MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

68th Annual

AMC 12B

American Mathematics Competition 12B

Wednesday, February 15, 2017

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 12 were prepared by MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (E):** After m months, Kymbrea's collection will have $30+2m$ comic books and LaShawn's collection will have $10+6m$ comic books. Solving $10+6m = 2(30+2m)$ yields $m = 25$, so LaShawn's collection will have twice as many comic books as Kymbrea's after 25 months.

2. **Answer (E):** Adding the inequalities $y > -1$ and $z > 1$ yields $y+z > 0$. The other four choices give negative values if, for example, $x = \frac{1}{8}$, $y = -\frac{1}{4}$, and $z = \frac{3}{2}$.

3. **Answer (D):** The given equation implies that $3x+y = -2(x-3y)$, which is equivalent to $x = y$. Therefore
$$\frac{x+3y}{3x-y} = \frac{4y}{2y} = 2.$$

4. **Answer (C):** Let $2d$ be the distance in kilometers to the friend's house. Then Samia bicycled distance d at rate 17 and walked distance d at rate 5, for a total time of
$$\frac{d}{17} + \frac{d}{5} = \frac{44}{60}$$
hours. Solving this equation yields $d = \frac{17}{6} = 2.833\ldots$. Therefore Samia walked about 2.8 kilometers.

5. **Answer (B):** Because 1.5 times the interquartile range for this data set is $1.5\cdot(43-33) = 15$, outliers are data values less than $33-15 = 18$ or greater than $43+15 = 58$. Only the value 6 meets this condition, so there is 1 outlier.

6. **Answer (D):** The center of the circle is the midpoint of the diameter, which is $(4,3)$, and the radius is $\sqrt{4^2+3^2} = 5$. Therefore the equation of the circle is $(x-4)^2 + (y-3)^2 = 25$. If $y = 0$, then $(x-4)^2 = 16$, so $x = 0$ or $x = 8$. The circle intersects the x-axis at $(8,0)$.

OR

Any diameter of a circle is a line of symmetry. Because the line $x = 4$ goes through the center of the circle, $(4,3)$, it contains a diameter. The reflection of $(0,0)$ in this line is $(8,0)$. Alternatively, $(8,6)$ can be reflected in the line $y = 3$, resulting in the same point.

7. **Answer (B):** Because $\cos(\sin(x+\pi)) = \cos(-\sin(x)) = \cos(\sin(x))$, the function is periodic with period π. Furthermore, $\cos(\sin(x)) = 1$ if and only if $\sin(x) = 0$, which occurs if and only if x is a multiple of π, so the period cannot be less than π. Therefore the function $\cos(\sin(x))$ has least period π.

8. **Answer (C):** Let x be the length of the short side of the rectangle, and let y be the length of the long side. Then the length of the diagonal is $\sqrt{x^2+y^2}$, and

$$\frac{x^2}{y^2} = \frac{y^2}{x^2+y^2}, \qquad \text{so} \qquad \frac{y^2}{x^2} = \frac{x^2+y^2}{y^2} = \frac{x^2}{y^2} + 1.$$

Let $r = \frac{x^2}{y^2}$ be the requested squared ratio. Then $\frac{1}{r} = r + 1$, so $r^2 + r - 1 = 0$. By the quadratic formula, the positive solution is $r = \frac{\sqrt{5}-1}{2}$.

9. **Answer (A):** The first circle has equation $(x+10)^2+(y+4)^2 = 169$, and the second circle has equation $(x-3)^2+(y-9)^2 = 65$. Expanding these two equations, subtracting, and simplifying yields $x + y = 3$. Because the points of intersection of the two circles must satisfy this new equation, it must be the required equation of the line through those points, so $c = 3$. In fact, the circles intersect at $(2, 1)$ and $(-5, 8)$.

10. **Answer (D):** The students who like dancing but say they dislike it constitute $60\% \cdot (100\% - 80\%) = 12\%$ of the students. Similarly, the students who dislike dancing and say they dislike it constitute $(100\% - 60\%) \cdot 90\% = 36\%$ of the students. Therefore the requested fraction is $\frac{12}{12+36} = \frac{1}{4} = 25\%$.

11. **Answer (B):** The monotonous positive integers with one digit or increasing digits can be put into a one-to-one correspondence with the nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The number of such subsets is $2^9 - 1 = 511$. The monotonous positive integers with one digit or decreasing digits can be put into a one-to-one correspondence with the subsets of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ other than $\emptyset$ and $\{0\}$. The number of these is $2^{10} - 2 = 1022$. The single-digit numbers are included in both sets, so there are $511 + 1022 - 9 = 1524$ monotonous positive integers.

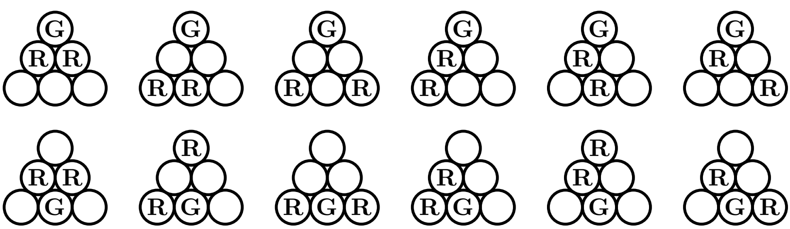
12. **Answer (D):** The principal root of the equation $z^{12} = 64$ is

$$z = 64^{\frac{1}{12}} \cdot \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{2} \cdot \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right).$$

The 12 roots lie in the complex plane on the circle of radius $\sqrt{2}$ centered at the origin. The roots with positive real part make angles of 0, $\pm\frac{\pi}{6}$, and $\pm\frac{\pi}{3}$ with the positive real axis. When these five numbers are added, the imaginary parts cancel out and the sum is

$$\sqrt{2} + 2\sqrt{2} \cdot \cos\frac{\pi}{6} + 2\sqrt{2} \cdot \cos\frac{\pi}{3} = \sqrt{2} \cdot (1 + \sqrt{3} + 1) = 2\sqrt{2} + \sqrt{6}.$$

13. **Answer (D):** By symmetry, there are just two cases for the position of the green disk: corner or non-corner. If a corner disk is painted green, then there is 1 case in which both red disks are adjacent to the green disk, there are 2 cases in which neither red disk is adjacent to the green disk, and there are 3 cases in which exactly one of the red disks is adjacent to the green disk. Similarly, if a non-corner disk is painted green, then there is 1 case in which neither red disk is in a corner, there are 2 cases in which both red disks are in a corner, and there are 3 cases in which exactly one of the red disks is in a corner. The total number of paintings is $1 + 2 + 3 + 1 + 2 + 3 = 12$.



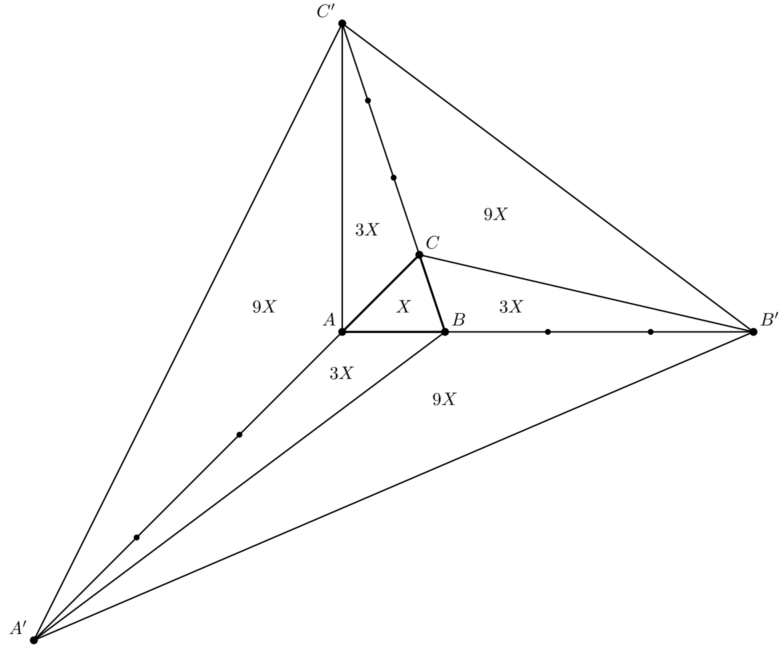
14. **Answer (E):** A frustum is constructed by removing a right circular cone from a larger right circular cone. The volume of the given frustum is the volume of a right circular cone with a 4-inch-diameter base and a height of 8 inches, minus the volume of a right circular cone with a 2-inch-diameter base and a height of 4 inches. (The stated heights come from considering similar right triangles.) Because the volume of a right circular cone is $\frac{1}{3}\pi r^2 h$, the volume of the frustum is

$$\frac{1}{3}\pi \cdot 2^2 \cdot 8 - \frac{1}{3}\pi \cdot 1^2 \cdot 4 = \frac{28}{3}\pi.$$

The volume of the top cone of the novelty is $\frac{1}{3}\pi \cdot 2^2 \cdot 4 = \frac{16}{3}\pi$. The requested volume of ice cream is the sum of the volume of each part of the novelty, namely $\frac{28}{3}\pi + \frac{16}{3}\pi = \frac{44}{3}\pi$.

Note: In general, the volume of a frustum with height h and base radii R and r is $\frac{1}{3}\pi h(r^2 + rR + R^2)$.

15. **Answer (E):** Draw segments $\overline{CB'}$, $\overline{AC'}$, and $\overline{BA'}$. Let X be the area of $\triangle ABC$. Because $\triangle BB'C$ has a base 3 times as long and the same altitude, its area is $3X$. Similarly, the areas of $\triangle AA'B$ and $\triangle CC'A$ are also $3X$. Furthermore, $\triangle AA'C'$ has 3 times the base and the same height as $\triangle ACC'$, so its area is $9X$. The areas of $\triangle CC'B'$ and $\triangle BB'A'$ are also $9X$ by the same reasoning. Therefore the area of $\triangle A'B'C'$ is $X + 3(3X) + 3(9X) = 37X$, and the requested ratio is $37 : 1$. Note that nothing in this argument requires $\triangle ABC$ to be equilateral.



OR

Let $s = AB$. Applying the Law of Cosines to $\triangle B'BC'$ gives

$$\begin{aligned}(B'C')^2 &= (3s)^2 + (4s)^2 - 2 \cdot 3s \cdot 4s \cdot \cos 120^\circ \\ &= s^2\left(25 - 24\left(-\frac{1}{2}\right)\right) = 37s^2.\end{aligned}$$

By symmetry, $\triangle A'B'C'$ is also equilateral and therefore is similar to

$\triangle ABC$ with similarity ratio $\sqrt{37}$. Hence the ratio of their areas is $37 : 1$.

OR

Let $s = AB$. The areas of $\triangle B'BC'$, $\triangle C'CA'$, and $\triangle A'AB'$ are all

$$\frac{1}{2}(3s)(4s)\sin 120^\circ = 3\sqrt{3}s^2.$$

Therefore the requested ratio is

$$\frac{3\left(3\sqrt{3}s^2\right) + \frac{1}{4}\sqrt{3}s^2}{\frac{1}{4}\sqrt{3}s^2} = \frac{37}{1}.$$

16. **Answer (B):** There are $\left\lfloor \frac{21}{2} \right\rfloor + \left\lfloor \frac{21}{4} \right\rfloor + \left\lfloor \frac{21}{8} \right\rfloor + \left\lfloor \frac{21}{16} \right\rfloor = 10+5+2+1 = 18$ powers of 2 in the prime factorization of 21!. Thus $21! = 2^{18}k$, where k is odd. A divisor of 21! must be of the form $2^i b$ where $0 \le i \le 18$ and b is a divisor of k. For each choice of b, there is one odd divisor of 21! and 18 even divisors. Therefore the probability that a randomly chosen divisor is odd is $\frac{1}{19}$. In fact, $21! = 2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, so it has $19 \cdot 10 \cdot 5 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 60{,}800$ positive integer divisors, of which $10 \cdot 5 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 3{,}200$ are odd.

17. **Answer (D):** Let p be the probability of heads. To win Game A requires that all three tosses be heads, which occurs with probability p^3, or all three tosses be tails, which occurs with probability $(1-p)^3$. To win Game B requires that the first two tosses be the same, the probability of which is $p^2+(1-p)^2$, and that the last two tosses be the same, which occurs with the same probability. Therefore the probability of winning Game A minus the probability of winning Game B is

$$\left(p^3 + (1-p)^3\right) - \left(p^2 + (1-p)^2\right)^2.$$

As $p = \frac{2}{3}$, this gives

$$\left(\left(\frac{2}{3}\right)^3 + \left(\frac{1}{3}\right)^3\right) - \left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right)^2 = \frac{1}{3} - \frac{25}{81} = \frac{2}{81}.$$

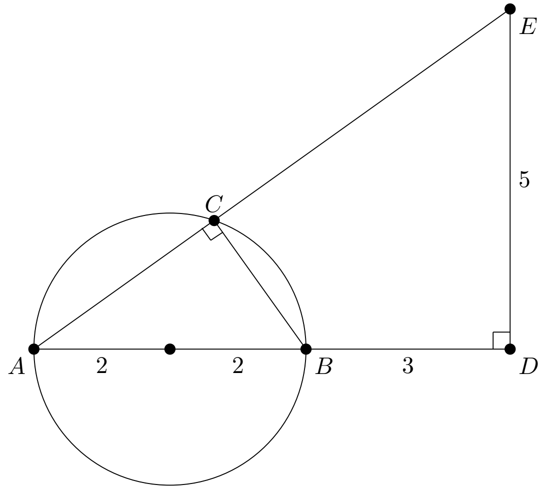
Thus the probability of winning Game A is $\frac{2}{81}$ greater than the probability of winning Game B.

Note: Expanding and then factoring the general expression above for the probability of winning Game A minus the probability of winning Game B yields $p(1-p)(2p-1)^2$. This value is always nonnegative, so the player should never choose Game B. It equals 0 if and only if $p = 0$, $\frac{1}{2}$, or 1. It is maximized when $p = \frac{2\pm\sqrt{2}}{4}$, which is about 85% or 15%, and in this case winning Game A is 6.25 percentage points more likely than winning Game B.

18. **Answer (D):** Because $\angle ACB$ is inscribed in a semicircle, it is a right angle. Therefore $\triangle ABC$ is similar to $\triangle AED$, so their areas are related as AB^2 is to AE^2. Because $AB^2 = 4^2 = 16$ and, by the Pythagorean Theorem,

$$AE^2 = (4+3)^2 + 5^2 = 74,$$

this ratio is $\frac{16}{74} = \frac{8}{37}$. The area of $\triangle AED$ is $\frac{35}{2}$, so the area of $\triangle ABC$ is $\frac{35}{2} \cdot \frac{8}{37} = \frac{140}{37}$.

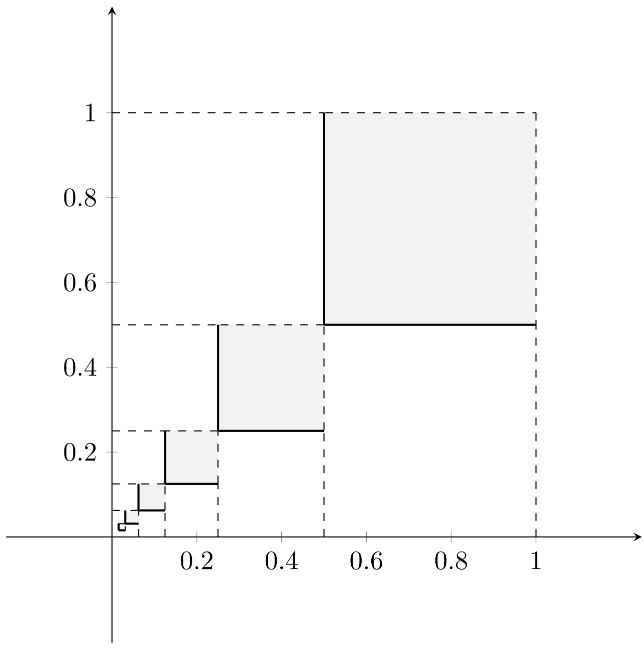


19. **Answer (C):** The remainder when N is divided by 5 is clearly 4. A positive integer is divisible by 9 if and only if the sum of its digits is divisible by 9. The sum of the digits of N is $4(0+1+2+\cdots+9) + 10\cdot1 + 10\cdot2 + 10\cdot3 + (4+0) + (4+1) + (4+2) + (4+3) + (4+4) = 270$, so N must be a multiple of 9. Then $N-9$ must also be a multiple of 9, and the last digit of $N-9$ is 5, so it is also a multiple of 5. Thus $N-9$ is a multiple of 45, and N leaves a remainder of 9 when divided by 45.

20. **Answer (D):** The set of all possible ordered pairs (x, y) is bounded by the unit square in the coordinate plane with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$. For each positive integer n, $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor = -n$ if and only if $\frac{1}{2^n} \leq x < \frac{1}{2^{n-1}}$ and $\frac{1}{2^n} \leq y < \frac{1}{2^{n-1}}$. Thus the set of ordered pairs (x, y) such that $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor = -n$ is bounded by a square with side length $\frac{1}{2^n}$ and therefore area $\frac{1}{4^n}$. The union of these squares over all positive integers n has area

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3},$$

and therefore the requested probability is $\frac{1}{3}$. (It is also clear from the diagram that one third of the square is shaded.)



OR

The problem can be modeled with Xerxes and Yolanda each repeatedly flipping a fair coin to determine the binary (base two "decimal") expansions of x and y, respectively. If Xerxes flips a head, he writes down a 0 as the next binary digit; if he flips a tail, he writes down a 1. Yolanda does the same. Then $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor$ if and only if the first time that either of them flips a tail, so does the other. There are three equally likely outcomes: tail-tail, tail-head, and head-tail. Therefore the requested probability is $\frac{1}{3}$.

21. **Answer (E):** Let S be the sum of Isabella's 7 scores. Then S is a multiple of 7, and

$$658 = 91 + 92 + 93 + \cdots + 97 \leq S \leq 94 + 95 + 96 + \cdots + 100 = 679,$$

so S is one of 658, 665, 672, or 679. Because $S - 95$ is a multiple of 6, it follows that $S = 665$. Thus the sum of Isabella's first 6 scores was $665 - 95 = 570$, which is a multiple of 5, and the sum of her first 5 scores was also a multiple of 5. Therefore her sixth score must have been a multiple of 5. Because her seventh score was 95 and her scores were all different, her sixth score was 100. One possible sequence of scores is $91, 93, 92, 96, 98, 100, 95$.

22. **Answer (B):** There are $4 \cdot 3 = 12$ outcomes for each set of draws and therefore 12^4 outcomes in all. To count the number of outcomes in which each player will end up with four coins, note that this can happen in four ways:

• For some permutation (w, x, y, z) of $\{\text{Abby}, \text{Bernardo}, \text{Carl}, \text{Debra}\}$, the outcomes of the four draws are that w gives a coin to x, x gives a coin to y, y gives a coin to z, and z gives a coin to w, in one of $4! = 24$ orders. There are 3 ways to choose whom Abby gives her coin to and 2 ways to choose whom that person gives his or her coin to, which makes 6 ways to choose the givers and receivers for these transaction. Therefore there are $24 \cdot 6 = 144$ ways for this to happen.

• One pair of the players exchange coins, and the other two players also exchange coins, in one of $4! = 24$ orders. There are 3 ways to choose the pairings. Therefore there are $24 \cdot 3 = 72$ ways for this to happen.

• Two of the players exchange coins twice. There are $\binom{4}{2} = 6$ ways to choose those players and $\binom{4}{2} = 6$ ways to choose the orders of the exchanges, for a total of $6 \cdot 6 = 36$ ways for this to happen.

• One of the players is involved in all four transactions, giving and receiving a coin from each of two others. There are 4 ways to choose this player, 3 ways to choose the other two players, and $4! = 24$ ways to choose the order in which the transactions will take place. Therefore there are $4 \cdot 3 \cdot 24 = 288$ ways for this to happen.

In all, there are $144 + 72 + 36 + 288 = 540$ outcomes that will result in each player having four coins. The requested probability is $\frac{540}{12^4} = \frac{5}{192}$.

23. **Answer (D):** Let $g(x) = f(x) - x^2$. Then $g(2) = g(3) = g(4) = 0$, so for some constant $a \neq 0$, $g(x) = a(x-2)(x-3)(x-4)$. Thus the coefficients of x^3 and x^2 in $f(x)$ are a and $1-9a$, respectively, so the sum of the roots of $f(x)$ is $9 - \frac{1}{a}$. If $L(x)$ is any linear function, then the roots of $f(x) - L(x)$ have the same sum. The given information implies that the sets of roots for three such functions are $\{2, 3, x_1\}$, $\{2, 4, x_2\}$, and $\{3, 4, x_3\}$, where

$$24 = x_1 + x_2 + x_3 = 3\left(9 - \frac{1}{a}\right) - 2(2+3+4) = 9 - \frac{3}{a},$$

so $a = -\frac{1}{5}$. Therefore $f(x) = x^2 - \frac{1}{5}(x-2)(x-3)(x-4)$, and $f(0) = \frac{24}{5}$. (In fact, $D = (9, 39)$, $E = (8, 40)$, $F = (7, 37)$, and the roots of f are 12, $1+i$, and $1-i$.)

24. **Answer (D):** Let F lie on $\overline{AB}$ so that $\overline{DF} \perp \overline{AB}$. Because $BCDF$ is a rectangle, $\angle FCB \cong \angle DBC \cong \angle CAB \cong \angle BCE$, so E lies on $\overline{CF}$ and it is the foot of the altitude to the hypotenuse in $\triangle CBF$. Therefore $\triangle BEF \sim \triangle CBF \cong \triangle BCD \sim \triangle ABC$. Because

$$\overline{DF} \perp \overline{AB}, \quad \overline{FE} \perp \overline{EB}, \quad \text{and} \quad \frac{AB}{DF} = \frac{AB}{BC} = \frac{BE}{FE},$$

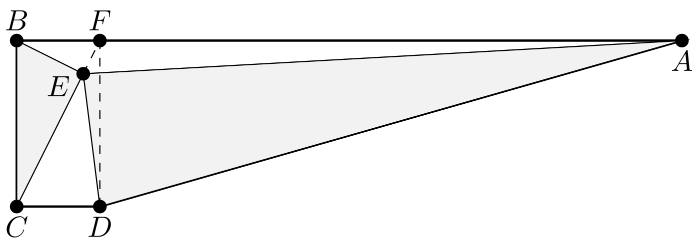
it follows that $\triangle ABE \sim \triangle DFE$. Thus $\angle DEA = \angle DEF - \angle AEF = \angle AEB - \angle AEF = \angle FEB = 90^\circ$. Furthermore,

$$\frac{AE}{ED} = \frac{BE}{EF} = \frac{AB}{BC},$$

so $\triangle AED \sim \triangle ABC$. Assume without loss of generality that $BC = 1$, and let $AB = r > 1$. Because $\frac{AB}{BC} = \frac{BC}{CD}$, it follows that $BF = CD = \frac{1}{r}$. Then

$$17 = \frac{\text{Area}(\triangle AED)}{\text{Area}(\triangle CEB)} = AD^2 = FD^2 + AF^2 = 1 + \left(r - \frac{1}{r}\right)^2,$$

and because $r > 1$ this yields $r^2 - 4r - 1 = 0$, with positive solution $r = 2 + \sqrt{5}$.



OR

Without loss of generality, assume that $BC = 1$. The given conditions imply that the quadrilateral can be placed in the coordinate plane with $C = (0, 0)$, $B = (0, 1)$, $A = (r, 1)$, and $D = (\frac{1}{r}, 0)$. Let E have positive coordinates (x, y). Because $\triangle ABC \sim \triangle CEB$, these coordinates must satisfy

$$\frac{x}{y} = \tan(\angle ECB) = \tan(\angle BAC) = \frac{1}{r}$$

and

$$\sqrt{x^2 + y^2} = \frac{CE}{1} = \frac{r}{\sqrt{1 + r^2}}.$$

Solving this system of equations gives

$$x = \frac{r}{1 + r^2} \quad \text{and} \quad y = \frac{r^2}{1 + r^2}.$$

The area of $\triangle CEB$ is $\frac{x}{2}$. The area of $\triangle AED$ can be computed using the fact that the area of a polygon with vertices (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n) in counterclockwise order is

$$\frac{1}{2}\big((x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_{n-1}x_n + y_nx_1)\big).$$

In this case,

$$\text{Area}(\triangle AED) = \frac{1}{2}\left(y \cdot r + \frac{1}{r} - x - \frac{y}{r}\right).$$

Substituting in the expressions for x and y in terms of r, setting $\text{Area}(\triangle AED) = 17 \cdot \text{Area}(\triangle CEB)$, and simplifying yields the equation $r^4 - 18r^2 + 1 = 0$. Applying the quadratic formula, and noting that $r > 1$, gives $r^2 = 9 + 4\sqrt{5} = (2 + \sqrt{5})^2$, so $r = 2 + \sqrt{5}$.

OR

Let $\theta = \angle ACB$, and without loss of generality assume $BC = 1$. Let F lie on $\overline{AB}$ so that $\overline{DF} \perp \overline{AB}$. Then the requested fraction is $AB = \tan\theta$. Because $\triangle ABC \sim \triangle BCD \sim \triangle CEB \sim \triangle BEF$, it follows that $CD = \cot\theta$, $BE = \cos\theta$, and $CE = \sin\theta$. Then the area of quadrilateral $ABCD$ is $[ABCD] = \frac{1}{2}(\tan\theta + \cot\theta) = \frac{1}{2\sin\theta\cos\theta}$; and the areas of three of the four triangles into which that area can be decomposed are $[ABE] = \frac{1}{2}\tan\theta\cos^2\theta = \frac{1}{2}\sin\theta\cos\theta$, $[BCE] = \frac{1}{2}\sin\theta\cos\theta$, and $[CDE] = \frac{1}{2}\sin^2\theta\cot\theta = \frac{1}{2}\sin\theta\cos\theta$. (Interestingly, the three triangles all have the same area.) Then

$$[AED] = \frac{1}{2\sin\theta\cos\theta} - \frac{3}{2}\sin\theta\cos\theta = 17\cdot\frac{1}{2}\sin\theta\cos\theta.$$

This last equation simplifies to $20\sin^2\theta\cos^2\theta = 1$, so $(2\sin\theta\cos\theta)^2 = \frac{1}{5}$. Then $\sin(2\theta) = \frac{1}{\sqrt{5}}$, $\cos(2\theta) = \frac{-2}{\sqrt{5}}$ (because $AB > BC$ implies $\frac{\pi}{4} < \theta < \frac{\pi}{2}$), and

$$\tan\theta = \frac{\sin(2\theta)}{\cos(2\theta)+1} = \frac{1}{-2+\sqrt{5}} = 2+\sqrt{5}.$$

25. **Answer (D):** Let T be the number of teams participating in the tournament, and let P be the set of participants. For every $A \subseteq P$ let $f(A)$ be the number of teams whose 5 players are in A. According to the described property,

$$\left(\frac{1}{\binom{n}{9}}\sum_{\substack{A\subseteq P\\|A|=9}} f(A)\right)\cdot\left(\frac{1}{\binom{n}{8}}\sum_{\substack{A\subseteq P\\|A|=8}} f(A)\right) = 1.$$

Note that each of the T teams is counted exactly $\binom{n-5}{4}$ times in the sum $\sum_{\substack{A\subseteq P\\|A|=9}} f(A)$. Indeed, once a particular team is fixed, there are exactly $\binom{n-5}{4}$ ways of choosing the remaining 4 persons to determine a set A of size 9. Thus the sum in the first factor is equal to $\binom{n-5}{4}T$; similarly, the sum in the second factor is equal to $\binom{n-5}{3}T$. The described property is now equivalent to

$$\frac{\binom{n-5}{4}\binom{n-5}{3}T^2}{\binom{n}{9}\binom{n}{8}} = 1.$$

Therefore

$$T^2 = \frac{(n!)^2 4!3!}{((n-5)!)^2 9!8!} = \frac{n^2(n-1)^2(n-2)^2(n-3)^2(n-4)^2}{9\cdot 8^2\cdot 7^2\cdot 6^2\cdot 5^2\cdot 4},$$

so

$$T = \frac{n(n-1)(n-2)(n-3)(n-4)}{8\cdot 7\cdot 6\cdot 5\cdot 3\cdot 2} = \frac{n(n-1)(n-2)(n-3)(n-4)}{2^5\cdot 3^2\cdot 5\cdot 7}.$$

Thus a number n has the required property if and only if T is an integer and $n \geq 9$. Let $N = n(n-1)(n-2)(n-3)(n-4)$; because

N consists of the product of five consecutive integers, it is always a multiple of 5. Similarly, $N \equiv 0 \pmod 7$ if and only if $n \equiv 0, 1, 2, 3, 4 \pmod 7$, $N \equiv 0 \pmod 9$ if and only if $n \equiv 0, 1, 2, 3, 4, 6, 7 \pmod 9$, and $N \equiv 0 \pmod{32}$ if and only if $n \equiv 0, 1, 2, 3, 4, 8, 10, 12 \pmod{16}$. Therefore by the Chinese Reminder Theorem there are exactly $5 \cdot 7 \cdot 8 = 280$ residue-class solutions mod $16 \cdot 9 \cdot 7 = 1008$. Thus there are $2 \cdot 280 = 560$ values of n with the desired property in the interval $1 \leq n \leq 2 \cdot 1008 = 2016$. The numbers 1, 2, 3, and 4 are among them, and 5, 6, 7, and 8 are not. In addition, $2017 \equiv 1 \pmod{1008}$; thus 2017 is also a valid value of n. Therefore there are $560 - 4 + 1 = 557$ possible values of n in the required range.

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