MAA AMC

American Mathematics Competitions

SPRING

Official Solutions

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AMC 12 A

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This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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The problems and solutions for this AMC 12 A were prepared by the MAA AMC 10/12 Editorial Board under the direction of Azar Khosravani and Carl Yerger, co-Editors-in-Chief.

1. What is the value of

$$2^{1+2+3} - (2^1 + 2^2 + 2^3)\,?$$

(A) 0 **(B)** 50 **(C)** 52 **(D)** 54 **(E)** 57

Answer (B): The expression in the problem simplifies to

$$2^6 - (2 + 4 + 8) = 64 - 14 = 50.$$

2. Under what conditions is $\sqrt{a^2 + b^2} = a + b$ true, where a and b are real numbers?

(A) It is never true. **(B)** It is true if and only if $ab = 0$.

(C) It is true if and only if $a + b \geq 0$.

(D) It is true if and only if $ab = 0$ and $a + b \geq 0$.

(E) It is always true.

Answer (D): Squaring both sides produces

$$a^2 + b^2 = (a + b)^2 = a^2 + 2ab + b^2,$$

which is equivalent to $ab = 0$. Thus $ab = 0$ is a necessary condition for $\sqrt{a^2 + b^2} = a + b$ to be true. But $ab = 0$ if and only if $a = 0$ or $b = 0$, in which case the given equation becomes either $\sqrt{b^2} = b$ or $\sqrt{a^2} = a$, which are true if and only if $b \geq 0$ or $a \geq 0$, respectively. It follows that $\sqrt{a^2 + b^2} = a + b$ if and only if $ab = 0$ and $a + b \geq 0$. In other words, for the given equation to be true, one of the variables must be 0 and the other must be nonnegative.

3. The sum of two natural numbers is 17,402. One of the two numbers is divisible by 10. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?

(A) 10,272 **(B)** 11,700 **(C)** 13,362 **(D)** 14,238 **(E)** 15,426

Answer (D): Let n be the number that is known to be a multiple of 10. Then the other number is $\frac{n}{10}$. The sum condition says that $n + \frac{n}{10} = 17{,}402$, which can be rewritten as $\frac{11n}{10} = 17{,}402$. Multiplying both sides by $\frac{10}{11}$ gives $n = 15{,}820$. The requested difference is $15{,}820 - 1{,}582 = 14{,}238$.

OR

Let s be the lesser number. Then the greater number is $10s$ and the sum of the two numbers is $11s = 17{,}402$. This gives $s = 1{,}582$, and the difference of the two numbers is $9s = 14{,}238$.

4. Tom has a collection of 13 snakes, 4 of which are purple and 5 of which are happy. He observes that

- all of his happy snakes can add,
- none of his purple snakes can subtract, and
- all of his snakes that can't subtract also can't add.

Which of these conclusions can be drawn about Tom's snakes?

(A) Purple snakes can add.

(B) Purple snakes are happy.

(C) Snakes that can add are purple.

(D) Happy snakes are not purple.

(E) Happy snakes can't subtract.

Answer (D): To see that choices **(A)**, **(B)**, **(C)**, and **(E)** do not follow from the given informtion, consider the following two snakes that may be part of Tom's collection. One snake is happy but not purple and can both add and subtract. The second is purple but not happy and can neither add nor subtract. Then each of the three bulleted statements is true, but each of these choices is false.

To show that answer choice **(D)** is correct, first observe that the third bulleted statement is equivalent to "Snakes that can add also can subtract." The second bulleted statement is equivalent to "Snakes that can subtract are not purple." The three bulleted statements combined then lead to the conclusion "Happy snakes are not purple."

OR

Let H, P, A, and S denote the statements that a snake is happy, is purple, can add, and can subtract, respectively. Let $\rightarrow$ denote "implies" and $\sim$ denote "not". Recall that an implication $X \rightarrow Y$ is logically equivalent to its contrapositive

$$(\sim Y) \rightarrow (\sim X).$$

Then the three bulleted statements can be exactly summarized as

$$H \rightarrow A \rightarrow S \rightarrow (\sim P).$$

Choice **(D)**, which is

$$H \rightarrow (\sim P),$$

follows from the transitivity of the "implies" relation. However, choices **(A)**, which is $P \rightarrow A$; **(B)**, which is $P \rightarrow H$; **(C)**, which is $A \rightarrow P$; and **(E)**, which is $H \rightarrow (\sim S)$, do not follow from those three implications.

5. When a student multiplied the number 66 by the repeating decimal,

$$\underline{1}.\underline{a}\,\underline{b}\,\underline{a}\,\underline{b}\ldots = \underline{1}.\overline{\underline{a}\,\underline{b}},$$

where a and b are digits, he did not notice the notation and just multiplied 66 times $\underline{1}.\underline{a}\,\underline{b}$. Later he found that his answer is 0.5 less than the correct answer. What is the 2-digit integer $\underline{a}\,\underline{b}$?

(A) 15 **(B)** 30 **(C)** 45 **(D)** 60 **(E)** 75

Answer (E): The given condition is

$$66 \cdot \underline{1}.\underline{a}\,\underline{b} = 66 \cdot \underline{1}.\overline{\underline{a}\,\underline{b}} - 0.5,$$

which is equivalent to

$$0.5 = 66 \cdot \underline{1}.\overline{\underline{a}\,\underline{b}} - 66 \cdot \underline{1}.\underline{a}\,\underline{b} = 66(\underline{1}.\overline{\underline{a}\,\underline{b}} - \underline{1}.\underline{a}\,\underline{b}) = 66 \cdot \underline{0}.\underline{0}\,\underline{0}\,\overline{\underline{a}\,\underline{b}}.$$

Because

$$\underline{0}.\overline{\underline{a}\,\underline{b}} = \frac{\underline{a}\,\underline{b}}{99},$$

this is equivalent to

$$\frac{1}{2} = 66 \cdot \frac{\underline{a}\,\underline{b}}{9900},$$

from which $\underline{a}\,\underline{b} = 75$.

6. A deck of cards has only red cards and black cards. The probability of a randomly chosen card being red is $\frac{1}{3}$. When 4 black cards are added to the deck, the probability of choosing red becomes $\frac{1}{4}$. How many cards were in the deck originally?

(A) 6 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 18

Answer (C): Let r and b be the numbers of red cards and black cards, respectively. Before adding the additional cards, the probability of choosing a red card is $\frac{r}{r+b} = \frac{1}{3}$. This implies $3r = r + b$, so $2r = b$. After 4 black cards are added, the probability of choosing a red card is

$$\frac{r}{r+b+4} = \frac{r}{3r+4} = \frac{1}{4}.$$

Therefore $4r = 3r + 4$, so $r = 4$. Then $b = 2r = 8$, and the deck has $4 + 8 = 12$ cards in all.

OR

Let t be the number of cards and r be the number of red cards in the deck originally. Then the ratio of the original probability of choosing a red card to the probability of choosing a red card after the 4 black cards have been added is

$$\frac{\frac{r}{t}}{\frac{r}{t+4}} = \frac{\frac{1}{3}}{\frac{1}{4}}.$$

This simplifies to

$$\frac{t+4}{t} = \frac{4}{3}.$$

Therefore $3(t + 4) = 4t$ and $t = 12$.

7. What is the least possible value of $(xy-1)^2 + (x+y)^2$ for real numbers x and y ?

(A) 0 **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** 2

Answer (D): Because

$$(xy-1)^2 + (x+y)^2 = x^2y^2 + x^2 + y^2 + 1 = \left(x^2+1\right)\left(y^2+1\right)$$

and both factors are at least 1, the least possible value of the expression is 1. It occurs when $x = 0$ and $y = 0$.

8. A sequence of numbers is defined by $D_0 = 0$, $D_1 = 0$, $D_2 = 1$, and $D_n = D_{n-1} + D_{n-3}$ for $n \geq 3$. What are the parities (evenness or oddness) of the triple of numbers $(D_{2021}, D_{2022}, D_{2023})$, where E denotes even and O denotes odd?

(A) (O, E, O) **(B)** (E, E, O) **(C)** (E, O, E) **(D)** (O, O, E) **(E)** (O, O, O)

Answer (C): The parities of the first 10 terms of the sequence (D_n) are as follows:

n	0	1	2	3	4	5	6	7	8	9
Parity	E	E	O	O	O	E	O	E	E	O

The terms D_7, D_8, and D_9 have the same parities as D_0, D_1, D_2, respectively, so the sequence is periodic with period 7. The remainder when 2021 is divided by 7 is 5, so the parities of $(D_{2021}, D_{2022}, D_{2023})$ are the same as the parities of (D_5, D_6, D_7), namely (E, O, E).

9. Which of the following is equivalent to

$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})\,?$$

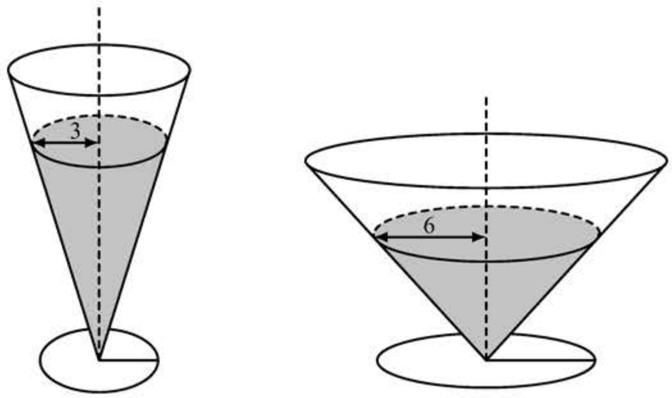
(A) $3^{127} + 2^{127}$ **(B)** $3^{127} + 2^{127} + 2 \cdot 3^{63} + 3 \cdot 2^{63}$ **(C)** $3^{128} - 2^{128}$ **(D)** $3^{128} + 2^{128}$ **(E)** 5^{127}

Answer (C): The product telescopes as follows:

$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)\cdots(2^{64}+3^{64})$$

$$
\begin{aligned}
&= (3-2)(3+2)(3^2+2^2)(3^4+2^4)(3^8+2^8)\cdots(3^{64}+2^{64})\\
&= (3^2-2^2)(3^2+2^2)(3^4+2^4)(3^8+2^8)\cdots(3^{64}+2^{64})\\
&= (3^4-2^4)(3^4+2^4)(3^8+2^8)\cdots(3^{64}+2^{64})\\
&\vdots\\
&= (3^{64}-2^{64})(3^{64}+2^{64})\\
&= 3^{128}-2^{128}.
\end{aligned}
$$

10. Two right circular cones with vertices facing down as shown in the figure below contain the same amount of liquid. The radii of the tops of the liquid surfaces are 3 cm and 6 cm. Into each cone is dropped a spherical marble of radius 1 cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?



(A) 1 : 1 **(B)** 47 : 43 **(C)** 2 : 1 **(D)** 40 : 13 **(E)** 4 : 1

Answer (E): If two cones have the same volume and the radius of the narrow cone is one-half the radius of the wide cone, then because the volume of a cone varies directly with both the square of the radius and the height, it follows that the height of the narrow cone is 4 times the height of the wide cone. The volumes of liquid are the same both before and after the marble is added, so both heights for the narrow cone are 4 times the corresponding heights for the wide cone. Hence the rise of the level in the narrow cone is also 4 times the rise of the level in the wide cone.

OR

Let h_1 and h_2 be the heights of the liquid in the narrow and wide cones, respectively, before the marble is dropped in. The two equal volumes of liquid are

$$\frac{1}{3}\pi \cdot 3^2 \cdot h_1 = \frac{1}{3}\pi \cdot 6^2 \cdot h_2,$$

which implies $h_1 = 4h_2$. When the marble of volume $\frac{4}{3}\pi$ is added, both the heights and the radii of the conical volumes of water change, but the volumes remain equal. Let R_1 and H_1 be the new radius and height for the new narrow cone, and R_2 and H_2 be the new radius and height for the new wide cone. Then

$$\frac{1}{3}\pi R_1^2 H_1 = 3\pi h_1 + \frac{4}{3}\pi.$$

By similar triangles, $R_1 = \frac{3H_1}{h_1}$. Substituting and solving for H_1 gives

$$H_1 = \sqrt[3]{h_1^3 + \frac{4}{9}h_1^2}.$$

Letting $h_1 = 4h_2$ gives

$$H_1 = \sqrt[3]{64h_2^3 + \frac{64}{9}h_2^2},$$

which is equivalent to

$$H_1 = 4\sqrt[3]{h_2^3 + \frac{1}{9}h_2^2}.$$

Similarly,

$$\frac{1}{3}\pi R_2^2 H_2 = 12\pi h_2 + \frac{4}{3}\pi,$$

and by similar triangles, $R_2 = \frac{6H_2}{h_2}$. Substituting and solving for H_2 gives

$$H_2 = \sqrt[3]{h_2^3 + \frac{1}{9}h_2^2}.$$

It follows that the ratio of the rise of the water levels is

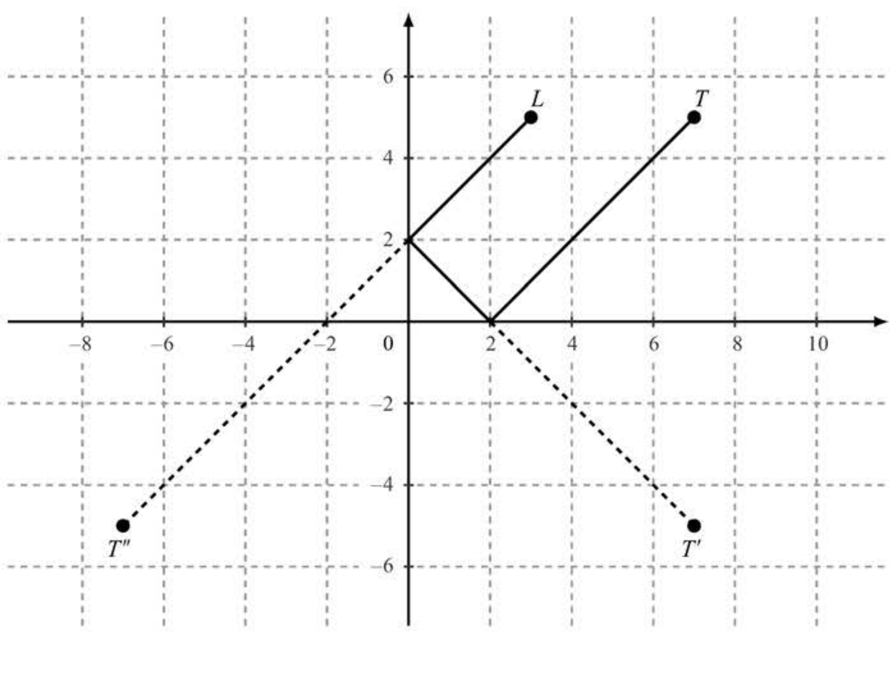
$$\frac{H_1 - h_1}{H_2 - h_2} = \frac{4\sqrt[3]{h_2^3 + \frac{1}{9}h_2^2} - 4h_2}{\sqrt[3]{h_2^3 + \frac{1}{9}h_2^2} - h_2} = 4.$$

11. A laser is placed at the point $(3, 5)$. The laser beam travels in a straight line. Larry wants the beam to hit and bounce off the y-axis, then hit and bounce off the x-axis, then hit the point $(7, 5)$. What is the total distance the beam will travel along this path?

(A) $2\sqrt{10}$ **(B)** $5\sqrt{2}$ **(C)** $10\sqrt{2}$ **(D)** $15\sqrt{2}$ **(E)** $10\sqrt{5}$

Answer (C): Reflect $T(7, 5)$ about the x-axis to get $T'(7, -5)$. Reflecting this point about the y-axis gives $T''(-7, -5)$. Because reflection is an isometry and the angle of incidence equals the angle of reflection, the segment of the beam's travel from $(0, 2)$ to $(2, 0)$ has the same length as the line segment from $(0, 2)$ to $(-2, 0)$, and the segment of the beam's travel from $(2, 0)$ to $T(7, 5)$ has the same length as the line segment from $(-2, 0)$ to $T''(-7, -5)$. Therefore the total distance of the path from $L(3, 5)$ to $T(7, 5)$ is equal to the distance

$$LT'' = \sqrt{\left(3 - (-7)\right)^2 + \left(5 - (-5)\right)^2} = 10\sqrt{2}.$$



12. All the roots of polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers, possibly repeated. What is the value of B ?

(A) -88 **(B)** -80 **(C)** -64 **(D)** -41 **(E)** -40

Answer (A): Because this polynomial has degree 6, there are 6 roots, counting multiplicities. By Vieta's formulas, the sum of the roots is 10 and their product is 16. The only way this can happen is for the roots, listed with repetitions, to be $1, 1, 2, 2, 2, 2$. Thus the polynomial is $(z-1)^2(z-2)^4$. By the Binomial Theorem, this polynomial equals

$$(z^2 - 2z + 1) \cdot (z^4 - 8z^3 + 24z^2 - 32z + 16).$$

When this product is expanded, the coefficient of z^3 is $B = -32 - 48 - 8 = -88$.

OR

Proceed as in the first solution. Then, using Vieta's formulas, observe that $-B$ is the sum of the products of the roots taken 3 at a time, namely

$$\binom{4}{3}(2 \cdot 2 \cdot 2) + \binom{4}{2}\binom{2}{1}(2 \cdot 2 \cdot 1) + \binom{4}{1}\binom{2}{2}(2 \cdot 1 \cdot 1).$$

This gives a total of $32 + 48 + 8 = 88$. Therefore $B = -88$.

13. Of the following complex numbers z, which one has the property that z^5 has the greatest real part?

(A) -2 **(B)** $-\sqrt{3} + i$ **(C)** $-\sqrt{2} + \sqrt{2}i$ **(D)** $-1 + \sqrt{3}i$

(E) $2i$

Answer (B): Each of the five given numbers has the same modulus, 2. In polar form, the five numbers are, in the order of the answer choices, $2\operatorname{cis}(\pi)$, $2\operatorname{cis}\left(\frac{5}{6}\pi\right)$, $2\operatorname{cis}\left(\frac{3}{4}\pi\right)$, $2\operatorname{cis}\left(\frac{2}{3}\pi\right)$, and $2\operatorname{cis}\left(\frac{1}{2}\pi\right)$. Thus their fifth powers are, respectively,

$$\begin{aligned}
32\operatorname{cis}(5\pi) &= 32\operatorname{cis}(\pi), \\
32\operatorname{cis}\left(\frac{25}{6}\pi\right) &= 32\operatorname{cis}\left(\frac{1}{6}\pi\right), \\
32\operatorname{cis}\left(\frac{15}{4}\pi\right) &= 32\operatorname{cis}\left(-\frac{1}{4}\pi\right), \\
32\operatorname{cis}\left(\frac{10}{3}\pi\right) &= 32\operatorname{cis}\left(-\frac{2}{3}\pi\right), \text{ and} \\
32\operatorname{cis}\left(\frac{5}{2}\pi\right) &= 32\operatorname{cis}\left(\frac{1}{2}\pi\right).
\end{aligned}$$

All of the arguments have been rewritten here to lie between $-\pi$ and π, inclusive. Therefore the one with the greatest real part is the one whose argument is closest to 0. This is $32\operatorname{cis}\left(\frac{1}{6}\pi\right)$, the fifth power of $-\sqrt{3} + i$.

14. What is the value of

$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right) \cdot \left(\sum_{k=1}^{100} \log_{9^k} 25^k\right)?$$

(A) 21 **(B)** $100\log_5 3$ **(C)** $200\log_3 5$ **(D)** 2,200

(E) 21,000

Answer (E): Note that

$$\log_{5^k} 3^{k^2} = \frac{\log 3^{k^2}}{\log 5^k} = \frac{k^2 \log 3}{k \log 5} = \frac{\log 3}{\log 5} \cdot k$$

and

$$\log_{9^k} 25^k = \frac{\log 25^k}{\log 9^k} = \frac{2k \log 5}{2k \log 3} = \frac{\log 5}{\log 3}.$$

Thus

$$\sum_{k=1}^{20} \log_{5^k} 3^{k^2} = \frac{\log 3}{\log 5} \sum_{k=1}^{20} k = \frac{\log 3}{\log 5} \cdot \frac{20 \cdot 21}{2} = 210 \cdot \frac{\log 3}{\log 5}$$

and

$$\sum_{k=1}^{100} \log_{9^k} 25^k = 100 \cdot \frac{\log 5}{\log 3}.$$

The requested product is $210 \cdot 100 = 21{,}000$.

15. A choir director must select a group of singers from among his 6 tenors and 8 basses. The only requirements are that the difference between the numbers of tenors and basses must be a multiple of 4, and the group must have at least one singer. Let N be the number of groups that could be selected. What is the remainder when N is divided by 100 ?

(A) 47 **(B)** 48 **(C)** 83 **(D)** 95 **(E)** 96

Answer (D): Suppose the number of tenors selected is t and the number of basses selected is b. Then the conditions require that $t \equiv b \pmod 4$, and $(t, b) \neq (0, 0)$. Instead, consider having the director select the tenors who will sing and the basses who will not sing. Then the director will be selecting t tenors and $b' = 8 - b$ basses, where $t \equiv b \pmod 4$, so $t + b' = t + 8 - b = 8 \equiv 0 \pmod 4$. Thus the director selects k singers from the 14 singers, where $k \equiv 0 \pmod 4$. The number of ways this can be done with at least one singer is

$$\begin{aligned} N &= \binom{14}{4} + \binom{14}{8} + \binom{14}{12} \\ &= \binom{14}{4} + \binom{14}{6} + \binom{14}{2} \\ &= \frac{14 \cdot 13 \cdot 12 \cdot 11}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \frac{14 \cdot 13}{2 \cdot 1} \\ &= 7 \cdot 13 \cdot 11 + 7 \cdot 13 \cdot 11 \cdot 3 + 7 \cdot 13 \\ &= 91 \cdot (11 + 33 + 1) \\ &= 91 \cdot 45 \\ &= 4095. \end{aligned}$$

The requested remainder is 95.

OR

The calculation above can be done without messy arithmetic by using the properties of Pascal's Triangle:

$$\begin{aligned} N &= \binom{14}{0} + \binom{14}{4} + \binom{14}{8} + \binom{14}{12} - \binom{14}{0} \\ &= \frac{1}{2} \sum_{k=0}^{7} \binom{14}{2k} - 1 \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{k=1}^{7}\binom{14}{2k-1} - 1 \\
&= \frac{1}{4}\sum_{k=0}^{14}\binom{14}{k} - 1 \\
&= \frac{1}{4}\cdot 2^{14} - 1 \\
&= 2^{12} - 1 \\
&= 4095.
\end{aligned}
$$

OR

Let t be the number of tenors selected and b the number of basses selected. Then there are 15 choices for (t, b): $(0,4)$, $(0,8)$, $(1,1)$, $(1,5)$, $(2,2)$, $(2,6)$, $(3,3)$, $(3,7)$, $(4,0)$, $(4,4)$, $(4,8)$, $(5,1)$, $(5,5)$, $(6,2)$, and $(6,6)$. The number of ways to select the singers is therefore the sum of the following products of binomial coefficients:

$$
\begin{aligned}
\binom{6}{0}\cdot\binom{8}{4} &= 1\cdot 70 = 70 \\
\binom{6}{0}\cdot\binom{8}{8} &= 1\cdot 1 = 1 \\
\binom{6}{1}\cdot\binom{8}{1} &= 6\cdot 8 = 48 \\
\binom{6}{1}\cdot\binom{8}{5} &= 6\cdot 56 = 336 \\
\binom{6}{2}\cdot\binom{8}{2} &= 15\cdot 28 = 420 \\
\binom{6}{2}\cdot\binom{8}{6} &= 15\cdot 28 = 420 \\
\binom{6}{3}\cdot\binom{8}{3} &= 20\cdot 56 = 1120 \\
\binom{6}{3}\cdot\binom{8}{7} &= 20\cdot 8 = 160 \\
\binom{6}{4}\cdot\binom{8}{0} &= 15\cdot 1 = 15 \\
\binom{6}{4}\cdot\binom{8}{4} &= 15\cdot 70 = 1050 \\
\binom{6}{4}\cdot\binom{8}{8} &= 15\cdot 1 = 15 \\
\binom{6}{5}\cdot\binom{8}{1} &= 6\cdot 8 = 48 \\
\binom{6}{5}\cdot\binom{8}{5} &= 6\cdot 56 = 336 \\
\binom{6}{6}\cdot\binom{8}{2} &= 1\cdot 28 = 28 \\
\binom{6}{6}\cdot\binom{8}{6} &= 1\cdot 28 = 28
\end{aligned}
$$

The sum of the numbers calculated above is 4095, and the requested remainder is 95.

16. In the following list of numbers, the integer n appears n times in the list for $1 \le n \le 200$.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots, 200, 200, \ldots, 200$$

What is the median of the numbers in this list?

(A) 100.5 **(B)** 134 **(C)** 142 **(D)** 150.5 **(E)** 167

Answer (C): There are

$$1 + 2 + 3 + \cdots + 200 = \frac{200 \cdot 201}{2} = 20{,}100$$

numbers in the list, so the median is the average (mean) of the two middle numbers, the 10,050th and 10,051st entries. The number of entries less than or equal to n is

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Setting

$$\frac{n(n+1)}{2} = 10{,}050$$

and solving for n suggests that n is a little less than

$$\sqrt{20{,}100} \approx 100 \cdot 2.01^{\frac{1}{2}} \approx 100\left(\sqrt{2} + \frac{1}{2} \cdot 0.01\right) \approx 100(1.414 + 0.005) \approx 142.$$

A little arithmetic shows that

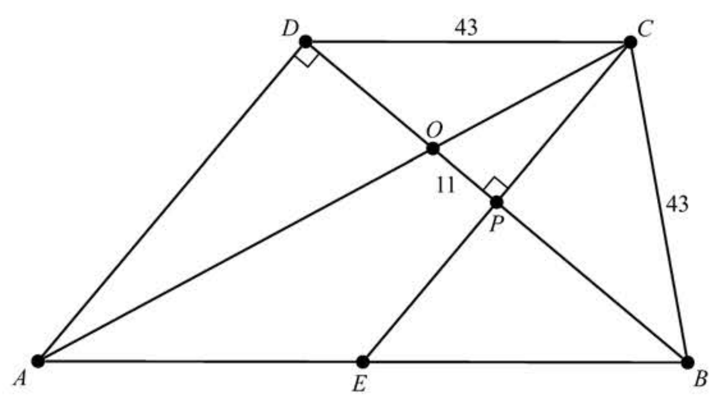
$$\frac{141(141+1)}{2} = 10{,}011 < 10{,}050 \quad \text{and} \quad \frac{142(142+1)}{2} = 10{,}153 > 10{,}051,$$

so both of the middle numbers in the list are 142, and this is the median.

17. Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, $BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$. Given that $OP = 11$, the length AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$?

(A) 65 **(B)** 132 **(C)** 157 **(D)** 194 **(E)** 215

Answer (D): Note that $\triangle BCD$ is isosceles with vertex angle C, and $\overline{CP}$ is a median. Thus $\overline{CP} \perp \overline{BD}$. Also, because $\overline{AD} \perp \overline{BD}$, it follows that $\overline{CP} \parallel \overline{AD}$.



Let E be the intersection of the lines CP and AB. Then $AECD$ is a parallelogram, so $AE = CD = 43$ and $\angle DAE \cong \angle DCE$. But $\angle DCE \cong \angle ECB$, because $\overline{CP}$ is also an angle bisector in $\triangle BCD$, and $\angle DAE \cong$

$\angle CEB$, as corresponding angles, because $\overline{CE} \parallel \overline{AD}$. By transitivity, $\angle ECB \cong \angle CEB$, so $\triangle BCE$ is isosceles with $EB = CB = 43$. Thus $AB = AE + EB = 43 + 43 = 86$.

Now $\triangle COD \sim \triangle AOB$, so $\frac{DO}{BO} = \frac{CD}{AB} = \frac{1}{2}$. Hence $DO = \frac{1}{3}BD$. This implies that

$$11 = OP = DP - DO = \frac{1}{2}BD - \frac{1}{3}BD = \frac{1}{6}BD,$$

so $BD = 66$.

The Pythagorean Theorem in $\triangle ABD$ gives

$$AD = \sqrt{AB^2 - BD^2} = \sqrt{86^2 - 66^2} = \sqrt{20 \cdot 152} = \sqrt{4 \cdot 5 \cdot 8 \cdot 19} = 4\sqrt{190}.$$

Thus $m = 4$, $n = 190$, and $m + n = 194$.

OR

Note that $\triangle AOD$ and $\triangle COP$ are similar, so $AD : CP = DO : 11$. Also, $\triangle CDB$ is isosceles, so $\angle PBC = \angle CDB = \angle DBA$. Therefore $\triangle DBA$ and $\triangle PBC$ are also similar. Thus $AD : CP = DB : PB = 2 : 1$. It follows that $DO = 22$, $DP = 33$, and $BD = 66$. Because $\triangle CDO$ and $\triangle ABO$ are similar with $DO : OB = 1 : 2$ it follows that $AB = 86$. The solution concludes as above.

18. Let f be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b. Suppose that f also has the property that $f(p) = p$ for every prime number p. For which of the following numbers x is $f(x) < 0$?

(A) $\frac{17}{32}$ **(B)** $\frac{11}{16}$ **(C)** $\frac{7}{9}$ **(D)** $\frac{7}{6}$ **(E)** $\frac{25}{11}$

Answer (E): If n is a positive integer whose prime factorization is $n = p_1 p_2 \cdots p_k$, then $f(n) = f(p_1) + f(p_2) + \cdots + f(p_k) = p_1 + p_2 + \cdots + p_k$. Because $f(1) = f(1 \cdot 1) = f(1) + f(1)$, it follows that $f(1) = 0$. If r is a positive rational number, then $0 = f(1) = f\left(r \cdot \frac{1}{r}\right) = f(r) + f\left(\frac{1}{r}\right)$, which implies that $f\left(\frac{1}{r}\right) = -f(r)$ for all positive rational numbers r. Hence

$$\begin{aligned} f\left(\frac{17}{32}\right) &= 17 - 5 \cdot 2 = 7 > 0, \\ f\left(\frac{11}{16}\right) &= 11 - 4 \cdot 2 = 3 > 0, \\ f\left(\frac{7}{9}\right) &= 7 - 2 \cdot 3 = 1 > 0, \\ f\left(\frac{7}{6}\right) &= 7 - 2 - 3 = 2 > 0, \text{ and} \\ f\left(\frac{25}{11}\right) &= 2 \cdot 5 - 11 = -1 < 0. \end{aligned}$$

Of the choices, only $x = \frac{25}{11}$ has the property that $f(x) < 0$.

19. How many solutions does the equation $\sin\left(\frac{\pi}{2}\cos x\right) = \cos\left(\frac{\pi}{2}\sin x\right)$ have in the closed interval $[0, \pi]$?

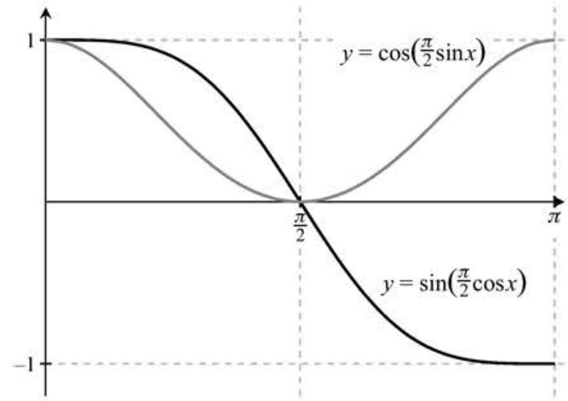
(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Answer (C): Let x be a solution to the given equation in the interval $[0, \pi]$. Then $\frac{\pi}{2}\sin x \in \left[0, \frac{\pi}{2}\right]$, so $\cos\left(\frac{\pi}{2}\sin x\right) \geq 0$, and therefore by the given equation, $\sin\left(\frac{\pi}{2}\cos x\right) \geq 0$. Likewise, $x \in [0, \pi]$ implies that $\frac{\pi}{2}\cos x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but because $\sin\left(\frac{\pi}{2}\cos x\right) \geq 0$, it follows that $\frac{\pi}{2}\cos x$ is also in $\left[0, \frac{\pi}{2}\right]$.

Because $\cos\alpha = \sin\left(\frac{\pi}{2} - \alpha\right)$, the given equation is equivalent to

$$\sin\left(\frac{\pi}{2}\cos x\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{2}\sin x\right),$$

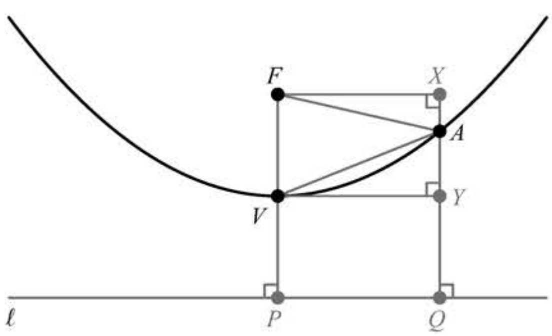
so $\cos x = 1 - \sin x$. Squaring both sides of this last equation gives $\cos^2 x = 1 - 2\sin x + \sin^2 x$, so $2\sin^2 x - 2\sin x = 2(\sin x - 1)\sin x = 0$. From this it follows that $x = 0$ or $x = \frac{\pi}{2}$ or $x = \pi$. The first two of these values satisfy the original equation, but the third does not. (Squaring introduced this extraneous solution.) Therefore there are exactly 2 solutions in the interval $[0, \pi]$.

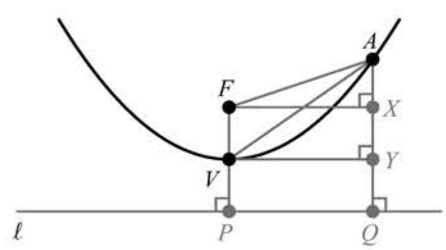


20. Suppose that on a parabola with vertex V and focus F there exists a point A such that $AF = 20$ and $AV = 21$. What is the sum of all possible values of the length FV?

(A) 13 **(B)** $\frac{40}{3}$ **(C)** $\frac{41}{3}$ **(D)** 14 **(E)** $\frac{43}{3}$

Answer (B): Let ℓ be the directrix of such a parabola. By definition, the parabola is the set of points T such that the distance from T to ℓ is equal to TF. Let P and Q be the orthogonal projections of F and A, respectively, onto ℓ, and let X and Y be the orthogonal projections of F and V, respectively, onto line AQ. Because $AF < AV$, there are two possible configurations that may arise, and they are shown below.





Let $d = FV = VP$. Because $AQ = AF = 20$, it follows that $AY = 20 - d$ and $AX = |20 - 2d|$. Because $FXYV$ is a rectangle, $FX = VY$, so applying the Pythagorean Theorem to $\triangle AFX$ and $\triangle AVY$ gives

$$\begin{aligned} 21^2 - (20-d)^2 &= AV^2 - AY^2 = VY^2 \\ &= FX^2 = AF^2 - AX^2 = 20^2 - (20-2d)^2. \end{aligned}$$

This equation simplifies to $3d^2 - 40d + 41 = 0$, which has solutions

$$d = \frac{20 \pm \sqrt{277}}{3}.$$

Specifically, the lesser solution is the value of d with the right configuration and the greater solution is the value of d with the left configuration. The requested answer is $\frac{40}{3}$.

OR

Let V be the origin, let F lie on the positive y-axis, and let $d = FV$. The equation of the parabola is then $x^2 = 4dy$. If (x, y) are the coordinates of A, then $x^2 + y^2 = 441$ and $y = 20 - d$. Substituting for x and y gives $4d(20 - d) + (20 - d)^2 = 441$, which simplifies to $3d^2 - 40d + 41 = 0$, and the solution proceeds as above.

21. The five solutions to the equation

$$(z-1)(z^2+2z+4)(z^2+4z+6) = 0$$

may be written in the form $x_k + y_k i$ for $1 \le k \le 5$, where x_k and y_k are real. Let $\mathcal{E}$ be the unique ellipse that passes through the points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), and (x_5, y_5). The eccentricity of $\mathcal{E}$ can be written in the form $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. What is $m + n$? (Recall that the *eccentricity* of an ellipse $\mathcal{E}$ is the ratio $\frac{c}{a}$, where $2a$ is the length of the major axis of $\mathcal{E}$ and $2c$ is the distance between its two foci.)

(A) 7 **(B)** 9 **(C)** 11 **(D)** 13 **(E)** 15

Answer (A): The solutions to this equation are $z = 1$, $z = -1 \pm \sqrt{3}\,i$, and $z = -2 \pm \sqrt{2}\,i$. Consider the five points $(1, 0)$, $(-1, \pm\sqrt{3})$, and $(-2, \pm\sqrt{2})$. These are five points that lie on $\mathcal{E}$. Note that because these five points are symmetric about the x-axis, $\mathcal{E}$ must also have this property. Therefore the equation of the ellipse is of the form

$$\frac{(x-h)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which is equivalent to

$$b^2(x-h)^2 + a^2y^2 = a^2b^2,$$

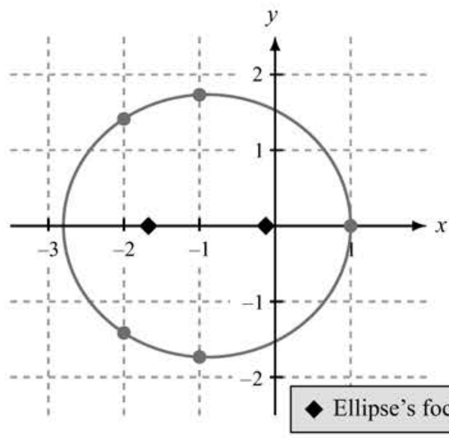
where the center of the ellipse is at $(h, 0)$ with $h < 1$, a is the length of the semi-major axis, and b is the length of the semi-minor axis (or vice versa, but it will it turn out that $a > b$).

Evaluating at the point $(1, 0)$ yields $a = 1 - h$. Evaluating at the points $(-1, \sqrt{3})$ and $(-2, \sqrt{2})$ yields the equations

$$b^2(1+h)^2 + 3(1-h)^2 = (1-h)^2b^2,$$
$$b^2(2+h)^2 + 2(1-h)^2 = (1-h)^2b^2.$$

Multiplying the first equation by -2 and the second equation by 3 and then adding leads to $h = -\frac{9}{10}$, so $a = \frac{19}{10}$. It then follows from either of the above equations that $b^2 = \frac{19^2}{120}$. Note that $a > b$, as promised. Letting c denote the distance from the center of the ellipse to a focus, and using the fact that $c^2 = a^2 - b^2$ for an ellipse, it follows that $c = \frac{19}{10\sqrt{6}}$. Then the eccentricity of the ellipse is $\frac{c}{a} = \sqrt{\frac{1}{6}}$, and therefore $m + n = 1 + 6 = 7$.

The graph of the ellipse is shown below, with the foci marked.



22. Suppose that the roots of the polynomial $P(x) = x^3 + ax^2 + bx + c$ are $\cos \frac{2\pi}{7}$, $\cos \frac{4\pi}{7}$, and $\cos \frac{6\pi}{7}$, where angles are in radians. What is abc ?

(A) $-\dfrac{3}{49}$ **(B)** $-\dfrac{1}{28}$ **(C)** $\dfrac{\sqrt[3]{7}}{64}$ **(D)** $\dfrac{1}{32}$ **(E)** $\dfrac{1}{28}$

Answer (D): By Vieta's formulas,

$$a = -\left(\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7}\right) \tag{1}$$

$$b = \cos \frac{2\pi}{7} \cdot \cos \frac{4\pi}{7} + \cos \frac{4\pi}{7} \cdot \cos \frac{6\pi}{7} + \cos \frac{6\pi}{7} \cdot \cos \frac{2\pi}{7} \tag{2}$$

$$c = -\cos \frac{2\pi}{7} \cdot \cos \frac{4\pi}{7} \cdot \cos \frac{6\pi}{7}. \tag{3}$$

The roots of the polynomial $Q(x) = x^6 + x^5 + \cdots + x + 1$ are the six primitive seventh roots of 1, namely the numbers $\cos \frac{2k\pi}{7} + i \sin \frac{2k\pi}{7}$ for $k = 1, 2, 3, 4, 5, 6$. These roots form three complex-conjugate pairs, so $Q(x)$ can be written in factored form as

$$\begin{aligned}
Q(x) &= \left(x - \left(\cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}\right)\right)\left(x - \left(\cos \frac{2\pi}{7} - i \sin \frac{2\pi}{7}\right)\right) \\
&\quad \cdot \left(x - \left(\cos \frac{4\pi}{7} + i \sin \frac{4\pi}{7}\right)\right)\left(x - \left(\cos \frac{4\pi}{7} - i \sin \frac{4\pi}{7}\right)\right) \\
&\quad \cdot \left(x - \left(\cos \frac{6\pi}{7} + i \sin \frac{6\pi}{7}\right)\right)\left(x - \left(\cos \frac{6\pi}{7} - i \sin \frac{6\pi}{7}\right)\right) \\
&= \left(x^2 - \left(2\cos \frac{2\pi}{7}\right)x + 1\right) \cdot \left(x^2 - \left(2\cos \frac{4\pi}{7}\right)x + 1\right) \qquad (4) \\
&\quad \cdot \left(x^2 - \left(2\cos \frac{6\pi}{7}\right)x + 1\right).
\end{aligned}$$

By expanding the last expression above for $Q(x)$, the coefficient of x^5 in $Q(x)$ is

$$1 = -2\cos \frac{2\pi}{7} - 2\cos \frac{4\pi}{7} - 2\cos \frac{6\pi}{7}.$$

It then follows from (1) that $a = \frac{1}{2}$. Similarly, the coefficient of x^4 in $Q(x)$ is

$$1 = 3 + 4\left(\cos \frac{2\pi}{7} \cos \frac{4\pi}{7} + \cos \frac{4\pi}{7} \cos \frac{6\pi}{7} + \cos \frac{6\pi}{7} \cos \frac{2\pi}{7}\right),$$

and it follows from (2) that $b = -\frac{1}{2}$. Once more expanding (4) and using (3) gives the coefficient of x^3 in $Q(x)$:

$$1 = -4\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right) - 8\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7} = 4a + 8c.$$

Thus $c = -\frac{1}{8}$. The requested product is $abc = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{1}{8}\right) = \frac{1}{32}$.

OR

Applying the identity $\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$ to the expressions for a, b, and c from the first solution gives

$$\begin{aligned} b &= \cos\frac{2\pi}{7}\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7}\cos\frac{6\pi}{7} + \cos\frac{4\pi}{7}\cos\frac{6\pi}{7} \\ &= \frac{1}{2}\left(\cos\frac{6\pi}{7} + \cos\frac{2\pi}{7}\right) + \frac{1}{2}\left(\cos\frac{8\pi}{7} + \cos\frac{4\pi}{7}\right) + \frac{1}{2}\left(\cos\frac{10\pi}{7} + \cos\frac{2\pi}{7}\right) \\ &= \frac{1}{2}\left(\cos\frac{6\pi}{7} + \cos\frac{2\pi}{7}\right) + \frac{1}{2}\left(\cos\frac{6\pi}{7} + \cos\frac{4\pi}{7}\right) + \frac{1}{2}\left(\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7}\right) \\ &= -a \end{aligned}$$

and

$$\begin{aligned} c &= -\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7} \\ &= -\frac{1}{2}\left(\cos\frac{6\pi}{7} + \cos\frac{2\pi}{7}\right)\cos\frac{6\pi}{7} \\ &= -\frac{1}{2}\cos^2\frac{6\pi}{7} - \frac{1}{4}\left(\cos\frac{8\pi}{7} + \cos\frac{4\pi}{7}\right) \\ &= -\frac{1}{4}\left(1 + \cos\frac{12\pi}{7}\right) - \frac{1}{4}\left(\cos\frac{6\pi}{7} + \cos\frac{4\pi}{7}\right) \\ &= \frac{1}{4}(a-1). \end{aligned}$$

To compute a, use a double angle identity to obtain

$$\begin{aligned} \cos\frac{6\pi}{7} &= \cos\frac{8\pi}{7} = 2\cos^2\frac{4\pi}{7} - 1 \\ \cos\frac{2\pi}{7} &= \cos\frac{16\pi}{7} = 2\cos^2\frac{8\pi}{7} - 1 = 2\cos^2\frac{6\pi}{7} - 1 \\ \cos\frac{4\pi}{7} &= 2\cos^2\frac{2\pi}{7} - 1. \end{aligned}$$

Then

$$\begin{aligned} -a &= 2\left(\cos^2\frac{4\pi}{7} + \cos^2\frac{6\pi}{7} + \cos^2\frac{2\pi}{7}\right) - 3 \\ &= 2(a^2 - 2b) - 3 = 2(a^2 + 2a) - 3. \end{aligned}$$

So $2a^2 + 5a - 3 = 0$, from which it follows that $a = \frac{1}{2}$ or $a = -3$. However, $-3 < a < 3$, so $a = \frac{1}{2}$, $b = -\frac{1}{2}$, $c = -\frac{1}{8}$, and $abc = \frac{1}{32}$.

23. Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving one square on each hop and choosing at random the direction of each hop—up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example if Frieda begins in the center square and makes two hops "up", the first hop would place her in the top

row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square. Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

(A) $\dfrac{9}{16}$ **(B)** $\dfrac{5}{8}$ **(C)** $\dfrac{3}{4}$ **(D)** $\dfrac{25}{32}$ **(E)** $\dfrac{13}{16}$

Answer (D): Let m denote the middle square, c denote a corner square, and e denote an edge square not in the corner. There are four ways to reach a corner in at most four moves starting from m:

$$\begin{aligned} ec&\text{, with probability } 1 \cdot \tfrac{1}{2} \cdot = \tfrac{1}{2},\\ emec&\text{, with probability } 1 \cdot \tfrac{1}{4} \cdot 1 \cdot \tfrac{1}{2} = \tfrac{1}{8},\\ eec&\text{, with probability } 1 \cdot \tfrac{1}{4} \cdot \tfrac{1}{2} = \tfrac{1}{8}, \text{ and}\\ eeec&\text{, with probability } 1 \cdot \tfrac{1}{4} \cdot \tfrac{1}{4} \cdot \tfrac{1}{2} = \tfrac{1}{32}, \end{aligned}$$

for a total probability of $\frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{32} = \frac{25}{32}$.

OR

The matrices below show the number of ways to reach each square after 1, 2, 3, and 4 hops.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 5 & 2 \\ 5 & 4 & 5 \\ 2 & 5 & 2 \end{bmatrix} \qquad \begin{bmatrix} 10 & 9 & 10 \\ 9 & 20 & 9 \\ 10 & 9 & 10 \end{bmatrix}$$

The first matrix shows that after one hop Frieda will be on one of the side edge squares. There is exactly one way to reach each of those squares in one hop.

The second matrix shows the number of ways to reach each square in two hops. For example to reach the upper left corner, Frieda can come from the right or from below so the two 1s adjacent to the corner in the first matrix are added to make 2 in the second matrix. It follows that Frieda can reach one of the corner squares in two hops in $4 \cdot 2 = 8$ different ways out of $4^2 = 16$ possible two-hop sequences.

Now consider the third hop assuming that a corner square has not yet been reached, making sure to count the wrap-around hops. For example the top row middle square can be reached from the center square or from the bottom row middle square (but not from the adjacent corner squares). The values 4 and 1 are added from the second matrix to make 5 in the third matrix. The matrix shows that Frieda can reach a corner square on her third hop in $4 \cdot 2 = 8$ different ways out of $4^3 = 64$ possible three-hop sequences.

Finally the fourth matrix shows that Frieda can reach a corner square on her fourth hop in $4 \cdot 10 = 40$ different ways out of $4^4 = 256$ possible four-hop sequences. The probability of landing on a corner square on one of the four hops is therefore

$$\frac{8}{16} + \frac{8}{64} + \frac{40}{256} = \frac{25}{32}.$$

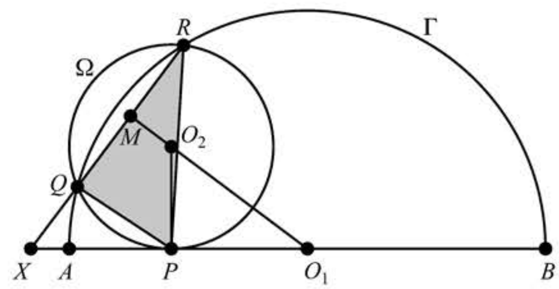
24. Semicircle Γ has diameter $\overline{AB}$ of length 14. Circle Ω lies tangent to $\overline{AB}$ at a point P and intersects Γ at points Q and R. If $QR = 3\sqrt{3}$ and $\angle QPR = 60^\circ$, then the area of $\triangle PQR$ is $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. What is $a + b + c$?

(A) 110 **(B)** 114 **(C)** 118 **(D)** 122 **(E)** 126

Answer (D): Let O_1 and O_2 be the centers of Γ and Ω, respectively, and let lines QR and AB intersect at point X. Without loss of generality, let Q lie between X and R. Then

$$\begin{aligned} [\triangle PQR] &= [\triangle XPR] - [\triangle XPQ] \\ &= \tfrac{1}{2} \cdot XP \cdot XR \cdot \sin(\angle PXQ) - \tfrac{1}{2} \cdot XP \cdot XQ \cdot \sin(\angle PXQ) \\ &= \frac{1}{2} \cdot XP \cdot QR \cdot \sin(\angle PXQ), \end{aligned}$$

where the brackets indicate area. Let M be the midpoint of $\overline{QR}$, which lies on line O_1O_2.



Triangle QO_2R is isosceles with base length $3\sqrt{3}$ and vertex angle $120°$, so $O_2M = \frac{3}{2}$ and $O_2Q = O_2R = 3$. Also

$$O_1M = \sqrt{(O_1Q)^2 - (MQ)^2} = \sqrt{7^2 - \left(\frac{3\sqrt{3}}{2}\right)^2} = \frac{13}{2},$$

so

$$O_1O_2 = O_1M - O_2M = \frac{13}{2} - \frac{3}{2} = 5.$$

Hence $O_1P = \sqrt{(O_1O_2)^2 - (O_2P)^2} = \sqrt{5^2 - 3^2} = 4$.

The similarity $\triangle XMO_1 \sim \triangle O_2PO_1$ implies

$$\sin(\angle PXQ) = \sin(\angle PO_2O_1) = \frac{4}{5},$$

so

$$O_1X = \frac{O_1M}{\sin(\angle PXQ)} = \frac{13}{2} \div \frac{4}{5} = \frac{65}{8}.$$

Therefore

$$XP = O_1X - O_1P = \frac{65}{8} - 4 = \frac{33}{8},$$

so

$$[\triangle PQR] = \tfrac{1}{2} \cdot XP \cdot QR \cdot \sin(\angle PXQ) = \frac{1}{2} \cdot \frac{33}{8} \cdot 3\sqrt{3} \cdot \frac{4}{5} = \frac{99\sqrt{3}}{20}.$$

The requested sum is $99 + 3 + 20 = 122$.

25. Let $d(n)$ denote the number of positive integers that divide n, including 1 and n. For example, $d(1) = 1$, $d(2) = 2$, and $d(12) = 6$. (This function is known as the *divisor function*.) Let

$$f(n) = \frac{d(n)}{\sqrt[3]{n}}.$$

There is a unique positive integer N such that $f(N) > f(n)$ for all positive integers $n \neq N$. What is the sum of the digits of N ?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer (E): Let $n = \prod_p p^{\alpha_p}$ be the prime factorization of n. Then $d(n) = \prod_p (\alpha_p + 1)$, so

$$f(n) = \prod_p \frac{\alpha_p + 1}{\sqrt[3]{p^{\alpha_p}}} = \prod_p f_p(\alpha_p),$$

where

$$f_p(\alpha) = \frac{\alpha + 1}{\sqrt[3]{p^{\alpha}}}.$$

For each prime p, the exponent α for which $f_p(\alpha)$ is greatest must be found.

Note that $f_p(\alpha+1) \neq f_p(\alpha)$ because

$$\left(\frac{\alpha+2}{\alpha+1}\right)^3 \neq p\,.$$

Here the left-hand side is a decreasing function of α that tends to 1 as $\alpha \to \infty$. Hence the set of α such that $f_p(\alpha+1) > f_p(\alpha)$ is finite (and possibly empty). It follows that if

$$p > 2^3 = 8, \tag{5}$$

then the optimal choice is $\alpha_p = 0$. Similarly, if

$$8 > p > \left(\frac{3}{2}\right)^3 = \frac{27}{8}, \tag{6}$$

then the optimal choice is $\alpha_p = 1$. If

$$\frac{27}{8} > p > \left(\frac{4}{3}\right)^3 = \frac{64}{27}, \tag{7}$$

then the optimal choice is $\alpha_p = 2$. Finally, if

$$\frac{64}{27} > p > \left(\frac{5}{4}\right)^3 = \frac{125}{64}, \tag{8}$$

then the optimal choice is $\alpha_p = 3$. Note that (5) holds for all $p > 7$, that (6) holds for $p = 5$ and $p = 7$, that (7) holds for $p = 3$, and that (8) holds for $p = 2$. Thus the extremal N is $N = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$. The requested sum of digits is $2+5+2+0 = 9$.

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