MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

69th Annual

AMC 12B

American Mathematics Competition 12B

Thursday, February 15, 2018

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 12 were prepared by
MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the
co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (A):** The total area of cornbread is $20 \cdot 18 = 360$ in^2. Because each piece of cornbread has area $2 \cdot 2 = 4$ in^2, the pan contains $360 \div 4 = 90$ pieces of cornbread.

OR

When cut, there are $20 \div 2 = 10$ pieces of cornbread along a long side of the pan and $18 \div 2 = 9$ pieces along a short side, so there are $10 \cdot 9 = 90$ pieces.

2. **Answer (D):** Sam covered $\frac{1}{2} \cdot 60 = 30$ miles during the first 30 minutes and $\frac{1}{2} \cdot 65 = 32.5$ miles during the second 30 minutes, so he needed to cover $96 - 30 - 32.5 = 33.5$ miles during the last 30 minutes. Thus his average speed during the last 30 minutes was

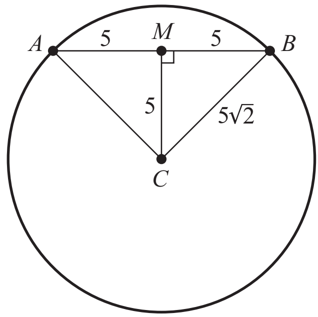
$$\frac{33.5 \text{ miles}}{\frac{1}{2} \text{ hour}} = 67 \text{ mph}.$$

3. **Answer (B):** The line with slope 2 containing the point $(40, 30)$ has the equation $y - 30 = 2(x - 40)$. Similarly, the line with slope 6 containing the point $(40, 30)$ has the equation $y - 30 = 6(x - 40)$. To find the x-intercepts of these two lines, let $y = 0$ and solve for x separately in each of these two equations. With the first equation the x-intercept is 25, and with the second equation the x-intercept is 35. Thus the distance between the two x-intercepts is $|25 - 35| = 10$.

OR

As the line with slope 2 rises from $y = 0$ to $y = 30$, x increases by 15. As the line with slope 6 rises from $y = 0$ to $y = 30$, x increases by 5. Thus the distance between the x-intercepts is $|15 - 5| = 10$.

4. **Answer (B):** Let the chord have endpoints A and B, and let C be the center of the circle. The segment from C to the midpoint M of $\overline{AB}$ is perpendicular to $\overline{AB}$ and has length 5. This creates the $45-45-90^\circ$ triangle CMB, whose sides are 5, 5, and $CB = 5\sqrt{2}$. Therefore the radius of the circle is $5\sqrt{2}$, and the area of the circle is $\pi \cdot \left(5\sqrt{2}\right)^2 = 50\pi$.



5. **Answer (D):** The number of qualifying subsets equals the difference between the total number of subsets of $\{2,3,4,5,6,7,8,9\}$ and the number of such subsets containing no prime numbers, which is the number of subsets of $\{4,6,8,9\}$. A set with n elements has 2^n subsets, so the requested number is $2^8 - 2^4 = 256 - 16 = 240$.

OR

A subset meeting the condition must be the union of a nonempty subset of $\{2,3,5,7\}$ and a subset of $\{4,6,8,9\}$. There are $2^4 - 1 = 15$ of the former and $2^4 = 16$ of the latter, which gives $15 \cdot 16 = 240$ choices in all.

6. **Answer (B):** The cost of 1 can is $\frac{Q}{S}$ quarters, which is $\frac{Q}{4S}$ dollars. Hence the number of cans that can be purchased with D dollars is

$$\frac{D}{\left(\frac{Q}{4S}\right)} = \frac{4DS}{Q}.$$

7. **Answer (C):** The change of base formula states that $\log_a b = \frac{\log b}{\log a}$. Thus the product telescopes:

$$\begin{aligned}\frac{\log 7}{\log 3} \cdot \frac{\log 9}{\log 5} \cdot \frac{\log 11}{\log 7} \cdot \frac{\log 13}{\log 9} \cdots \frac{\log 25}{\log 21} \cdot \frac{\log 27}{\log 23} &= \frac{\log 25}{\log 3} \cdot \frac{\log 27}{\log 5} \\ &= \frac{\log 5^2}{\log 3} \cdot \frac{\log 3^3}{\log 5} \\ &= \frac{2\log 5}{\log 3} \cdot \frac{3\log 3}{\log 5}. \\ &= 6.\end{aligned}$$

OR

Let

$$a = \log_3 7 \cdot \log_7 11 \cdot \log_{11} 15 \cdot \log_{15} 19 \cdot \log_{19} 23 \cdot \log_{23} 27$$

and

$$b = \log_5 9 \cdot \log_9 13 \cdot \log_{13} 17 \cdot \log_{17} 21 \cdot \log_{21} 25.$$

The required product is ab. Now

$$\begin{aligned}
b &= \log_5 9 \cdot \log_9 13 \cdot \log_{13} 17 \cdot \log_{17} 21 \cdot \log_{21} 25 \\
&= \log_5 9^{\log_9 13} \cdot \log_{13} 17 \cdot \log_{17} 21 \cdot \log_{21} 25 \\
&= \log_5 13 \cdot \log_{13} 17 \cdot \log_{17} 21 \cdot \log_{21} 25 \\
&= \log_5 13^{\log_{13} 17} \cdot \log_{17} 21 \cdot \log_{21} 25 \\
&= \log_5 17 \cdot \log_{17} 21 \cdot \log_{21} 25 \\
&= \log_5 17^{\log_{17} 21} \cdot \log_{21} 25 \\
&= \log_5 21 \cdot \log_{21} 25 \\
&= \log_5 21^{\log_{21} 25} \\
&= \log_5 25 \\
&= 2.
\end{aligned}$$

Similarly, $a = \log_3 27 = 3$, so $ab = 2 \cdot 3 = 6$.

8. **Answer (C):** Let O be the center of the circle. Triangle ABC is a right triangle, and O is the midpoint of the hypotenuse $\overline{AB}$. Then $\overline{OC}$ is a radius, and it is also one of the medians of the triangle. The centroid is located one third of the way along the median from O to C, so the centroid traces out a circle with center O and radius $\frac{1}{3} \cdot 12 = 4$ (except for the two missing points corresponding to $C = A$ or $C = B$). The area of this smaller circle is then $\pi \cdot 4^2 = 16\pi \approx 16 \cdot 3.14 \approx 50$.

9. **Answer (E):** Note that the sum of the first 100 positive integers is $\frac{1}{2} \cdot 100 \cdot 101 = 5050$. Then

$$\begin{aligned}
\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j) &= \sum_{i=1}^{100}\sum_{j=1}^{100} i + \sum_{i=1}^{100}\sum_{j=1}^{100} j \\
&= \sum_{j=1}^{100}\sum_{i=1}^{100} i + \sum_{i=1}^{100}\sum_{j=1}^{100} j
\end{aligned}$$

$$= 100 \sum_{i=1}^{100} i + 100 \sum_{j=1}^{100} j$$
$$= 100(5050 + 5050)$$
$$= 1{,}010{,}000.$$

OR

Note that the sum of the first 100 positive integers is $\frac{1}{2} \cdot 100 \cdot 101 = 5050$. Then

$$\begin{aligned}\sum_{i=1}^{100} \sum_{j=1}^{100} (i+j) &= \sum_{i=1}^{100} \big((i+1) + (i+2) + \cdots + (i+100)\big) \\ &= \sum_{i=1}^{100} (100i + 5050) \\ &= 100 \cdot 5050 + 100 \cdot 5050 \\ &= 1{,}010{,}000.\end{aligned}$$

OR

The sum contains 10,000 terms, and the average value of both i and j is $\frac{101}{2}$, so the sum is equal to

$$10{,}000 \left(\frac{101}{2} + \frac{101}{2} \right) = 1{,}010{,}000.$$

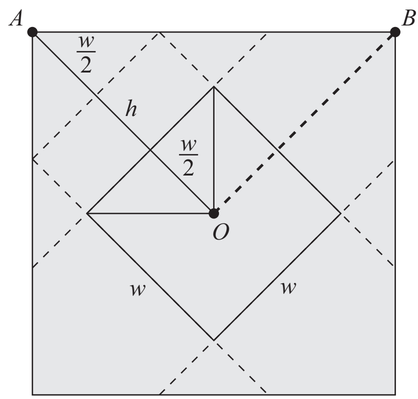
10. **Answer (D):** The list has $2018 - 10 = 2008$ entries that are not equal to the mode. Because the mode is unique, each of these 2008 entries can occur at most 9 times. There must be at least $\left\lceil \frac{2008}{9} \right\rceil = 224$ distinct values in the list that are different from the mode, because if there were fewer than this many such values, then the size of the list would be at most $9 \cdot \left(\left\lceil \frac{2008}{9} \right\rceil - 1 \right) + 10 = 2017 < 2018$. (The ceiling function notation $\lceil x \rceil$ represents the least integer greater than or equal to x.) Therefore the least possible number of distinct values that can occur in the list is 225. One list satisfying the conditions of the problem contains 9 instances of each of the numbers 1 through 223, 10 instances of the number 224, and one instance of 225.

11. **Answer (A):** The figure shows that the distance AO from a corner of the wrapping paper to the center is

$$\frac{w}{2} + h + \frac{w}{2} = w + h.$$

The side of the wrapping paper, $\overline{AB}$ in the figure, is the hypotenuse of a $45-45-90^\circ$ right triangle, so its length is $\sqrt{2} \cdot AO = \sqrt{2}(w+h)$. Therefore the area of the wrapping paper is

$$\left(\sqrt{2}(w+h)\right)^2 = 2(w+h)^2.$$



OR

The area of the wrapping paper, excluding the four small triangles indicated by the dashed lines, is equal to the surface area of the box, which is $2w^2 + 4wh$. The four triangles are isosceles right triangles with leg length h, so their combined area is $4 \cdot \frac{1}{2}h^2 = 2h^2$. Thus the total area of the wrapping paper is $2w^2 + 4wh + 2h^2 = 2(w+h)^2$.

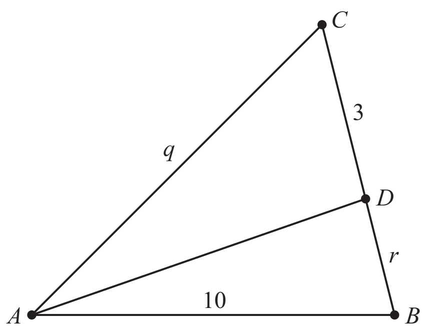
12. **Answer (C):** Let $q = AC$ and $r = BD$. By the Angle Bisector Theorem,

$$\frac{AC}{CD} = \frac{AB}{BD}, \quad \text{which means} \quad \frac{q}{3} = \frac{10}{r}, \quad \text{so} \quad r = \frac{30}{q}.$$

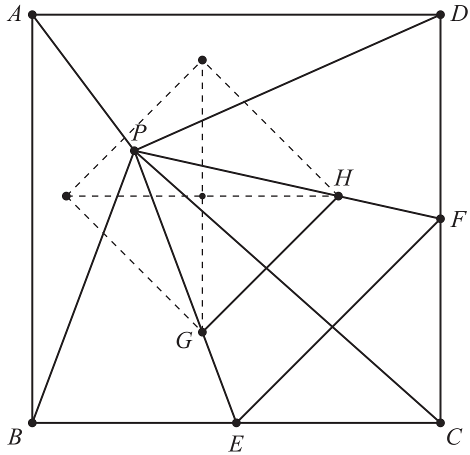
The possible values of AC can be determined by considering the three Triangle Inequalities in $\triangle ABC$.

- $AC + BC > AB$, which means $q + 3 + r > 10$. Substituting for r and simplifying gives $q^2 - 7q + 30 > 0$, which always holds because $q^2 - 7q + 30 = \left(q - \frac{7}{2}\right)^2 + \frac{71}{4}$.
- $BC + AB > AC$, which means $3 + r + 10 > q$. Substituting $r = \frac{30}{q}$, simplifying, and factoring gives $(q - 15)(q + 2) < 0$, which holds if and only if $-2 < q < 15$.
- $AB + AC > BC$, which means $10 + q > 3 + r$. Substituting $r = \frac{30}{q}$, simplifying, and factoring gives $(q + 10)(q - 3) > 0$, which holds if and only if $q > 3$ or $q < -10$.

Combining these inequalities shows that the set of possible values of q is the open interval $(3, 15)$, and the requested sum of the endpoints of the interval is $3 + 15 = 18$.



13. **Answer (C):** Let E and F be the midpoints of sides $\overline{BC}$ and $\overline{CD}$, respectively. Let G and H be the centroids of $\triangle BCP$ and $\triangle CDP$, respectively. Then G is on $\overline{PE}$, a median of $\triangle BCP$, a distance $\frac{2}{3}$ of the way from P to E. Similarly, H is on $\overline{PF}$ a distance $\frac{2}{3}$ of the way from P to F. Thus $\overline{GH}$ is parallel to $\overline{EF}$ and $\frac{2}{3}$ the length of $\overline{EF}$. Because $BC = 30$, it follows that $EC = 15$, $EF = 15\sqrt{2}$, and $GH = 10\sqrt{2}$. The midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ form a square, so the centroids of $\triangle ABP$, $\triangle BCP$, $\triangle CDP$, and $\triangle DAP$ also form a square, and that square has side length $10\sqrt{2}$. The requested area is $\left(10\sqrt{2}\right)^2 = 200$.



OR

Place the figure in the coordinate plane with $A = (0, 30)$, $B = (0, 0)$, $C = (30, 0)$, $D = (30, 30)$, and $P = (3x, 3y)$. Then the coordinates of the centroids of the four triangles are found by averaging the coordinates of the vertices: $(x, y + 10)$, $(x + 10, y)$, $(x + 20, y + 10)$, and $(x + 10, y + 20)$. It can be seen that the quadrilateral formed by the centroids is a square with center $(x + 10, y + 10)$ and vertices aligned vertically and horizontally. Its area is half the product of the lengths of its diagonals, $\frac{1}{2} \cdot 20 \cdot 20 = 200$.

Note: As the solutions demonstrate, the inner quadrilateral is always a square, and its size is independent of the location of point P. The location of the square within square $ABCD$ does depend on the location of P.

14. **Answer (E):** Let Chloe be n years old today, so she is $n - 1$ years older than Zoe. For integers $y \geq 0$, Chloe's age will be a multiple of Zoe's age y years from now if and only if

$$\frac{n + y}{1 + y} = 1 + \frac{n - 1}{1 + y}$$

is an integer, that is, $1 + y$ is a divisor of $n - 1$. Thus $n - 1$ has exactly 9 positive integer divisors, so the prime factorization of $n - 1$ has one of the two forms p^2q^2 or p^8. There are no two-digit integers of the form p^8, and the only one of the form p^2q^2 is $2^2 \cdot 3^2 = 36$. Therefore Chloe is 37 years old today, and Joey is 38. His age will be a multiple of Zoe's age in y years if and only if $1 + y$ is a divisor of

$38 - 1 = 37$. The nonnegative integer solutions for y are 0 and 36, so the only other time Joey's age will be a multiple of Zoe's age will be when he is $38 + 36 = 74$ years old. The requested sum is $7 + 4 = 11$.

15. **Answer (A):** Let $\underline{a}\,\underline{b}\,\underline{c}$ be a 3-digit positive odd multiple of 3 that does not include the digit 3. There are 8 possible values for a, namely 1, 2, 4, 5, 6, 7, 8, and 9, and 4 possible values for c, namely 1, 5, 7, and 9. The possible values of b can be put into three groups of the same size: $\{0, 6, 9\}$, $\{1, 4, 7\}$, and $\{2, 5, 8\}$. Recall that an integer is divisible by 3 if and only if the sum of its digits is divisible by 3. Thus for every possible pair of digits (a, c), the choices for b such that $\underline{a}\,\underline{b}\,\underline{c}$ is divisible by 3 constitute one of those groups. Hence the answer is $8 \cdot 4 \cdot 3 = 96$.

OR

There are $\frac{1}{2} \cdot \frac{1}{3} \cdot 900 = 150$ odd 3-digit multiples of 3. Those including the digit 3 have the form $\underline{a}\,\underline{b}\,\underline{3}$, $\underline{a}\,\underline{3}\,\underline{b}$, or $\underline{3}\,\underline{a}\,\underline{b}$. There are 30 of the first type, where the number $\underline{a}\,\underline{b}$ is one of 12, 15, 18, ..., 99. There are 15 of the second type, where the number $\underline{a}\,\underline{b}$ is one of 15, 21, 27, ..., 99. There are 17 of the third type, where the number $\underline{a}\,\underline{b}$ is one of 03, 09, 15, ..., 99. The numbers 303, 339, 363, 393, 633, and 933 are each counted twice, and 333 is counted 3 times. By the Inclusion–Exclusion Principle there are $150 - (30 + 15 + 17) + (1 \cdot 6 + 2 \cdot 1) = 96$ such numbers.

16. **Answer (B):** The answer would be the same if the equation were $z^8 = 81$, resulting from a horizontal translation of 6 units. The solutions to this equation are the 8 eighth roots of 81, each of which is $\sqrt[8]{3^4} = \sqrt{3}$ units from the origin. These 8 points form a regular octagon. The triangle of minimum area occurs when the vertices of the triangle are consecutive vertices of the octagon, so without loss of generality they have coordinates $A\left(\frac{1}{2}\sqrt{6}, \frac{1}{2}\sqrt{6}\right)$, $B\left(\sqrt{3}, 0\right)$, and $C\left(\frac{1}{2}\sqrt{6}, -\frac{1}{2}\sqrt{6}\right)$. This triangle has base $AC = \sqrt{6}$ and height $\sqrt{3} - \frac{1}{2}\sqrt{6}$, so its area is

$$\frac{1}{2} \cdot \sqrt{6} \cdot \left(\sqrt{3} - \frac{1}{2}\sqrt{6}\right) = \frac{3}{2}\sqrt{2} - \frac{3}{2}.$$

OR

The complex solutions form a regular octagon centered at $z = -6$. The distance from the center to any one of the vertices is $\sqrt[8]{81} =$

$\sqrt[8]{3^4} = \sqrt{3}$. By the Law of Cosines, the side length s of the octagon satisfies

$$s^2 = \left(\sqrt{3}\right)^2 + \left(\sqrt{3}\right)^2 - 2 \cdot \sqrt{3} \cdot \sqrt{3} \cdot \cos 45^\circ = 6 - 6 \cdot \frac{\sqrt{2}}{2} = 6 - 3\sqrt{2}.$$

The least possible area of $\triangle ABC$ occurs when two of the sides of $\triangle ABC$ are adjacent sides of the octagon; the angle between these two sides is 135°. The sine formula for area gives

$$\frac{1}{2} \cdot \left(6 - 3\sqrt{2}\right) \cdot \sin 135^\circ = \frac{1}{2} \cdot \left(6 - 3\sqrt{2}\right) \cdot \frac{\sqrt{2}}{2} = \frac{3}{2}\sqrt{2} - \frac{3}{2}.$$

17. **Answer (A):** The first inequality is equivalent to $9p > 5q$, and because both sides are integers, it follows that $9p - 5q \geq 1$. Similarly, $4q - 7p \geq 1$. Now

$$\begin{aligned} \frac{1}{63} = \frac{4}{7} - \frac{5}{9} &= \left(\frac{p}{q} - \frac{5}{9}\right) + \left(\frac{4}{7} - \frac{p}{q}\right) \\ &= \frac{9p - 5q}{9q} + \frac{4q - 7p}{7q} \\ &\geq \frac{1}{9q} + \frac{1}{7q} \\ &= \frac{16}{63q}. \end{aligned}$$

Thus $q \geq 16$. Because

$$\frac{8}{16} < \frac{5}{9} < \frac{9}{16} < \frac{4}{7} < \frac{10}{16},$$

the fraction $\frac{9}{16}$ lies in the required interval, but $\frac{8}{16}$ and $\frac{10}{16}$ do not. Therefore when q is as small as possible, $q = 16$ and $p = 9$, and the requested difference is $16 - 9 = 7$.

Note: A theorem in the study of Farey fractions states that if $\frac{a}{p} < \frac{b}{q}$ and $bp - aq = 1$, then the rational number with least denominator between $\frac{a}{p}$ and $\frac{b}{q}$ is $\frac{a+b}{p+q}$.

18. **Answer (B):** Applying the recursion for several steps leads to the conjecture that

$$f(n) = \begin{cases} n+2 & \text{if } n \equiv 0 \pmod{6}, \\ n & \text{if } n \equiv 1 \pmod{6}, \\ n-1 & \text{if } n \equiv 2 \pmod{6}, \\ n & \text{if } n \equiv 3 \pmod{6}, \\ n+2 & \text{if } n \equiv 4 \pmod{6}, \\ n+3 & \text{if } n \equiv 5 \pmod{6}. \end{cases}$$

The conjecture can be verified using the strong form of mathematical induction with two base cases and six inductive steps. For example, if $n \equiv 2 \pmod{6}$, then $n = 6k+2$ for some nonnegative integer k and

$$\begin{aligned} f(n) &= f(6k+2) \\ &= f(6k+1) - f(6k) + 6k + 2 \\ &= (6k+1) - (6k+2) + 6k + 2 \\ &= 6k+1 \\ &= n-1. \end{aligned}$$

Therefore $f(2018) = f(6 \cdot 336 + 2) = 2018 - 1 = 2017$.

OR

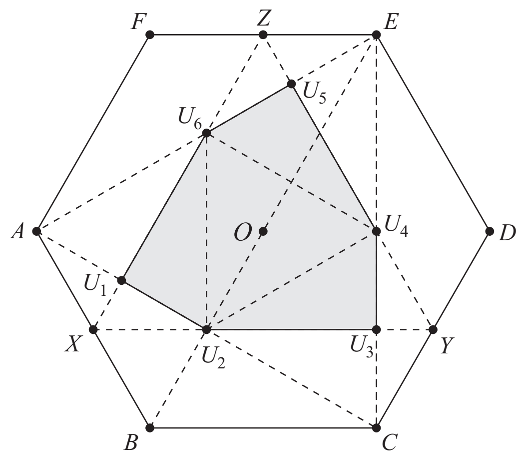
Note that

$$\begin{aligned} f(n) &= f(n-1) - f(n-2) + n \\ &= [f(n-2) - f(n-3) + (n-1)] - f(n-2) + n \\ &= -[f(n-4) - f(n-5) + (n-3)] + 2n - 1 \\ &= -[f(n-5) - f(n-6) + (n-4)] + f(n-5) + n + 2 \\ &= f(n-6) + 6. \end{aligned}$$

It follows that $f(2018) = f(2) + 2016 = 2017$.

19. **Answer (C):** Let d be the next divisor of n after 323. Then $\gcd(d, 323) \neq 1$, because otherwise $n \geq 323d > 323^2 > 100^2 = 10000$, contrary to the fact that n is a 4-digit number. Therefore $d - 323 \geq \gcd(d, 323) > 1$. The prime factorization of 323 is $17 \cdot 19$. Thus the next divisor of n is at least $323 + 17 = 340 = 17 \cdot 20$. Indeed, 340 will be the next number in Mary's list when $n = 17 \cdot 19 \cdot 20 = 6460$.

20. **Answer (C):** Let O be the center of the regular hexagon. Points B, O, E are collinear and $BE = BO + OE = 2$. Trapezoid $FABE$ is isosceles, and $\overline{XZ}$ is its midline. Hence $XZ = \frac{3}{2}$ and analogously $XY = ZY = \frac{3}{2}$.



Denote by U_1 the intersection of $\overline{AC}$ and $\overline{XZ}$ and by U_2 the intersection of $\overline{AC}$ and $\overline{XY}$. It is easy to see that $\triangle AXU_1$ and $\triangle U_2XU_1$ are congruent $30-60-90°$ right triangles.

By symmetry the area of the convex hexagon enclosed by the intersection of $\triangle ACE$ and $\triangle XYZ$, shaded in the figure, is equal to the area of $\triangle XYZ$ minus 3 times the area of $\triangle U_2XU_1$. The hypotenuse of $\triangle U_2XU_1$ is $XU_2 = AX = \frac{1}{2}$, so the area of $\triangle U_2XU_1$ is

$$\frac{1}{2} \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{32}\sqrt{3}.$$

The area of the equilateral triangle XYZ with side length $\frac{3}{2}$ is equal to $\frac{1}{4}\sqrt{3} \cdot \left(\frac{3}{2}\right)^2 = \frac{9}{16}\sqrt{3}$. Hence the area of the shaded hexagon is

$$\frac{9}{16}\sqrt{3} - 3 \cdot \frac{1}{32}\sqrt{3} = 3\sqrt{3}\left(\frac{3}{16} - \frac{1}{32}\right) = \frac{15}{32}\sqrt{3}.$$

OR

Let U_1 and U_2 be as above, and continue labeling the vertices of the shaded hexagon counterclockwise with U_3, U_4, U_5, and U_6 as shown. The area of $\triangle ACE$ is half the area of hexagon $ABCDEF$. Triangle $U_2U_4U_6$ is the midpoint triangle of $\triangle ACE$, so its area is $\frac{1}{4}$ of the area of $\triangle ACE$, and thus $\frac{1}{8}$ of the area of $ABCDEF$. Each of $\triangle U_2U_3U_4$, $\triangle U_4U_5U_6$, and $\triangle U_6U_1U_2$ is congruent to half of $\triangle U_2U_4U_6$, so the total shaded area is $\frac{5}{2}$ times the area of $\triangle U_2U_4U_6$ and therefore $\frac{5}{2}\cdot\frac{1}{8} = \frac{5}{16}$ of the area of $ABCDEF$. The area of $ABCDEF$ is $6\cdot\frac{\sqrt{3}}{4}\cdot 1^2$, so the requested area is $\frac{15}{32}\sqrt{3}$.

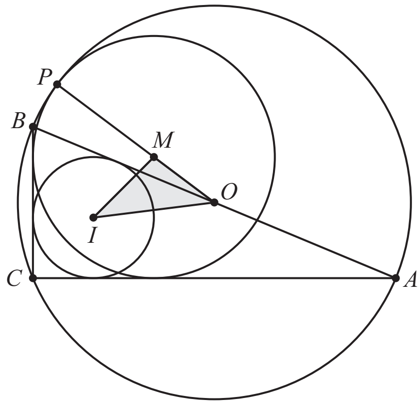
21. **Answer (E):** Place the figure on coordinate axes with coordinates $A(12,0)$, $B(0,5)$, and $C(0,0)$. The center of the circumscribed circle is the midpoint of the hypotenuse of right triangle ABC, so the coordinates of O are $\left(6, \frac{5}{2}\right)$. The radius r of the inscribed circle equals the area of the triangle divided by its semiperimeter, which here is $30 \div 15 = 2$, so the center of the inscribed circle is $I(2,2)$. Because the circle with center M is tangent to both coordinate axes, its center has coordinates (ρ,ρ), where ρ is its radius. Let P be the point of tangency of this circle and the circumscribed circle. Then M, P, and O are collinear because $\overline{MP}$ and $\overline{OP}$ are perpendicular to the common tangent line at P. Thus $MO = OP - MP = \frac{13}{2} - \rho$. By the distance formula, $MO = \sqrt{(\rho-6)^2 + (\rho - \frac{5}{2})^2}$. Equating these expressions and solving for ρ shows that $\rho = 4$. The area of $\triangle MOI$ can now be computed using the shoelace formula:

$$\left|\frac{4\cdot\frac{5}{2} + 6\cdot 2 + 2\cdot 4 - (4\cdot 6 + \frac{5}{2}\cdot 2 + 2\cdot 4)}{2}\right| = \frac{7}{2}.$$

Alternatively, the area can be computed as $\frac{1}{2}$ times MI, which by the distance formula is $\sqrt{(4-2)^2+(4-2)^2} = 2\sqrt{2}$, times the distance from point O to the line MI, whose equation is $x - y + 0 = 0$. This last value is

$$\frac{\left|1\cdot 6 + (-1)\cdot\frac{5}{2} + 0\right|}{\sqrt{1^2 + (-1)^2}} = \frac{7}{4}\sqrt{2},$$

so the area is $\frac{1}{2}\cdot(2\sqrt{2})\cdot\frac{7}{4}\sqrt{2} = \frac{7}{2}$.



22. **Answer (D):** Let $P(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are integers between 0 and 9, inclusive. The condition $P(-1) = -9$ is equivalent to $-a + b - c + d = -9$. Adding 18 to both sides gives $(9 - a) + b + (9 - c) + d = 9$ where $0 \leq 9 - a, b, 9 - c, d \leq 9$. By the stars and bars argument, there are $\binom{9+4-1}{4-1} = \binom{12}{3} = 220$ nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 = 9$. Each of these give rise to one of the desired polynomials.

OR

With the notation above, note that $(a + c) - (b + d) = 9$ can occur in several ways: $b+d = k$, $a+c = 9+k$ where $k = 0, 1, 2, \ldots, 9$. There are $k+1$ solutions to $b+d = k$ and $10-k$ solutions to $a+c = 9+k$ under the restrictions on a, b, c, and d, yielding $\sum_{k=0}^{9}(k+1)(10-k) = 220$ solutions in all.

23. **Answer (C):** To travel from A to B, one could circle $135°$ east along the equator and then $45°$ north. Construct an x-y-z coordinate system with origin at Earth's center C, the positive x-axis running through A, the positive y-axis running through the equator at $160°$ west longitude, and the positive z-axis running through the North Pole. Set Earth's radius to be 1. The coordinates of A are $(1, 0, 0)$. Let b be the y-coordinate of B; note that $b > 0$. Then the x-coordinate of B will be $-b$, and the z-coordinate will be $\sqrt{2}b$. Because the distance from the center of Earth is 1,

$$\sqrt{(-b)^2 + b^2 + \left(\sqrt{2}b\right)^2} = 1,$$

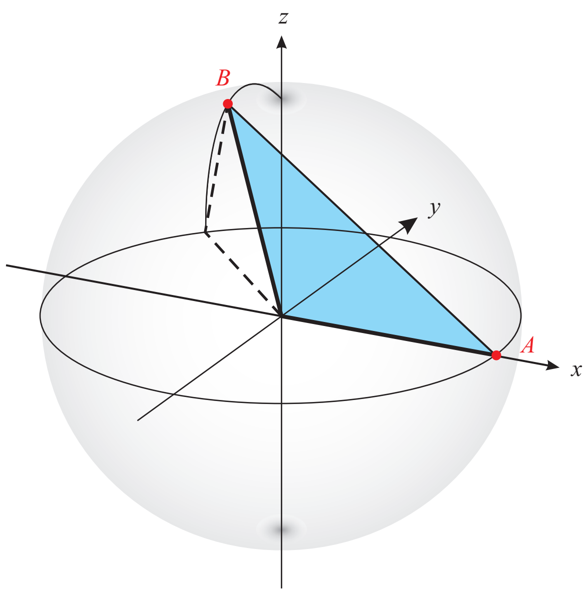
so $b = \frac{1}{2}$, and the coordinates are $\left(-\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$. The distance AB is therefore

$$\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{3}.$$

Applying the Law of Cosines to $\triangle ACB$ gives

$$3 = 1 + 1 - 2 \cdot 1 \cdot 1 \cdot \cos \angle ACB,$$

so $\cos \angle ACB = -\frac{1}{2}$ and $\angle ACB = 120°$. An alternative to using the Law of Cosines to find $\cos \angle ACB$ is to compute the dot product of the unit vectors $(1, 0, 0)$ and $\left(-\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$.



24. **Answer (C):** Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Then $0 \leq \{x\} < 1$. The given equation is equivalent to $x^2 = 10{,}000\{x\}$, that is,

$$\frac{x^2}{10{,}000} = \{x\}.$$

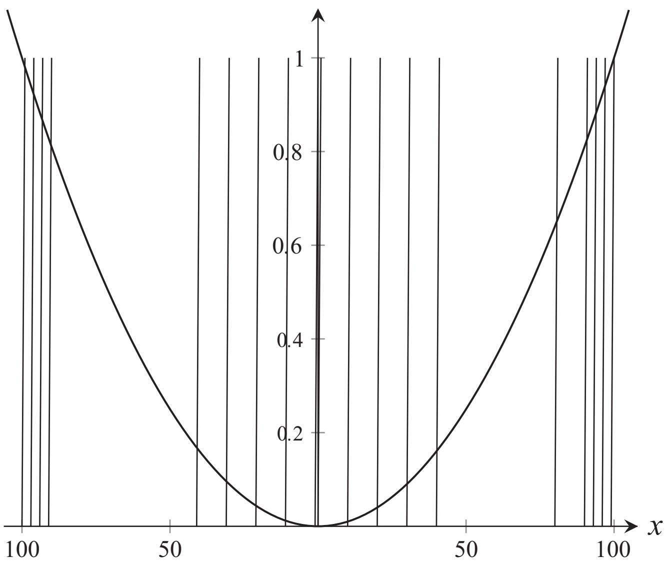
Therefore if x satisfies the equation, then

$$0 \leq \frac{x^2}{10{,}000} < 1.$$

This implies that $x^2 < 10{,}000$, so $-100 < x < 100$. The figure shows a sketch of the graphs of

$$f(x) = \frac{x^2}{10{,}000} \qquad \text{and} \qquad g(x) = \{x\}$$

for $-100 < x < 100$ on the same coordinate axes. The graph of g consists of the 200 half-open line segments with slope 1 connecting the points $(k, 0)$ and $(k+1, 1)$ for $k = -100, -99, \ldots, 98, 99$. (The endpoints of these intervals that lie on the x-axis are part of the graph, but the endpoints with y-coordinate 1 are not.) It is clear that there is one intersection point for x lying in each of the intervals $[-100, -99)$, $[-99, -98)$, $[-98, -97)$, $\ldots$, $[-1, 0)$, $[0, 1)$, $[1, 2)$, $\ldots$, $[97, 98)$, $[98, 99)$ but no others. Thus the equation has 199 solutions.



OR

The solutions to the equation correspond to points of intersection of the graphs $y = 10000\lfloor x \rfloor$ and $y = 10000x - x^2$. There will be a point of intersection any time the parabola intersects the half-open horizontal segment from the point $(a, 10000a)$ to the point $(a+1, 10000a)$, where a is an integer. This occurs for every integer value of a for which

$$10000a - a^2 \le 10000a < 10000(a+1) - (a+1)^2.$$

This is equivalent to $(a+1)^2 < 10000$, which occurs if and only if $-101 < a < 99$. Thus points of intersection occur on the intervals $[a, a+1)$ for $a = -100, -99, -98, \ldots, -1, 0, 1, \ldots, 97, 98$, resulting in 199 points of intersection.

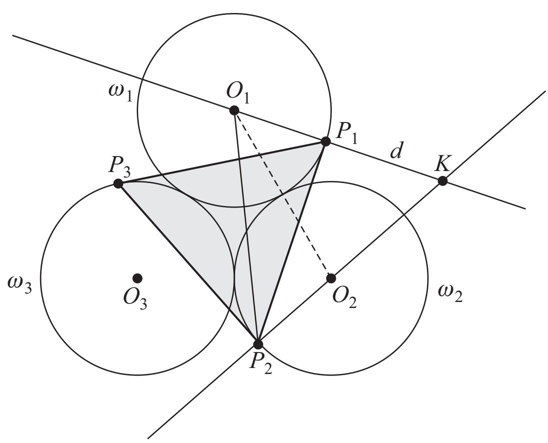
25. **Answer (D):** Let O_i be the center of circle ω_i for $i = 1, 2, 3$, and let K be the intersection of lines O_1P_1 and O_2P_2. Because $\angle P_1P_2P_3 = 60^\circ$, it follows that $\triangle P_2KP_1$ is a $30-60-90^\circ$ triangle. Let $d = P_1K$; then $P_2K = 2d$ and $P_1P_2 = \sqrt{3}d$. The Law of Cosines in $\triangle O_1KO_2$ gives

$$8^2 = (d+4)^2 + (2d-4)^2 - 2(d+4)(2d-4)\cos 60^\circ,$$

which simplifies to $3d^2 - 12d - 16 = 0$. The positive solution is $d = 2 + \frac{2}{3}\sqrt{21}$. Then $P_1P_2 = \sqrt{3}d = 2\sqrt{3} + 2\sqrt{7}$, and the required area is

$$\frac{\sqrt{3}}{4} \cdot \left(2\sqrt{3} + 2\sqrt{7}\right)^2 = 10\sqrt{3} + 6\sqrt{7} = \sqrt{300} + \sqrt{252}.$$

The requested sum is $300 + 252 = 552$.



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