2 (B) 4 (C) 5 (D) 6 (E) 8

9. Country $\mathcal{A}$ has $c\%$ of the world's population and owns $d\%$ of the world's wealth. Country $\mathcal{B}$ has $e\%$ of the world's population and $f\%$ of its wealth. Assume that the citizens of $\mathcal{A}$ share the wealth of $\mathcal{A}$ equally, and assume that those of $\mathcal{B}$ share the wealth of $\mathcal{B}$ equally. Find the ratio of the wealth of a citizen of $\mathcal{A}$ to the wealth of a citizen of $\mathcal{B}$.

(A) $\frac{cd}{ef}$ (B) $\frac{ce}{df}$ (C) $\frac{cf}{de}$ (D) $\frac{de}{cf}$ (E) $\frac{df}{ce}$

10. Let r be the number that results when both the base and the exponent of a^b are tripled, where $a, b > 0$. If r equals the product of a^b and x^b where $x > 0$, then $x =$

(A) 3 (B) $3a^2$ (C) $27a^2$ (D) $2a^{3b}$ (E) $3a^{2b}$

11. If $\log_2(\log_2(\log_2(x))) = 2$, then how many digits are in the base-ten representation for x?

(A) 5 (B) 7 (C) 9 (D) 11 (E) 13

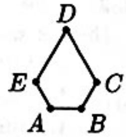
12. If $f(2x) = \frac{2}{2+x}$ for all $x > 0$, then $2f(x) =$

(A) $\frac{2}{1+x}$ (B) $\frac{2}{2+x}$ (C) $\frac{4}{1+x}$ (D) $\frac{4}{2+x}$ (E) $\frac{8}{4+x}$

13. A square of perimeter 20 is inscribed in a square of perimeter 28. What is the greatest distance between a vertex of the inner square and a vertex of the outer square?

(A) $\sqrt{58}$ (B) $\frac{7\sqrt{5}}{2}$ (C) 8 (D) $\sqrt{65}$ (E) $5\sqrt{3}$

14. The convex pentagon $ABCDE$ has $\angle A = \angle B = 120°$, $EA = AB = BC = 2$ and $CD = DE = 4$. What is the area of $ABCDE$?



(A) 10 **(B)** $7\sqrt{3}$ **(C)** 15
(D) $9\sqrt{3}$ **(E)** $12\sqrt{5}$

15. For how many values of n will an n-sided regular polygon have interior angles with integral degree measures?

(A) 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24

16. Consider the non-decreasing sequence of positive integers

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, ..$$

in which the n^{th} positive integer appears n times. The remainder when the 1993^{rd} term is divided by 5 is

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

17. Amy painted a dart board over a square clock face using the "hour positions" as boundaries. [See figure.] If t is the area of one of the eight triangular regions such as that between 12 o'clock and 1 o'clock, and q is the area of one of the four corner quadrilaterals such as that between 1 o'clock and 2 o'clock, then $\frac{q}{t} =$

(A) $2\sqrt{3} - 2$ **(B)** $\frac{3}{2}$ **(C)** $\frac{\sqrt{5}+1}{2}$ **(D)** $\sqrt{3}$ **(E)** 2

18. Al and Barb start their new jobs on the same day. Al's schedule is 3 work-days followed by 1 rest-day. Barb's schedule is 7 work-days followed by 3 rest-days. On how many of their first 1000 days do both have rest-days on the same day?

(A) 48 **(B)** 50 **(C)** 72 **(D)** 75 **(E)** 100

19. How many ordered pairs (m, n) of positive integers are solutions to $\frac{4}{m} + \frac{2}{n} = 1$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** more than 4

20. Consider the equation $10z^2 - 3iz - k = 0$, where z is a complex variable and $i^2 = -1$. Which of the following statements is true?

(A) For all positive real numbers k, both roots are pure imaginary.

(B) For all negative real numbers k, both roots are pure imaginary.

(C) For all pure imaginary numbers k, both roots are real and rational.

(D) For all pure imaginary numbers k, both roots are real and irrational.

(E) For all complex numbers k, neither root is real.

21. Let $a_1, a_2, \ldots, a_k$ be a finite arithmetic sequence with

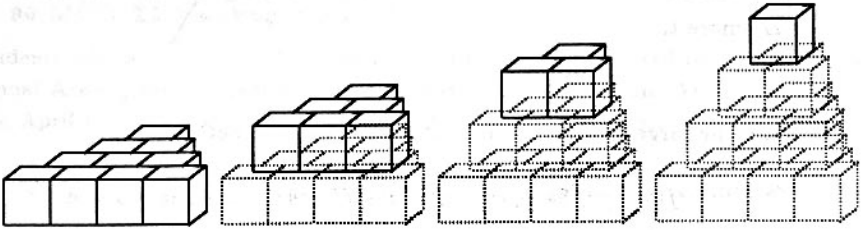
$$a_4 + a_7 + a_{10} = 17 \quad \text{and}$$

$$a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} = 77.$$

If $a_k = 13$, then $k =$

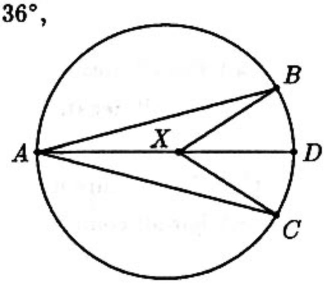
(A) 16 (B) 18 (C) 20 (D) 22 (E) 24

22. Twenty cubical blocks are arranged as shown. First, 10 are arranged in a triangular pattern; then a layer of 6, arranged in a triangular pattern, is centered on the 10; then a layer of 3, arranged in a triangular pattern, is centered on the 6; and finally one block is centered on top of the third layer. The blocks in the bottom layer are numbered 1 through 10 in some order. Each block in layers 2, 3 and 4 is assigned the number which is the sum of the numbers assigned to the three blocks on which it rests. Find the smallest possible number which could be assigned to the top block.



(A) 55 (B) 83 (C) 114 (D) 137 (E) 144

23. Points A, B, C and D are on a circle of diameter 1, and X is on diameter $\overline{AD}$. If $BX = CX$ and $3\angle BAC = \angle BXC = 36°$, then $AX =$



(A) $\cos 6° \cos 12° \sec 18°$

(B) $\cos 6° \sin 12° \csc 18°$

(C) $\cos 6° \sin 12° \sec 18°$

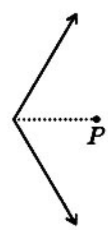
(D) $\sin 6° \sin 12° \csc 18°$

(E) $\sin 6° \sin 12° \sec 18°$

24. A box contains 3 shiny pennies and 4 dull pennies. One by one, pennies are drawn at random from the box and not replaced. If the probability is a/b that it will take more than four draws until the third shiny penny appears and a/b is in lowest terms, then $a + b =$

(A) 11 (B) 20 (C) 35 (D) 58 (E) 66

25. Let S be the set of points on the rays forming the sides of a 120° angle, and let P be a fixed point inside the angle on the angle bisector. Consider all distinct equilateral triangles PQR with Q and R in S. (Points Q and R may be on the same ray, and switching the names of Q and R does not create a distinct triangle.) There are



(A) exactly 2 such triangles.

(B) exactly 3 such triangles.

(C) exactly 7 such triangles.

(D) exactly 15 such triangles.

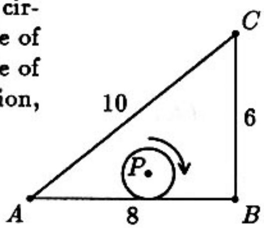
(E) more than 15 such triangles.

26. Find the largest positive value attained by the function

$$f(x) = \sqrt{8x - x^2} - \sqrt{14x - x^2 - 48}, \quad x \text{ a real number.}$$

(A) $\sqrt{7} - 1$ (B) 3 (C) $2\sqrt{3}$ (D) 4 (E) $\sqrt{55} - \sqrt{5}$

27. The sides of $\triangle ABC$ have lengths 6, 8 and 10. A circle with center P and radius 1 rolls around the inside of $\triangle ABC$, always remaining tangent to at least one side of the triangle. When P first returns to its original position, through what distance has P traveled?



(A) 10 (B) 12 (C) 14
(D) 15 (E) 17

28. How many triangles with positive area are there whose vertices are points in the xy-plane whose coordinates are integers (x, y) satisfying $1 \le x \le 4$ and $1 \le y \le 4$?

(A) 496 (B) 500 (C) 512 (D) 516 (E) 560

29. Which of the following sets could NOT be the lengths of the external diagonals of a right rectangular prism [a "box"]? (An *external diagonal* is a diagonal of one of the rectangular faces of the box.)

(A) $\{4, 5, 6\}$ (B) $\{4, 5, 7\}$ (C) $\{4, 6, 7\}$
(D) $\{5, 6, 7\}$ (E) $\{5, 7, 8\}$

30. Given $0 \le x_0 < 1$, let

$$x_n = \begin{cases} 2x_{n-1} & \text{if } 2x_{n-1} < 1 \\ 2x_{n-1} - 1 & \text{if } 2x_{n-1} \ge 1 \end{cases}$$

for all integers $n > 0$. For how many x_0 is it true that $x_0 = x_5$?

(A) 0 (B) 1 (C) 5 (D) 31 (E) infinitely many

SOLUTIONS

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WRITE TO US!

Questions and comments about the problems and solutions for this AHSME (but **not** requests for the Solutions Pamphlet) should be addressed to:

Prof Leo J Schneider, AHSME Chairman
Department of Mathematics and Computer Science
John Carroll University, University Heights, OH 44118-4581 USA

Comments about administrative arrangements and orders for any publications listed below should be addressed to:

Prof Walter E Mientka, AMC Executive Director
Department of Mathematics and Statistics
University of Nebraska, Lincoln, NE 68588-0658 USA

1993 AIME

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PUBLICATIONS

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Examinations: Each price is for one copy of an exam and its solutions for one year. Specify the years you want and how many copies of each. All prices effective to July 1, 1993.

- **AJHSME** (Junior High Exam), 1985-1992, $1 per copy per year.
- **AHSME** 1972-93, $1 per copy per year.
- **AIME** 1983-92, $2 per copy per year.
- USA and International Mathematical **Olympiads** (together), 1976-92, $4 per copy per year.
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- International Mathematical Olympiads, 1978-85, $11.00.

Journal

- The ARBELOS (short articles and challenging problems); recommended especially for AIME and USAMO qualifiers. Five volumes plus a Geometry volume, $7.00 each.

AMERICAN MATHEMATICS COMPETITIONS

45th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)
THURSDAY, FEBRUARY 24, 1994

Sponsored by

Mathematical Association of America
Society of Actuaries   Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society   American Statistical Association
American Mathematical Association of Two-Year Colleges
American Mathematical Society

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a thirty question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. SCORING RULES: You will receive 5 points for each correct answer, 2 points for each problem left unanswered, and 0 points for each incorrect answer.
4. No aids are permitted other than scratch paper, graph paper, ruler, compass, protractor and calculators that are accepted for use on the SAT. No problems on the test will *require* the use of a calculator.
5. Figures are not necessarily drawn to scale.
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1. $4^4 \cdot 9^4 \cdot 4^9 \cdot 9^9 =$

(A) 13^{13} **(B)** 13^{36} **(C)** 36^{13} **(D)** 36^{36} **(E)** 1296^{26}

2. A large rectangle is partitioned into four rectangles by two segments parallel to its sides. The areas of three of the resulting rectangles are shown. What is the area of the fourth rectangle?

6	14
?	35

(A) 10 **(B)** 15 **(C)** 20 **(D)** 21 **(E)** 25

3. How many of the following are equal to $x^x + x^x$ for all $x > 0$?

I: $2x^x$ **II**: x^{2x} **III**: $(2x)^x$ **IV**: $(2x)^{2x}$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

4. In the xy-plane, the segment with endpoints $(-5,0)$ and $(25,0)$ is the diameter of a circle. If the point $(x,15)$ is on the circle, then $x =$

(A) 10 **(B)** 12.5 **(C)** 15 **(D)** 17.5 **(E)** 20

5. Pat intended to multiply a number by 6 but instead divided by 6. Pat then meant to add 14 but instead subtracted 14. After these mistakes, the result was 16. If the correct operations had been used, the value produced would have been

(A) less than 400 **(B)** between 400 and 600 **(C)** between 600 and 800
(D) between 800 and 1000 **(E)** greater than 1000

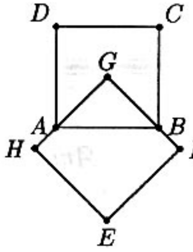
6. In the sequence

$$\ldots, a, b, c, d, 0, 1, 1, 2, 3, 5, 8, \ldots$$

each term is the sum of the two terms to its left. Find a.

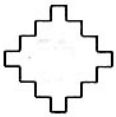
(A) -3 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 3

7. Squares $ABCD$ and $EFGH$ are congruent, $AB = 10$, and G is the center of square $ABCD$. The area of the region in the plane covered by these squares is



(A) 75 **(B)** 100 **(C)** 125
(D) 150 **(E)** 175

8. In the polygon shown, each side is perpendicular to its adjacent sides, and all 28 of the sides are congruent. The perimeter of the polygon is 56. The area of the region bounded by the polygon is



(A) 84 **(B)** 96 **(C)** 100 **(D)** 112 **(E)** 196

9. If $\angle A$ is four times $\angle B$, and the complement of $\angle B$ is four times the complement of $\angle A$, then $\angle B =$

(A) $10°$ **(B)** $12°$ **(C)** $15°$ **(D)** $18°$ **(E)** $22.5°$

10. For distinct real numbers x and y, let $M(x, y)$ be the larger of x and y and let $m(x, y)$ be the smaller of x and y. If $a < b < c < d < e$, then

$$M(M(a, m(b, c)), m(d, m(a, e))) =$$

(A) a **(B)** b **(C)** c **(D)** d **(E)** e

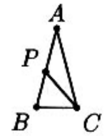
11. Three cubes of volume 1, 8 and 27 are glued together at their faces. The smallest possible surface area of the resulting configuration is

(A) 36 **(B)** 56 **(C)** 70 **(D)** 72 **(E)** 74

12. If $i^2 = -1$, then $\left(i - i^{-1}\right)^{-1} =$

(A) 0 **(B)** $-2i$ **(C)** $2i$ **(D)** $-i/2$ **(E)** $i/2$

13. In triangle ABC, $AB = AC$. If there is a point P strictly between A and B such that $AP = PC = CB$, then $\angle A =$



(A) 30° (B) 36° (C) 48° (D) 60° (E) 72°

14. Find the sum of the arithmetic series

$$20 + 20\frac{1}{5} + 20\frac{2}{5} + \cdots + 40.$$

(A) 3000 (B) 3030 (C) 3150 (D) 4100 (E) 6000

15. For how many n in $\{1, 2, 3, \ldots, 100\}$ is the tens digit of n^2 odd?

(A) 10 (B) 20 (C) 30 (D) 40 (E) 50

16. Some marbles in a bag are red and the rest are blue. If one red marble is removed, then one-seventh of the remaining marbles are red. If two blue marbles are removed instead of one red, then one-fifth of the remaining marbles are red. How many marbles were in the bag originally?

(A) 8 (B) 22 (C) 36 (D) 57 (E) 71

17. An 8 by $2\sqrt{2}$ rectangle has the same center as a circle of radius 2. The area of the region common to both the rectangle and the circle is

(A) 2π (B) $2\pi + 2$ (C) $4\pi - 4$ (D) $2\pi + 4$ (E) $4\pi - 2$

18. Triangle ABC is inscribed in a circle, and $\angle B = \angle C = 4\angle A$. If B and C are adjacent vertices of a regular polygon of n sides inscribed in this circle, then $n =$

(A) 5 (B) 7 (C) 9 (D) 15 (E) 18

19. Label one disk "**1**", two disks "**2**", three disks "**3**", ..., fifty disks "**50**". Put these $1+2+3+\cdots+50 = 1275$ labeled disks in a box. Disks are then drawn from the box at random without replacement. The minimum number of disks that must be drawn to guarantee drawing at least ten disks with the same label is

(A) 10 **(B)** 51 **(C)** 415 **(D)** 451 **(E)** 501

20. Suppose x, y, z is a geometric sequence with common ratio r and $x \neq y$. If $x, 2y, 3z$ is an arithmetic sequence, then r is

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** 2 **(E)** 4

21. Find the number of counterexamples to the statement:

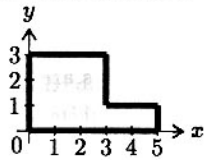
"If N is an odd positive integer the sum of whose digits is 4 *and none of whose digits is* 0, *then N is prime."*

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

22. Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students. In how many ways can Professors Alpha, Beta and Gamma choose their chairs?

(A) 12 **(B)** 36 **(C)** 60 **(D)** 84 **(E)** 630

23. In the xy-plane, consider the L-shaped region bounded by horizontal and vertical segments with vertices at $(0,0)$, $(0,3)$, $(3,3)$, $(3,1)$, $(5,1)$ and $(5,0)$. The slope of the line through the origin that divides the area of this region exactly in half is



(A) $\frac{2}{7}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{7}{9}$

24. A sample consisting of five observations has an arithmetic mean of 10 and a median of 12. The smallest value that the range (largest observation minus smallest) can assume for such a sample is

(A) 2 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 10

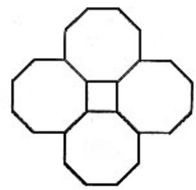
25. If x and y are non-zero real numbers such that

$$|x| + y = 3 \qquad \text{and} \qquad |x|\,y + x^3 = 0,$$

then the integer nearest to $x - y$ is

(A) -3 (B) -1 (C) 2 (D) 3 (E) 5

26. A regular polygon of m sides is exactly enclosed (no overlaps, no gaps) by m regular polygons of n sides each. (Shown here for $m = 4$, $n = 8$.) If $m = 10$, what is the value of n?



(A) 5 (B) 6 (C) 14 (D) 20 (E) 26

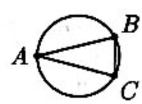
27. A bag of popping corn contains $2/3$ white kernels and $1/3$ yellow kernels. Only $1/2$ of the white kernels will pop, whereas $2/3$ of the yellow ones will pop. A kernel is selected at random from the bag, and pops when placed in the popper. What is the probability that the kernel selected was white?

(A) $\frac{1}{2}$ (B) $\frac{5}{9}$ (C) $\frac{4}{7}$ (D) $\frac{3}{5}$ (E) $\frac{2}{3}$

28. In the xy-plane, how many lines whose x-intercept is a positive prime number and whose y-intercept is a positive integer pass through the point (4,3)?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

29. Points A, B and C on a circle of radius r are situated so that $AB = AC$, $AB > r$, and the length of minor arc BC is r. If angles are measured in radians, then $AB/BC =$



(A) $\frac{1}{2}\csc\frac{1}{4}$ (B) $2\cos\frac{1}{2}$ (C) $4\sin\frac{1}{2}$ (D) $\csc\frac{1}{2}$ (E) $2\sec\frac{1}{2}$

30. When n standard 6-sided dice are rolled, the probability of obtaining a sum of 1994 is greater than zero and is the same as the probability of obtaining a sum of S. The smallest possible value of S is

(A) 333 (B) 335 (C) 337 (D) 339 (E) 341

SOLUTIONS

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Prof Leo J Schneider, AHSME Associate Chair
Department of Mathematics and Computer Science
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Prof Walter E Mientka, AMC Executive Director
Department of Mathematics and Statistics, University of Nebraska
Lincoln, NE 68588-0658 USA; Phone: 402-472-2257; Fax: 402-472-6087

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The Arbelos

- Short articles and challenging problems recommended especially for AIME and USAMO qualifiers. Six volumes, $7.00 each.

1994
American High School Mathematics Examination

(AHSME)

DO NOT OPEN UNTIL
THURSDAY, FEBRUARY 24, 1994

1. All information (Rules and Instructions) needed to administer the AHSME is contained in the AHSME TEACHERS' MANUAL, which is outside of this package. **PLEASE READ THE MANUAL BEFORE FEBRUARY 24.** Nothing is needed from inside this package until February 24.

2. Your PRINCIPAL or VICE PRINCIPAL must sign the AHSME Certification Form A found in the Teachers' Manual.

3. The Answer Forms must be mailed by First Class mail to Dr. Mientka no later than 24 hours following the examination.

AMERICAN MATHEMATICS COMPETITIONS

46th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)
THURSDAY, FEBRUARY 16, 1995

Sponsored by

Mathematical Association of America
Society of Actuaries   Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society   American Statistical Association
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Society of Pension Actuaries

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1. Kim earned scores of 87, 83 and 88 on her first three mathematics examinations. If Kim receives a 90 on the fourth exam, then her average will

(A) remain the same **(B)** increase by 1 **(C)** increase by 2
(D) increase by 3 **(E)** increase by 4

2. If $\sqrt{2+\sqrt{x}} = 3$, then $x =$

(A) 1 **(B)** $\sqrt{7}$ **(C)** 7 **(D)** 49 **(E)** 121

3. The total in-store price for an appliance is $99.99. A television commercial advertises the same product for three easy payments of $29.98 and a one-time shipping & handling charge of $9.98. How much is saved by buying the appliance from the television advertiser?

(A) 6 cents **(B)** 7 cents **(C)** 8 cents **(D)** 9 cents **(E)** 10 cents

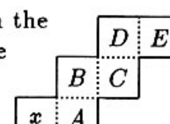
4. If M is 30% of Q, Q is 20% of P, and N is 50% of P, then $\frac{M}{N} =$

(A) $\frac{3}{250}$ **(B)** $\frac{3}{25}$ **(C)** 1 **(D)** $\frac{6}{5}$ **(E)** $\frac{4}{3}$

5. A rectangular field is 300 feet wide and 400 feet long. Random sampling indicates that there are, on the average, three ants per square inch throughout the field. [12 inches = 1 foot.] Of the following, the number that most closely approximates the number of ants in the field is

(A) 500 thousand **(B)** 5 million **(C)** 50 million **(D)** 500 million
(E) 5 billion

6. The figure shown can be folded into the shape of a cube. In the resulting cube, which of the lettered faces is opposite the face marked x?

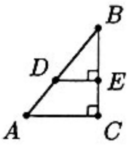


(A) A **(B)** B **(C)** C **(D)** D **(E)** E

7. The radius of Earth at the equator is approximately 4000 miles. Suppose a jet flies once around Earth at a speed of 500 miles per hour relative to Earth. If the flight path is a negligible height above the equator, then, among the following choices, the best estimate of the number of hours of flight is

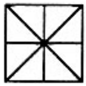
(A) 8 **(B)** 25 **(C)** 50 **(D)** 75 **(E)** 100

8. In triangle ABC, $\angle C = 90°$, $AC = 6$ and $BC = 8$. Points D and E are on $\overline{AB}$ and $\overline{BC}$, respectively, and $\angle BED = 90°$. If $DE = 4$, then $BD =$



(A) 5 **(B)** $\frac{16}{3}$ **(C)** $\frac{20}{3}$ **(D)** $\frac{15}{2}$ **(E)** 8

9. Consider the figure consisting of a square, its diagonals, and the segments joining the midpoints of opposite sides. The total number of triangles of any size in the figure is



(A) 10 **(B)** 12 **(C)** 14 **(D)** 16 **(E)** 18

10. The area of the triangle bounded by the lines $y = x$, $y = -x$ and $y = 6$ is

(A) 12 **(B)** $12\sqrt{2}$ **(C)** 24 **(D)** $24\sqrt{2}$ **(E)** 36

11. How many four-digit numbers, $N = \underline{a}\,\underline{b}\,\underline{c}\,\underline{d}_{\text{base ten}}$, satisfy all three of the following conditions?

(i) $4,000 \le N < 6,000$; (ii) N is a multiple of 5; (iii) $3 \le b < c \le 6$.

(A) 10 **(B)** 18 **(C)** 24 **(D)** 36 **(E)** 48

12. Let f be a linear function with the properties that $f(1) \le f(2)$, $f(3) \ge f(4)$, and $f(5) = 5$. Which of the following statements is true?

(A) $f(0) < 0$ **(B)** $f(0) = 0$ **(C)** $f(1) < f(0) < f(-1)$

(D) $f(0) = 5$ **(E)** $f(0) > 5$

13. The addition below is incorrect. The display can be made correct by changing one digit d, wherever it occurs, to another digit e. Find the sum of d and e.

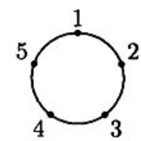
$$\begin{array}{r} 7\ 4\ 2\ 5\ 8\ 6 \\ +\ 8\ 2\ 9\ 4\ 3\ 0 \\ \hline 1\ 2\ 1\ 2\ 0\ 1\ 6 \end{array}$$

(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** more than 10

14. If $f(x) = ax^4 - bx^2 + x + 5$ and $f(-3) = 2$, then $f(3) =$

(A) -5 **(B)** -2 **(C)** 1 **(D)** 3 **(E)** 8

15. Five points on a circle are numbered 1, 2, 3, 4, and 5 in clockwise order. A bug jumps in a clockwise direction from one point to another around the circle; if it is on an odd-numbered point, it moves one point, and if it is on an even-numbered point, it moves two points. If the bug begins on point 5, after 1995 jumps it will be on point



(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

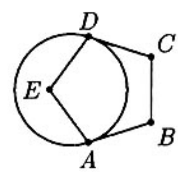
16. Anita attends a baseball game in Atlanta and estimates that there are 50,000 fans in attendance. Bob attends a baseball game in Boston and estimates that there are 60,000 in attendance. A league official who knows the actual numbers attending the two games notes that:

(i) The actual attendance in Atlanta is within 10% of Anita's estimate.
(ii) Bob's estimate is within 10% of the actual attendance in Boston.

To the nearest 1,000, the largest possible difference between the numbers attending the two games is

(A) 10,000 **(B)** 11,000 **(C)** 20,000 **(D)** 21,000 **(E)** 22,000

17. Given regular pentagon $ABCDE$, a circle can be drawn that is tangent to $\overline{DC}$ at D and to $\overline{AB}$ at A. The number of degrees in minor arc AD is

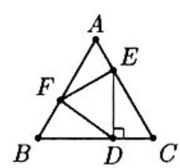


(A) 72 **(B)** 108 **(C)** 120
(D) 135 **(E)** 144

18. Two rays with common endpoint O form a 30° angle. Point A lies on one ray, point B on the other ray, and $AB = 1$. The maximum possible length of OB is

(A) 1 **(B)** $\frac{1+\sqrt{3}}{\sqrt{2}}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** $\frac{4}{\sqrt{3}}$

19. Equilateral triangle DEF is inscribed in equilateral triangle ABC as shown with $\overline{DE} \perp \overline{BC}$. The ratio of the area of $\triangle DEF$ to the area of $\triangle ABC$ is



(A) $\frac{1}{6}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{2}{5}$ **(E)** $\frac{1}{2}$

20. If a, b and c are three (not necessarily different) numbers chosen randomly and with replacement from the set $\{1,2,3,4,5\}$, the probability that $ab+c$ is even is

(A) $\frac{2}{5}$ **(B)** $\frac{59}{125}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{64}{125}$ **(E)** $\frac{3}{5}$

21. Two nonadjacent vertices of a rectangle are $(4,3)$ and $(-4,-3)$, and the coordinates of the other two vertices are integers. The number of such rectangles is

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

22. A pentagon is formed by cutting a triangular corner from a rectangular piece of paper. The five sides of the pentagon have lengths 13, 19, 20, 25 and 31, although this is not necessarily their order around the pentagon. The area of the pentagon is

(A) 459 **(B)** 600 **(C)** 680 **(D)** 720 **(E)** 745

23. The sides of a triangle have lengths $11, 15$, and k, where k is an integer. For how many values of k is the triangle obtuse?

(A) 5 **(B)** 7 **(C)** 12 **(D)** 13 **(E)** 14

24. There exist positive integers A, B and C, with no common factor greater than 1, such that

$$A\log_{200}5 + B\log_{200}2 = C.$$

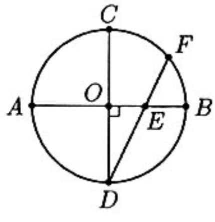
What is $A+B+C$?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

25. A list of five positive integers has mean 12 and range 18. The mode and median are both 8. How many different values are possible for the second largest element of the list?

(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

26. In the figure, $\overline{AB}$ and $\overline{CD}$ are diameters of the circle with center O, $\overline{AB}\perp\overline{CD}$, and chord $\overline{DF}$ intersects $\overline{AB}$ at E. If $DE=6$ and $EF=2$, then the area of the circle is



(A) 23π **(B)** $\frac{47}{2}\pi$ **(C)** 24π

(D) $\frac{49}{2}\pi$ **(E)** 25π

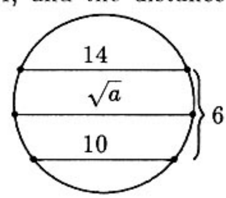
27. Consider the triangular array of numbers with $0, 1, 2, 3, \ldots$ along the sides and interior numbers obtained by adding the two adjacent numbers in the previous row. Rows 1 through 6 are shown.

$$\begin{array}{ccccccccccc} & & & & & 0 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 2 & & 2 & & 2 & & & \\ & & 3 & & 4 & & 4 & & 3 & & \\ & 4 & & 7 & & 8 & & 7 & & 4 & \\ 5 & & 11 & & 15 & & 15 & & 11 & & 5 \end{array}$$

Let $f(n)$ denote the sum of the numbers in row n. What is the remainder when $f(100)$ is divided by 100?

(A) 12 **(B)** 30 **(C)** 50 **(D)** 62 **(E)** 74

28. Two parallel chords in a circle have lengths 10 and 14, and the distance between them is 6. The chord parallel to these chords and midway between them is of length $\sqrt{a}$ where a is

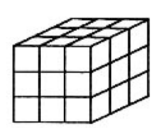


(A) 144 **(B)** 156 **(C)** 168
(D) 176 **(E)** 184

29. For how many three-element sets of positive integers $\{a, b, c\}$ is it true that $a \times b \times c = 2310$?

(A) 32 **(B)** 36 **(C)** 40 **(D)** 43 **(E)** 45

30. A large cube is formed by stacking 27 unit cubes. A plane is perpendicular to one of the internal diagonals of the large cube and bisects that diagonal. The number of unit cubes that the plane intersects is



(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

SOLUTIONS

Your School Examination Manager will be sent at least one copy of the 1995 AHSME Solutions Pamphlet. It is meant to be loaned or given to students (but not duplicated).

WRITE TO US!

Correspondence about the problems and solutions should be addressed to:

Prof Harold B. Reiter, AHSME Chair
Department of Mathematics
University of North Carolina at Charlotte, Charlotte, NC 28223 USA

Comments about administrative arrangements and orders for any publications listed below should be addressed to:

Prof Walter E Mientka, AMC Executive Director
Department of Mathematics and Statistics, University of Nebraska
Lincoln, NE 68588-0658 USA; Phone: 402-472-2257; Fax: 402-472-6087

1995 AIME

The AIME will be held on Thursday, March 23, 1995. It is a 15-question, 3-hour, integer-answer examination. You will be invited to participate only if you receive a score of 100 or above on this AHSME. (Your school must also agree to administer the AIME.) Top-scoring students on the AHSME/AIME will be selected to take the **USA Mathematical Olympiad** (USAMO) on Thursday, April 27, 1995; see the AHSME or AIME Teachers' Manual for the selection procedure. The best way to prepare for the AIME and USAMO is to study previous years of these exams. Copies may be ordered as indicated below.

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1995
American High School Mathematics Examination
(AHSME)

DO NOT OPEN UNTIL
THURSDAY, FEBRUARY 16, 1995

****Administration On An Earlier Date Will Disqualify Your School's Results****

1. All information (Rules and Instructions) needed to administer the AHSME is contained in the AHSME TEACHERS' MANUAL, which is outside of this package. **PLEASE READ THE MANUAL BEFORE FEBRUARY 16.** Nothing is needed from inside this package until February 16.

2. Your PRINCIPAL or VICE PRINCIPAL must sign the AHSME Certification Form A found in the Teachers' Manual.

3. The Answer Forms must be mailed by First Class mail to Dr. Mientka no later than 24 hours following the examination.

AMERICAN MATHEMATICS COMPETITIONS

47th ANNUAL AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION (AHSME)

THURSDAY, FEBRUARY 15, 1996

Sponsored by

Mathematical Association of America
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INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a thirty-question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. SCORING RULES: You will receive 5 points for each correct answer, 2 points for each problem left unanswered, and 0 points for each incorrect answer.
4. No aids are permitted other than scratch paper, graph paper, ruler, compass, protractor, erasers and calculators that are accepted for use on the SAT. No problems on the test will *require* the use of a calculator.
5. Figures are not necessarily drawn to scale.
6. The answers to the problems are to be marked on the AHSME ANSWER FORM with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
7. When your proctor gives the signal, begin working the problems. You will have **90 MINUTES** working time for the test.

Students who score 100 or above on this AHSME will be invited to take the 14th annual **American Invitational Mathematics Examination** (AIME) on Thursday, March 28, 1996. More details about the AIME and other information are on the back page of this test booklet.

The results of this AHSME are used to identify students with unusual mathematical ability. To assure that this purpose is served, the Committee on the American Mathematics Competitions reserves the right to re-examine students before deciding whether to grant official status to individual or team scores. Re-examination will be requested when, after an inquiry, there is reasonable basis to believe that scores have been obtained by extremely lucky guessing or dishonesty. Official status will not be granted if a student or school does not agree to a requested re-examination. The Committee also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

1. The addition below is incorrect. What is the largest digit that can be changed to make the addition correct?

$$\begin{array}{r} 641 \\ 852 \\ +\ 973 \\ \hline 2456 \end{array}$$

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

2. Each day Walter gets \$3 for doing his chores or \$5 for doing them exceptionally well. After 10 days of doing his chores daily, Walter has received a total of \$36. On how many days did Walter do them exceptionally well?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

3. $\dfrac{(3!)!}{3!} =$

(A) 1 **(B)** 2 **(C)** 6 **(D)** 40 **(E)** 120

4. Six numbers from a list of nine integers are 7, 8, 3, 5, 9, and 5. The largest possible value of the median of all nine numbers in this list is

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

5. Given that $0 < a < b < c < d$, which of the following is the largest?

(A) $\dfrac{a+b}{c+d}$ **(B)** $\dfrac{a+d}{b+c}$ **(C)** $\dfrac{b+c}{a+d}$ **(D)** $\dfrac{b+d}{a+c}$ **(E)** $\dfrac{c+d}{a+b}$

6. If $f(x) = x^{(x+1)}(x+2)^{(x+3)}$ then $f(0) + f(-1) + f(-2) + f(-3) =$

(A) $-8/9$ **(B)** 0 **(C)** $8/9$ **(D)** 1 **(E)** $10/9$

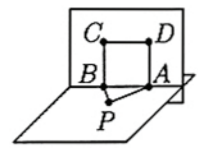
7. A father takes his twins and a younger child out to dinner on the twins' birthday. The restaurant charges \$4.95 for the father and \$0.45 for each year of a child's age, where age is defined as the age at the most recent birthday. If the bill is \$9.45, which of the following could be the age of the youngest child?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

8. If $3 = k \cdot 2^r$ and $15 = k \cdot 4^r$, then $r =$

(A) $-\log_2 5$ (B) $\log_5 2$ (C) $\log_{10} 5$ (D) $\log_2 5$ (E) $\frac{5}{2}$

9. Triangle PAB and square $ABCD$ are in perpendicular planes. Given that $PA = 3$, $PB = 4$, and $AB = 5$, what is PD?



(A) 5 (B) $\sqrt{34}$ (C) $\sqrt{41}$ (D) $2\sqrt{13}$ (E) 8

10. How many line segments have both their endpoints located at the vertices of a given cube?

(A) 12 (B) 15 (C) 24 (D) 28 (E) 56

11. Given a circle of radius 2, there are many line segments of length 2 that are tangent to the circle at their midpoints. Find the area of the region consisting of all such line segments.

(A) $\pi/4$ (B) $4 - \pi$ (C) $\pi/2$ (D) π (E) 2π

12. A function f from the integers to the integers is defined as follows:

$$f(n) = \begin{cases} n + 3 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$$

Suppose k is odd and $f(f(f(k))) = 27$. What is the sum of the digits of k?

(A) 3 (B) 6 (C) 9 (D) 12 (E) 15

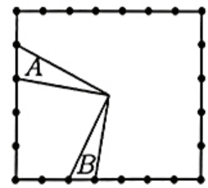
13. Sunny runs at a steady rate, and Moonbeam runs m times as fast, where m is a number greater than 1. If Moonbeam gives Sunny a head start of h meters, how many meters must Moonbeam run to overtake Sunny?

(A) hm (B) $\frac{h}{h+m}$ (C) $\frac{h}{m-1}$ (D) $\frac{hm}{m-1}$ (E) $\frac{h+m}{m-1}$

14. Let $E(n)$ denote the sum of the even digits of n. For example, $E(5681) = 6 + 8 = 14$. Find $E(1) + E(2) + E(3) + \cdots + E(100)$.

(A) 200 (B) 360 (C) 400 (D) 900 (E) 2250

15. Two opposite sides of a rectangle are each divided into n congruent segments, and the endpoints of one segment are joined to the center to form triangle A. The other sides are each divided into m congruent segments, and the endpoints of one of these segments are joined to the center to form triangle B. [See figure for $n = 5$, $m = 7$.] What is the ratio of the area of triangle A to the area of triangle B?

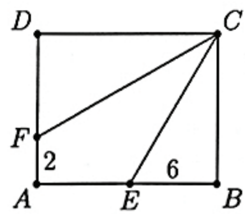


(A) 1 **(B)** m/n **(C)** n/m **(D)** $2m/n$ **(E)** $2n/m$

16. A fair standard six-sided die is tossed three times. Given that the sum of the first two tosses equals the third, what is the probability that at least one "2" is tossed?

(A) $\frac{1}{6}$ **(B)** $\frac{91}{216}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{8}{15}$ **(E)** $\frac{7}{12}$

17. In rectangle $ABCD$, angle C is trisected by $\overline{CF}$ and $\overline{CE}$, where E is on $\overline{AB}$, F is on $\overline{AD}$, $BE = 6$, and $AF = 2$. Which of the following is closest to the area of the rectangle $ABCD$?

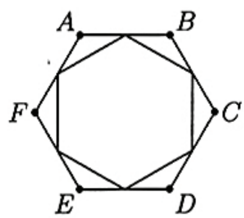


(A) 110 **(B)** 120 **(C)** 130 **(D)** 140 **(E)** 150

18. A circle of radius 2 has center at $(2, 0)$. A circle of radius 1 has center at $(5, 0)$. A line is tangent to the two circles at points in the first quadrant. Which of the following is closest to the y-intercept of the line?

(A) $\sqrt{2}/4$ **(B)** $8/3$ **(C)** $1 + \sqrt{3}$ **(D)** $2\sqrt{2}$ **(E)** 3

19. The midpoints of the sides of a regular hexagon $ABCDEF$ are joined to form a smaller hexagon. What fraction of the area of $ABCDEF$ is enclosed by the smaller hexagon?



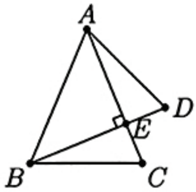
(A) $\frac{1}{2}$ **(B)** $\frac{\sqrt{3}}{3}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{\sqrt{3}}{2}$

20. In the xy-plane, what is the length of the shortest path from $(0,0)$ to $(12,16)$ that does not go inside the circle $(x-6)^2+(y-8)^2=25$?

(A) $10\sqrt{3}$ (B) $10\sqrt{5}$ (C) $10\sqrt{3}+\frac{5\pi}{3}$ (D) $40\frac{\sqrt{3}}{3}$ (E) $10+5\pi$

21. Triangles ABC and ABD are isosceles with $AB = AC = BD$, and $\overline{BD}$ intersects $\overline{AC}$ at E. If $\overline{BD} \perp \overline{AC}$, then $\angle C + \angle D$ is

(A) $115°$ (B) $120°$ (C) $130°$ (D) $135°$
(E) not uniquely determined



22. Four distinct points, A, B, C, and D, are to be selected from 1996 points evenly spaced around a circle. All quadruples are equally likely to be chosen. What is the probability that the chord $\overline{AB}$ intersects the chord $\overline{CD}$?

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) $\frac{3}{4}$

23. The sum of the lengths of the twelve edges of a rectangular box is 140, and the distance from one corner of the box to the farthest corner is 21. The total surface area of the box is

(A) 776 (B) 784 (C) 798 (D) 800 (E) 812

24. The sequence

$$1,2,1,2,2,1,2,2,2,1,2,2,2,2,1,2,2,2,2,2,1,2,\ldots$$

consists of 1's separated by blocks of 2's with n 2's in the n^{th} block. The sum of the first 1234 terms of this sequence is

(A) 1996 (B) 2419 (C) 2429 (D) 2439 (E) 2449

25. Given that $x^2+y^2=14x+6y+6$, what is the largest possible value that $3x+4y$ can have?

(A) 72 (B) 73 (C) 74 (D) 75 (E) 76

26. An urn contains marbles of four colors: red, white, blue, and green. When four marbles are drawn without replacement, the following events are equally likely:

(a) the selection of four red marbles;

(b) the selection of one white and three red marbles;

(c) the selection of one white, one blue, and two red marbles; and

(d) the selection of one marble of each color.

What is the smallest number of marbles satisfying the given condition?

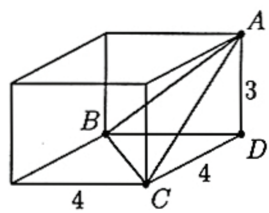
(A) 19 **(B)** 21 **(C)** 46 **(D)** 69 **(E)** more than 69

27. Consider two solid spherical balls, one centered at $(0,0,\frac{21}{2})$ with radius 6, and the other centered at $(0,0,1)$ with radius $\frac{9}{2}$. How many points (x,y,z) with only integer coordinates (lattice points) are there in the intersection of the balls?

(A) 7 **(B)** 9 **(C)** 11 **(D)** 13 **(E)** 15

28. On a $4 \times 4 \times 3$ rectangular parallelepiped, vertices A, B, and C are adjacent to vertex D. The perpendicular distance from D to the plane containing A, B, and C is closest to

(A) 1.6 **(B)** 1.9 **(C)** 2.1 **(D)** 2.7 **(E)** 2.9



29. If n is a positive integer such that $2n$ has 28 positive divisors and $3n$ has 30 positive divisors, then how many positive divisors does $6n$ have?

(A) 32 **(B)** 34 **(C)** 35 **(D)** 36 **(E)** 38

30. A hexagon inscribed in a circle has three consecutive sides each of length 3 and three consecutive sides each of length 5. The chord of the circle that divides the hexagon into two trapezoids, one with three sides each of length 3 and the other with three sides each of length 5, has length equal to m/n, where m and n are relatively prime positive integers. Find $m+n$.

(A) 309 **(B)** 349 **(C)** 369 **(D)** 389 **(E)** 409

SOLUTIONS

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Department of Mathematics
University of North Carolina at Charlotte, Charlotte, NC 28223 USA

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Prof Walter E Mientka, AMC Executive Director
Department of Mathematics and Statistics, University of Nebraska
Lincoln, NE 68588-0658 USA; Phone: 402-472-2257; Fax: 402-472-6087

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1996
American High School Mathematics Examination
(AHSME)

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AMERICAN MATHEMATICS COMPETITIONS

48th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)
THURSDAY, FEBRUARY 13, 1997

Sponsored by

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1. If a and b are digits for which

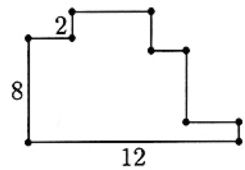
$$\begin{array}{r} 2\ a \\ \times\ b\ 3 \\ \hline 6\ 9 \\ 9\ 2\ \ \\ \hline 9\ 8\ 9 \end{array}$$

then a + b =

(A) 3 (B) 4 (C) 7 (D) 9 (E) 12

2. The adjacent sides of the decagon shown meet at right angles. What is its perimeter?

(A) 22 (B) 32 (C) 34 (D) 44 (E) 50



3. If x, y, and z are real numbers such that

$$(x-3)^2 + (y-4)^2 + (z-5)^2 = 0,$$

then $x + y + z =$

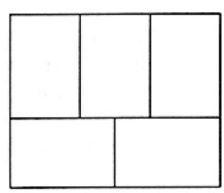
(A) -12 (B) 0 (C) 8 (D) 12 (E) 50

4. If a is 50% larger than c, and b is 25% larger than c, then a is what percent larger than b?

(A) 20% (B) 25% (C) 50% (D) 100% (E) 200%

5. A rectangle with perimeter 176 is divided into five congruent rectangles as shown in the diagram. What is the perimeter of one of the five congruent rectangles?

(A) 35.2 (B) 76 (C) 80 (D) 84 (E) 86



6. Consider the sequence

$$1, -2, 3, -4, 5, -6, \ldots,$$

whose n^{th} term is $(-1)^{n+1} \cdot n$. What is the average of the first 200 terms of the sequence?

(A) -1 (B) -0.5 (C) 0 (D) 0.5 (E) 1

7. The sum of seven integers is -1. What is the maximum number of the seven integers that can be larger than 13?

(A) 1 (B) 4 (C) 5 (D) 6 (E) 7

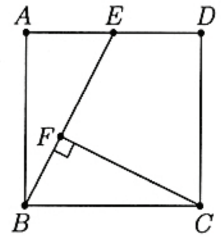
8. Mientka Publishing Company prices its best seller *Where's Walter?* as follows:

$$C(n) = \begin{cases} 12n, & \text{if } 1 \le n \le 24, \\ 11n, & \text{if } 25 \le n \le 48, \\ 10n, & \text{if } 49 \le n, \end{cases}$$

where n is the number of books ordered, and $C(n)$ is the cost in dollars of n books. Notice that 25 books cost less than 24 books. For how many values of n is it cheaper to buy more than n books than to buy exactly n books?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

9. In the figure, $ABCD$ is a 2×2 square, E is the midpoint of $\overline{AD}$, and F is on $\overline{BE}$. If $\overline{CF}$ is perpendicular to $\overline{BE}$, then the area of quadrilateral $CDEF$ is



(A) 2 **(B)** $3 - \frac{\sqrt{3}}{2}$ **(C)** $\frac{11}{5}$ **(D)** $\sqrt{5}$ **(E)** $\frac{9}{4}$

10. Two six-sided dice are fair in the sense that each face is equally likely to turn up. However, one of the dice has the 4 replaced by 3 and the other die has the 3 replaced by 4. When these dice are rolled, what is the probability that the sum is an odd number?

(A) $\frac{1}{3}$ **(B)** $\frac{4}{9}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{9}$ **(E)** $\frac{11}{18}$

11. In the sixth, seventh, eighth, and ninth basketball games of the season, a player scored $23, 14, 11$, and 20 points, respectively. Her points-per-game average was higher after nine games than it was after the first five games. If her average after ten games was greater than 18, what is the least number of points she could have scored in the tenth game?

(A) 26 **(B)** 27 **(C)** 28 **(D)** 29 **(E)** 30

12. If m and b are real numbers and $mb > 0$, then the line whose equation is $y = mx + b$ <u>cannot</u> contain the point

(A) $(0, 1997)$ **(B)** $(0, -1997)$ **(C)** $(19, 97)$ **(D)** $(19, -97)$ **(E)** $(1997, 0)$

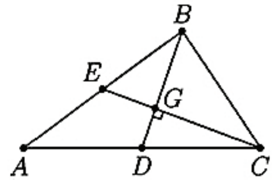
13. How many two-digit positive integers N have the property that the sum of N and the number obtained by reversing the order of the digits of N is a perfect square?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

14. The number of geese in a flock increases so that the difference between the populations in year $n+2$ and year n is directly proportional to the population in year $n+1$. If the populations in the years 1994, 1995, and 1997 were 39, 60, and 123, respectively, then the population in 1996 was

(A) 81 **(B)** 84 **(C)** 87 **(D)** 90 **(E)** 102

15. Medians BD and CE of triangle ABC are perpendicular, $BD = 8$, and $CE = 12$. The area of triangle ABC is



(A) 24 **(B)** 32 **(C)** 48 **(D)** 64 **(E)** 96

16. The three row sums and the three column sums of the array

$$\begin{bmatrix} 4 & 9 & 2 \\ 8 & 1 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$

are the same. What is the least number of entries that must be altered to make all six sums different from one another?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

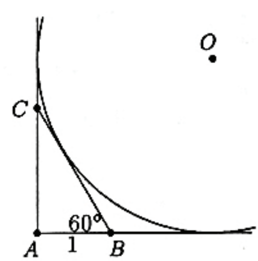
17. A line $x = k$ intersects the graph of $y = \log_5 x$ and the graph of $y = \log_5(x+4)$. The distance between the points of intersection is 0.5. Given that $k = a + \sqrt{b}$, where a and b are integers, what is $a + b$?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

18. A list of integers has mode 32 and mean 22. The smallest number in the list is 10. The median m of the list is a member of the list. If the list member m were replaced by $m + 10$, the mean and median of the new list would be 24 and $m + 10$, respectively. If m were instead replaced by $m - 8$, the median of the new list would be $m - 4$. What is m?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

19. A circle with center O is tangent to the coordinate axes and to the hypotenuse of the 30°-60°-90° triangle ABC as shown, where $AB = 1$. To the nearest hundredth, what is the radius of the circle?



(A) 2.18 **(B)** 2.24 **(C)** 2.31
(D) 2.37 **(E)** 2.41

20. Which one of the following integers can be expressed as the sum of 100 consecutive positive integers?

(A) 1,627,384,950 **(B)** 2,345,678,910 **(C)** 3,579,111,300

(D) 4,692,581,470 **(E)** 5,815,937,260

21. For any positive integer n, let

$$f(n) = \begin{cases} \log_8 n, & \text{if } \log_8 n \text{ is rational,} \\ 0, & \text{otherwise.} \end{cases}$$

What is $\sum_{n=1}^{1997} f(n)$?

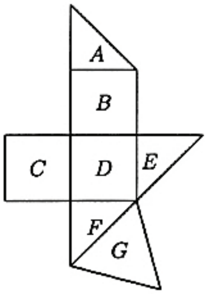
(A) $\log_8 2047$ **(B)** 6 **(C)** $\frac{55}{3}$ **(D)** $\frac{58}{3}$ **(E)** 585

22. Ashley, Betty, Carlos, Dick, and Elgin went shopping. Each had a whole number of dollars to spend, and together they had \$56. The absolute difference between the amounts Ashley and Betty had to spend was \$19. The absolute difference between the amounts Betty and Carlos had was \$7, between Carlos and Dick was \$5, between Dick and Elgin was \$4, and between Elgin and Ashley was \$11. How much did Elgin have?

(A) \$6 **(B)** \$7 **(C)** \$8 **(D)** \$9 **(E)** \$10

23. In the figure, polygons A, E, and F are isosceles right triangles; B, C, and D are squares with sides of length 1; and G is an equilateral triangle. The figure can be folded along its edges to form a polyhedron having the polygons as faces. The volume of this polyhedron is

(A) 1/2 **(B)** 2/3 **(C)** 3/4 **(D)** 5/6 **(E)** 4/3



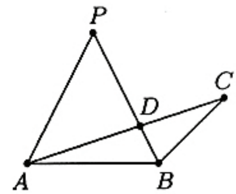
24. A *rising* number, such as 34689, is a positive integer each digit of which is larger than each of the digits to its left. There are $\binom{9}{5} = 126$ five-digit rising numbers. When these numbers are arranged from smallest to largest, the 97^{th} number in the list does not contain the digit

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

25. Let $ABCD$ be a parallelogram and let $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, $\overrightarrow{CC'}$, and $\overrightarrow{DD'}$ be parallel rays in space on the same side of the plane determined by $ABCD$. If $AA' = 10$, $BB' = 8$, $CC' = 18$, $DD' = 22$, and M and N are the midpoints of $\overline{A'C'}$ and $\overline{B'D'}$, respectively, then $MN =$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

26. Triangle ABC and point P in the same plane are given. Point P is equidistant from A and B, angle APB is twice angle ACB, and $\overline{AC}$ intersects $\overline{BP}$ at point D. If $PB = 3$ and $PD = 2$, then $AD \cdot CD =$



(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

27. Consider those functions f that satisfy $f(x+4) + f(x-4) = f(x)$ for all real x. Any such function is periodic, and there is a least common positive period p for all of them. Find p.

(A) 8 **(B)** 12 **(C)** 16 **(D)** 24 **(E)** 32

28. How many ordered triples of integers (a, b, c) satisfy

$$|a+b| + c = 19 \text{ and } ab + |c| = 97?$$

(A) 0 **(B)** 4 **(C)** 6 **(D)** 10 **(E)** 12

29. Call a positive real number *special* if it has a decimal representation that consists entirely of digits 0 and 7. For example, $\frac{700}{99} = 7.\overline{07} = 7.070707\ldots$ and 77.007 are special numbers. What is the smallest n such that 1 can be written as a sum of n special numbers?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10

(E) 1 cannot be represented as a sum of finitely many special numbers

30. For positive integers n, denote by $D(n)$ the number of pairs of different adjacent digits in the binary (base two) representation of n. For example, $D(3) = D(11_2) = 0$, $D(21) = D(10101_2) = 4$, and $D(97) = D(1100001_2) = 2$. For how many positive integers n less than or equal to 97 does $D(n) = 2$?

(A) 16 **(B)** 20 **(C)** 26 **(D)** 30 **(E)** 35

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(AHSME)
Tuesday, FEBRUARY 10, 1998

Sponsored by

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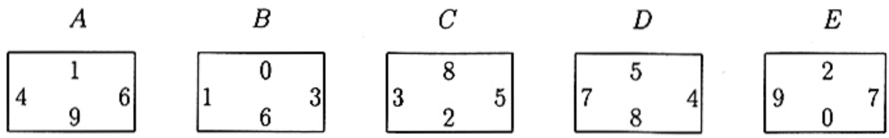
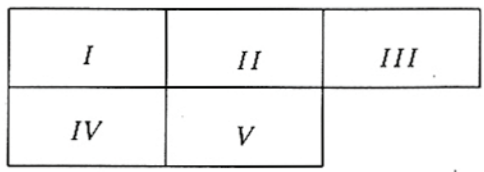
1. 

Each of the sides of five congruent rectangles is labeled with an integer, as shown above. These five rectangles are placed, without rotating or reflecting, in positions I through V so that the labels on coincident sides are equal.



Which of the rectangles is in position I?

(A) A **(B)** B **(C)** C **(D)** D **(E)** E

2. Letters A, B, C, and D represent four different digits selected from $0, 1, 2, \ldots, 9$. If $(A+B)/(C+D)$ is an integer that is as large as possible, what is the value of $A+B$?

(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

3. If $\mathtt{a}, \mathtt{b}$, and $\mathtt{c}$ are digits for which

$$\begin{array}{r} 7\ \mathtt{a}\ 2 \\ -\ 4\ 8\ \mathtt{b} \\ \hline \mathtt{c}\ 7\ 3 \end{array}$$

then $\mathtt{a} + \mathtt{b} + \mathtt{c} =$

(A) 14 **(B)** 15 **(C)** 16 **(D)** 17 **(E)** 18

4. Define $[a, b, c]$ to mean $\frac{a+b}{c}$, where $c \neq 0$. What is the value of

$$[[60, 30, 90], [2, 1, 3], [10, 5, 15]]?$$

(A) 0 **(B)** 0.5 **(C)** 1 **(D)** 1.5 **(E)** 2

5. If $2^{1998} - 2^{1997} - 2^{1996} + 2^{1995} = k \cdot 2^{1995}$, what is the value of k?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

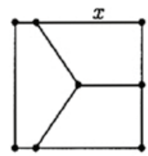
6. If 1998 is written as a product of two positive integers whose difference is as small as possible, then the difference is

(A) 8 (B) 15 (C) 17 (D) 47 (E) 93

7. If $N > 1$, then $\sqrt[3]{N\sqrt[3]{N\sqrt[3]{N}}} =$

(A) $N^{\frac{1}{27}}$ (B) $N^{\frac{1}{9}}$ (C) $N^{\frac{1}{3}}$ (D) $N^{\frac{13}{27}}$ (E) N

8. A square with sides of length 1 is divided into two congruent trapezoids and a pentagon, which have equal areas, by joining the center of the square with points on three of the sides, as shown. Find x, the length of the longer parallel side of each trapezoid.

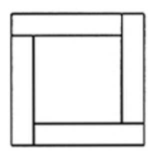


(A) $\frac{3}{5}$ (B) $\frac{2}{3}$ (C) $\frac{3}{4}$ (D) $\frac{5}{6}$ (E) $\frac{7}{8}$

9. A speaker talked for sixty minutes to a full auditorium. Twenty percent of the audience heard the entire talk and ten percent slept through the entire talk. Half of the remainder heard one third of the talk and the other half heard two thirds of the talk. What was the average number of minutes of the talk heard by members of the audience?

(A) 24 (B) 27 (C) 30 (D) 33 (E) 36

10. A large square is divided into a small square surrounded by four congruent rectangles as shown. The perimeter of each of the congruent rectangles is 14. What is the area of the large square?



(A) 49 (B) 64 (C) 100 (D) 121 (E) 196

11. Let R be a rectangle. How many circles in the plane of R have a diameter both of whose endpoints are vertices of R?

(A) 1 (B) 2 (C) 4 (D) 5 (E) 6

12. How many different prime numbers are factors of N if

$$\log_2(\log_3(\log_5(\log_7 N))) = 11?$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 7

13. Walter rolls four standard six-sided dice and finds that the product of the numbers on the upper faces is 144. Which of the following could **not** be the sum of the upper four faces?

(A) 14 **(B)** 15 **(C)** 16 **(D)** 17 **(E)** 18

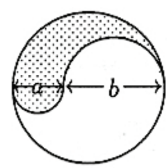
14. A parabola has vertex at $(4, -5)$ and has two x-intercepts, one positive and one negative. If this parabola is the graph of $y = ax^2 + bx + c$, which of a, b, and c must be positive?

(A) only a **(B)** only b **(C)** only c **(D)** a and b only **(E)** none

15. A regular hexagon and an equilateral triangle have equal areas. What is the ratio of the length of a side of the triangle to the length of a side of the hexagon?

(A) $\sqrt{3}$ **(B)** 2 **(C)** $\sqrt{6}$ **(D)** 3 **(E)** 6

16. The figure shown is the union of a circle and two semicircles of diameters a and b, all of whose centers are collinear. The ratio of the area of the shaded region to that of the unshaded region is



(A) $\sqrt{\frac{a}{b}}$ **(B)** $\frac{a}{b}$ **(C)** $\frac{a^2}{b^2}$ **(D)** $\frac{a+b}{2b}$ **(E)** $\frac{a^2+2ab}{b^2+2ab}$

17. Let $f(x)$ be a function with the two properties:

(a) for any two real numbers x and y, $f(x+y) = x + f(y)$, and

(b) $f(0) = 2$.

What is the value of $f(1998)$?

(A) 0 **(B)** 2 **(C)** 1996 **(D)** 1998 **(E)** 2000

18. A right circular cone of volume A, a right circular cylinder of volume M, and a sphere of volume C all have the same radius, and the common height of the cone and the cylinder is equal to the diameter of the sphere. Then

(A) $A - M + C = 0$ **(B)** $A + M = C$ **(C)** $2A = M + C$

(D) $A^2 - M^2 + C^2 = 0$ **(E)** $2A + 2M = 3C$

19. How many triangles have area 10 and vertices at $(-5, 0), (5, 0)$, and $(5\cos\theta, 5\sin\theta)$ for some angle θ?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

20. Three cards, each with a positive integer written on it, are lying face-down on a table. Casey, Stacy, and Tracy are told that

    (a) the numbers are all different,

    (b) they sum to 13, and

    (c) they are in increasing order, left to right.

    First, Casey looks at the number on the leftmost card and says, "I don't have enough information to determine the other two numbers." Then Tracy looks at the number on the rightmost card and says, "I don't have enough information to determine the other two numbers." Finally, Stacy looks at the number on the middle card and says, "I don't have enough information to determine the other two numbers." Assume that each person knows that the other two reason perfectly and hears their comments. What number is on the middle card?

    (A) 2 **(B)** 3 **(C)** 4 **(D)** 5

    (E) There is not enough information to determine the number.

21. In an h-meter race, Sunny is exactly d meters ahead of Windy when Sunny finishes the race. The next time they race, Sunny sportingly starts d meters behind Windy, who is at the starting line. Both runners run at the same constant speed as they did in the first race. How many meters ahead is Sunny when Sunny finishes the second race?

    (A) $\frac{d}{h}$ **(B)** 0 **(C)** $\frac{d^2}{h}$ **(D)** $\frac{h^2}{d}$ **(E)** $\frac{d^2}{h-d}$

22. What is the value of the expression

$$\frac{1}{\log_2 100!} + \frac{1}{\log_3 100!} + \frac{1}{\log_4 100!} + \cdots + \frac{1}{\log_{100} 100!}?$$

    (A) 0.01 **(B)** 0.1 **(C)** 1 **(D)** 2 **(E)** 10

23. The graphs of $x^2 + y^2 = 4 + 12x + 6y$ and $x^2 + y^2 = k + 4x + 12y$ intersect when k satisfies $a \le k \le b$, and for no other values of k. Find $b - a$.

    (A) 5 **(B)** 68 **(C)** 104 **(D)** 140 **(E)** 144

24. Call a 7-digit telephone number $d_1d_2d_3$-$d_4d_5d_6d_7$ *memorable* if the prefix sequence $d_1d_2d_3$ is exactly the same as either of the sequences $d_4d_5d_6$ or $d_5d_6d_7$ (possibly both). Assuming that each d_i can be any of the ten decimal digits $0, 1, 2, \ldots 9$, the number of different memorable telephone numbers is

    (A) 19,810 **(B)** 19,910 **(C)** 19,990 **(D)** 20,000 **(E)** 20,100

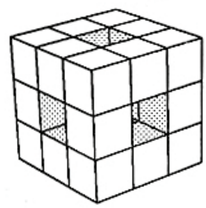
25. A piece of graph paper is folded once so that (0,2) is matched with (4,0), and (7,3) is matched with (m, n). Find $m + n$.

(A) 6.7 (B) 6.8 (C) 6.9 (D) 7.0 (E) 8.0

26. In quadrilateral $ABCD$, it is given that $\angle A = 120°$, angles B and D are right angles, $AB = 13$, and $AD = 46$. Then $AC =$

(A) 60 (B) 62 (C) 64 (D) 65 (E) 72

27. A $9 \times 9 \times 9$ cube is composed of twenty-seven $3 \times 3 \times 3$ cubes. The big cube is 'tunneled' as follows: First, the six $3 \times 3 \times 3$ cubes which make up the center of each face as well as the center $3 \times 3 \times 3$ cube are removed as shown. Second, each of the twenty remaining $3 \times 3 \times 3$ cubes is diminished in the same way. That is, the center facial unit cubes as well as each center cube are removed.



The surface area of the final figure is

(A) 384 (B) 729 (C) 864 (D) 1024 (E) 1056

28. In triangle ABC, angle C is a right angle and $CB > CA$. Point D is located on $\overline{BC}$ so that angle CAD is twice angle DAB. If $AC/AD = 2/3$, then $CD/BD = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

(A) 10 (B) 14 (C) 18 (D) 22 (E) 26

29. A point (x, y) in the plane is called a *lattice point* if both x and y are integers. The area of the largest square that contains exactly three lattice points in its interior is closest to

(A) 4.0 (B) 4.2 (C) 4.5 (D) 5.0 (E) 5.6

30. For each positive integer n, let

$$a_n = \frac{(n+9)!}{(n-1)!}$$

Let k denote the smallest positive integer for which the rightmost nonzero digit of a_k is odd. The rightmost nonzero digit of a_k is

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

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1998 AIME

The AIME will be held on Tuesday, March 17, 1998. It is a 15-question, 3-hour, integer-answer examination. You will be invited to participate only if you receive a score of 100 or above on this AHSME. Top-scoring students on the AHSME/AIME will be selected to take the **USA Mathematical Olympiad** (USAMO) on Tuesday, April 28, 1998. The best way to prepare for the AIME and USAMO is to study previous years of these exams. Copies may be ordered as indicated below.

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1998

American High School Mathematics Examination
(AHSME)

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1. All information (Rules and Instructions) needed to administer the AHSME is contained in the AHSME TEACHERS' MANUAL, which is outside of this package. **PLEASE READ THE MANUAL BEFORE FEBRUARY 10.** Nothing is needed from inside this package until February 10.

2. Your PRINCIPAL or VICE PRINCIPAL must sign the AHSME Certification Form A found in the Teachers' Manual.

3. The Answer Forms must be mailed by First Class mail to Dr. Mientka no later than 24 hours following the examination.

4. *The publication, reproduction or communications of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Duplication at any time via copier, telephone, eMail, World Wide Web or media of any type is a violation of the copyright law.*

AMERICAN MATHEMATICS COMPETITIONS

50th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION (AHSME)
TUESDAY, FEBRUARY 9, 1999

Sponsored by

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YOUR NAME:

NAME OF YOUR TEACHER:

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a thirty-question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. The answers to the problems are to be marked on the AHSME ANSWER FORM with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING RULES: You will receive 5 points for each correct answer, 2 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, ruler, compass, protractor, erasers and calculators that are accepted for use on the SAT. No problems on the test will *require* the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the Answer Form.
8. When your proctor gives the signal, begin working the problems. You will have **90 MINUTES** working time for the test.
9. When you finish the exam *sign your name* in the space on the Answer Form.

Students who score 100 or above on this AHSME will be invited to take the 17th annual **American Invitational Mathematics Examination** (AIME) on Tuesday, March 16, 1999. More details about the AIME and other information are on the back page of this test booklet.

This examination was prepared during the tenure of the American Mathematics Competitions Executive Director, Dr. Walter E. Mientka.

1. $1 - 2 + 3 - 4 + \cdots - 98 + 99 =$

(A) -50 **(B)** -49 **(C)** 0 **(D)** 49 **(E)** 50

2. Which one of the following statements is false?

(A) All equilateral triangles are congruent to each other.

(B) All equilateral triangles are convex.

(C) All equilateral triangles are equiangular.

(D) All equilateral triangles are regular polygons.

(E) All equilateral triangles are similar to each other.

3. The number halfway between $1/8$ and $1/10$ is

(A) $\frac{1}{80}$ **(B)** $\frac{1}{40}$ **(C)** $\frac{1}{18}$ **(D)** $\frac{1}{9}$ **(E)** $\frac{9}{80}$

4. Find the sum of all prime numbers between 1 and 100 that are simultaneously 1 greater than a multiple of 4 and 1 less than a multiple of 5.

(A) 118 **(B)** 137 **(C)** 158 **(D)** 187 **(E)** 245

5. The marked price of a book was 30% less than the suggested retail price. Alice purchased the book for half the marked price at a Fiftieth Anniversary sale. What percent of the suggested retail price did Alice pay?

(A) 25% **(B)** 30% **(C)** 35% **(D)** 60% **(E)** 65%

6. What is the sum of the digits of the decimal form of the product $2^{1999} \cdot 5^{2001}$?

(A) 2 **(B)** 4 **(C)** 5 **(D)** 7 **(E)** 10

7. What is the largest number of acute angles that a convex hexagon can have?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

8. At the end of 1994 Walter was half as old as his grandmother. The sum of the years in which they were born is 3838. How old will Walter be at the end of 1999?

(A) 48 **(B)** 49 **(C)** 53 **(D)** 55 **(E)** 101

9. Before Ashley started a three-hour drive, her car's odometer reading was 29792, a palindrome. (A palindrome is a number that reads the same way from left to right as it does from right to left.) At her destination, the odometer reading was another palindrome. If Ashley never exceeded the speed limit of 75 miles per hour, which of the following was her greatest possible average speed?

(A) $33\frac{1}{3}$ **(B)** $53\frac{1}{3}$ **(C)** $66\frac{2}{3}$ **(D)** $70\frac{1}{3}$ **(E)** $74\frac{1}{3}$

10. A sealed envelope contains a card with a single digit on it. Three of the following statements are true, and the other is false.

I. The digit is 1.

II. The digit is not 2.

III. The digit is 3.

IV. The digit is not 4.

Which one of the following must necessarily be correct?

(A) I is true. **(B)** I is false. **(C)** II is true. **(D)** III is true.

(E) IV is false.

11. The student lockers at Olympic High are numbered consecutively beginning with locker number 1. The plastic digits used to number the lockers cost two cents apiece. Thus, it costs two cents to label locker number 9 and four cents to label locker number 10. If it costs \$137.94 to label all the lockers, how many lockers are there at the school?

(A) 2001 **(B)** 2010 **(C)** 2100 **(D)** 2726 **(E)** 6897

12. What is the maximum number of points of intersection of the graphs of two different fourth degree polynomial functions $y = p(x)$ and $y = q(x)$, each with leading coefficient 1?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 8

13. Define a sequence of real numbers $a_1, a_2, a_3, \ldots$ by $a_1 = 1$ and $a_{n+1}^3 = 99a_n^3$ for all $n \geq 1$. Then a_{100} equals

(A) 33^{33} **(B)** 33^{99} **(C)** 99^{33} **(D)** 99^{99} **(E)** none of these

14. Four girls — Mary, Alina, Tina, and Hanna — sang songs in a concert as trios, with one girl sitting out each time. Hanna sang 7 songs, which was more than any other girl, and Mary sang 4 songs, which was fewer than any other girl. How many songs did these trios sing?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

15. Let x be a real number such that $\sec x - \tan x = 2$. Then $\sec x + \tan x =$

(A) 0.1 **(B)** 0.2 **(C)** 0.3 **(D)** 0.4 **(E)** 0.5

16. What is the radius of a circle inscribed in a rhombus with diagonals of length 10 and 24?

(A) 4 **(B)** 58/13 **(C)** 60/13 **(D)** 5 **(E)** 6

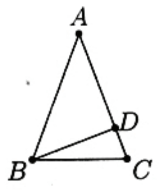
17. Let $P(x)$ be a polynomial such that when $P(x)$ is divided by $x - 19$, the remainder is 99, and when $P(x)$ is divided by $x - 99$, the remainder is 19. What is the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$?

(A) $-x + 80$ **(B)** $x + 80$ **(C)** $-x + 118$ **(D)** $x + 118$ **(E)** 0

18. How many zeros does $f(x) = \cos(\log(x))$ have on the interval $0 < x < 1$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 10 **(E)** infinitely many

19. Consider all triangles ABC satisfying the following conditions: $AB = AC$, D is a point on $\overline{AC}$ for which $\overline{BD} \perp \overline{AC}$, AD and CD are integers, and $BD^2 = 57$. Among all such triangles, the smallest possible value of AC is



(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

20. The sequence $a_1, a_2, a_3, \ldots$ satisfies $a_1 = 19$, $a_9 = 99$, and, for all $n \geq 3$, a_n is the arithmetic mean of the first $n-1$ terms. Find a_2.

(A) 29 **(B)** 59 **(C)** 79 **(D)** 99 **(E)** 179

21. A circle is circumscribed about a triangle with sides $20, 21$, and 29, thus dividing the interior of the circle into four regions. Let A, B, and C be the areas of the non-triangular regions, with C being the largest. Then

(A) $A + B = C$ **(B)** $A + B + 210 = C$ **(C)** $A^2 + B^2 = C^2$

(D) $20A + 21B = 29C$ **(E)** $\frac{1}{A^2} + \frac{1}{B^2} = \frac{1}{C^2}$

22. The graphs of $y = -|x-a| + b$ and $y = |x-c| + d$ intersect at points $(2,5)$ and $(8,3)$. Find $a + c$.

(A) 7 **(B)** 8 **(C)** 10 **(D)** 13 **(E)** 18

23. The equiangular convex hexagon $ABCDEF$ has $AB = 1, BC = 4, CD = 2$, and $DE = 4$. The area of the hexagon is

(A) $\frac{15}{2}\sqrt{3}$ **(B)** $9\sqrt{3}$ **(C)** 16 **(D)** $\frac{39}{4}\sqrt{3}$ **(E)** $\frac{43}{4}\sqrt{3}$

24. Six points on a circle are given. Four of the chords joining pairs of the six points are selected at random. What is the probability that the four chords form a convex quadrilateral?

(A) $\frac{1}{15}$ **(B)** $\frac{1}{91}$ **(C)** $\frac{1}{273}$ **(D)** $\frac{1}{455}$ **(E)** $\frac{1}{1365}$

25. There are unique integers $a_2, a_3, a_4, a_5, a_6, a_7$ such that

$$\frac{5}{7} = \frac{a_2}{2!} + \frac{a_3}{3!} + \frac{a_4}{4!} + \frac{a_5}{5!} + \frac{a_6}{6!} + \frac{a_7}{7!},$$

where $0 \leq a_i < i$ for $i = 2, 3, \ldots, 7$. Find $a_2 + a_3 + a_4 + a_5 + a_6 + a_7$.

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

26. Three non-overlapping regular plane polygons, at least two of which are congruent, all have sides of length 1. The polygons meet at a point A in such a way that the sum of the three interior angles at A is $360°$. Thus the three polygons form a new polygon with A as an interior point. What is the largest possible perimeter that this polygon can have?

    (A) 12 **(B)** 14 **(C)** 18 **(D)** 21 **(E)** 24

27. In triangle ABC, $3\sin A + 4\cos B = 6$ and $4\sin B + 3\cos A = 1$. Then $\angle C$ in degrees is

    (A) 30 **(B)** 60 **(C)** 90 **(D)** 120 **(E)** 150

28. Let $x_1, x_2, \ldots, x_n$ be a sequence of integers such that
    (i) $-1 \le x_i \le 2$, for $i = 1, 2, 3, \ldots, n$;
    (ii) $x_1 + x_2 + \cdots + x_n = 19$; and
    (iii) $x_1^2 + x_2^2 + \cdots + x_n^2 = 99$.
    Let m and M be the minimal and maximal possible values of $x_1^3 + x_2^3 + \cdots + x_n^3$, respectively. Then $M/m =$

    (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

29. A tetrahedron with four equilateral triangular faces has a sphere inscribed within it and a sphere circumscribed about it. For each of the four faces, there is a sphere tangent externally to the face at its center and to the circumscribed sphere. A point P is selected at random inside the circumscribed sphere. The probability that P lies inside one of the five small spheres is closest to

    (A) 0 **(B)** 0.1 **(C)** 0.2 **(D)** 0.3 **(E)** 0.4

30. The number of ordered pairs of integers (m, n) for which $mn \ge 0$ and

    $$m^3 + n^3 + 99mn = 33^3$$

    is equal to

    (A) 2 **(B)** 3 **(C)** 33 **(D)** 35 **(E)** 99

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Correspondence about the problems and solutions should be addressed to:

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- International Mathematical Olympiad Book I, 1959-77, $15.00
- International Mathematical Olympiad Book II, 1978-85, $18.00
- World Olympiad Problems/Solutions 1995-96, 1996-97 $15.00 each

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1999

American High School Mathematics Examination
(AHSME)

☞ DO NOT OPEN UNTIL TUESDAY, FEBRUARY 9, 1999

****Administration On An Earlier Date Will Disqualify Your School's Results****

1. All information (Rules and Instructions) needed to administer the AHSME is contained in the AHSME TEACHERS' MANUAL, which is outside of this package. **PLEASE READ THE MANUAL BEFORE FEBRUARY 9.** Nothing is needed from inside this package until February 9.

2. Your PRINCIPAL or VICE PRINCIPAL must sign the AHSME Certification Form A found in the Teachers' Manual.

3. The Answer Forms must be mailed by First Class mail to Dr. Mientka no later than 24 hours following the examination.

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