



**MAA**

MATHEMATICAL ASSOCIATION OF AMERICA

# Solutions Pamphlet

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# AMC 12A

American Mathematics Competition 12A

Tuesday, February 7, 2017

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

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The problems and solutions for this AMC 12 were prepared by MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the co-chairs Jerrold W. Grossman and Carl Yerger.

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1. **Answer (D):** The cheapest popsicles cost  $\$3.00 \div 5 = \$0.60$  each. Because  $14 \cdot \$0.60 = \$8.40$  and Pablo has just  $\$8$ , he could not pay for 14 popsicles even if he were allowed to buy partial boxes. The best he can hope for is 13 popsicles, and he can achieve that by buying two 5-popsicle boxes (for  $\$6$ ) and one 3-popsicle box (for  $\$2$ ).

**OR**

If Pablo buys two single popsicles for  $\$1$  each, he could have bought a 3-popsicle box for the same amount of money. Similarly, if Pablo buys three single popsicles or both one 3-popsicle box and one single popsicle, he could have bought a 5-popsicle box for the same amount of money. If Pablo buys two 3-popsicle boxes, he could have bought a 5-popsicle box and a single popsicle for the same amount of money. The previous statements imply that a maximum number of popsicles for a given amount of money can be obtained by buying either at most one single popsicle and the rest 5-popsicle boxes, or a single 3-popsicle box and the rest 5-popsicle boxes. When Pablo has  $\$8$ , he can obtain the maximum number of popsicles by buying two 5-popsicle boxes and one 3-popsicle box. This gives a total of  $2 \cdot 5 + 1 \cdot 3 = 13$  popsicles.

2. **Answer (C):** Let the two numbers be  $x$  and  $y$ . Then  $x + y = 4xy$ . Dividing this equation by  $xy$  gives  $\frac{1}{y} + \frac{1}{x} = 4$ . One such pair of numbers is  $x = \frac{1}{3}$ ,  $y = 1$ .
3. **Answer (B):** The given statement is logically equivalent to its contrapositive: If a student did not receive an A on the exam, then the student did not get all the multiple choice questions right, which means that he got at least one of them wrong. None of the other statements follows logically from the given implication; the teacher made no promises concerning students who did not get all the multiple choice questions right. In particular, a statement does not imply its inverse or its converse; and the negation of the statement that Lewis got all the questions right is not the statement that he got all the questions wrong.
4. **Answer (A):** If the square had side length  $x$ , then Jerry's path had length  $2x$ , and Silvia's path along the diagonal, by the Pythagorean Theorem, had length  $\sqrt{2}x$ . Therefore Silvia's trip was shorter by

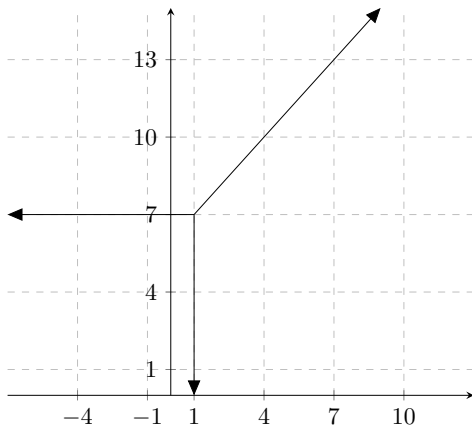
$2x - \sqrt{2}x$ , and the required percentage is

$$\frac{2x - \sqrt{2}x}{2x} = 1 - \frac{\sqrt{2}}{2} \approx 1 - 0.707 = 0.293 = 29.3\%.$$

The closest of the answer choices is 30%.

5. **Answer (B):** Each of the 20 people who know each other shakes hands with 10 people. Each of the 10 people who know no one shakes hands with 29 people. Because each handshake involves two people, the number of handshakes is  $\frac{1}{2}(20 \cdot 10 + 10 \cdot 29) = 245$ .
6. **Answer (B):** Four rods can form a quadrilateral with positive area if and only if the length of the longest rod is less than the sum of the lengths of the other three. Therefore if the fourth rod has length  $n$  cm, then  $n$  must satisfy the inequalities  $15 < 3 + 7 + n$  and  $n < 3 + 7 + 15$ , that is,  $5 < n < 25$ . Because  $n$  is an integer, it must be one of the 19 integers from 6 to 24, inclusive. However, the rods of lengths 7 cm and 15 cm have already been chosen, so the number of rods that Joy can choose is  $19 - 2 = 17$ .
7. **Answer (B):** It is clear after listing the first few values,  $f(1) = 2$ ,  $f(2) = f(1) + 1 = 3$ ,  $f(3) = f(1) + 2 = 4$ ,  $f(4) = f(3) + 1 = 5$ , and so on, that  $f(n) = n + 1$  for all positive integers  $n$ . Indeed, the function is uniquely determined by the recursive description, and the function defined by  $f(n) = n + 1$  fits the description. Therefore  $f(2017) = 2018$ .
8. **Answer (D):** Let  $h = AB$ . The region consists of a solid circular cylinder of radius 3 and height  $h$ , together with two solid hemispheres of radius 3 centered at  $A$  and  $B$ . The volume of the cylinder is  $\pi \cdot 3^2 \cdot h = 9\pi h$ , and the two hemispheres have a combined volume of  $\frac{4}{3}\pi \cdot 3^3 = 36\pi$ . Therefore  $9\pi h + 36\pi = 216\pi$ , and  $h = 20$ .
9. **Answer (E):** Suppose that the two larger quantities are the first and the second. Then  $3 = x + 2 \geq y - 4$ . This is equivalent to  $x = 1$  and  $y \leq 7$ , and its graph is the downward-pointing ray with endpoint  $(1, 7)$ . Similarly, if the two larger quantities are the first and third, then  $3 = y - 4 \geq x + 2$ . This is equivalent to  $y = 7$  and  $x \leq 1$ , and its graph is the leftward-pointing ray with endpoint  $(1, 7)$ . Finally, if the

two larger quantities are the second and third, then  $x + 2 = y - 4 \geq 3$ . This is equivalent to  $y = x + 6$  and  $x \geq 1$ , and its graph is the ray with endpoint  $(1, 7)$  that points upward and to the right. Thus the graph consists of three rays with common endpoint  $(1, 7)$ .



**Note:** This problem is related to a relatively new area of mathematics called tropical geometry.

10. **Answer (C):** Half of the time Laurent will pick a number between 2017 and 4034, in which case the probability that his number will be greater than Chloé's number is 1. The other half of the time, he will pick a number between 0 and 2017, and by symmetry his number will be the larger one in half of those cases. Therefore the requested probability is  $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$ .

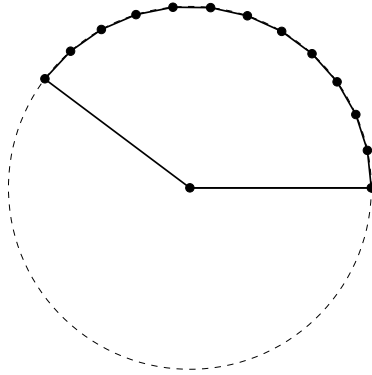
**OR**

The choices of numbers can be represented in the coordinate plane by points in the rectangle with vertices at  $(0, 0)$ ,  $(2017, 0)$ ,  $(2017, 4034)$ , and  $(0, 4034)$ . The portion of the rectangle representing the event that Laurent's number is greater than Chloé's number is the portion above the line segment with endpoints  $(0, 0)$  and  $(2017, 2017)$ . This area is  $\frac{3}{4}$  of the area of the entire rectangle, so the requested probability is  $\frac{3}{4}$ .

11. **Answer (D):** If the polygon has  $n$  sides and the degree measure of the forgotten angle is  $\alpha$ , then  $(n - 2)180 = 2017 + \alpha$ . Because  $0 < \alpha < 180$ ,

$$2017 < (n - 2)180 < 2197,$$

which implies that  $n = 14$ , the angle sum is 2160, and  $\alpha = 143$ . To see that such a polygon exists, draw a circle and a central angle of measure  $143^\circ$ , and divide the minor arc spanned by the angle into 12 small arcs. The polygon is then formed by the two radii and 12 small chords, as illustrated.



12. **Answer (B):** Horse  $k$  will again be at the starting point after  $t$  minutes if and only if  $k$  is a divisor of  $t$ . Let  $I(t)$  be the number of integers  $k$  with  $1 \leq k \leq 10$  that divide  $t$ . Then  $I(1) = 1$ ,  $I(2) = 2$ ,  $I(3) = 2$ ,  $I(4) = 3$ ,  $I(5) = 2$ ,  $I(6) = 4$ ,  $I(7) = 2$ ,  $I(8) = 4$ ,  $I(9) = 3$ ,  $I(10) = 4$ ,  $I(11) = 1$ , and  $I(12) = 5$ . Thus  $T = 12$  and the requested sum of digits is  $1 + 2 = 3$ .

13. **Answer (B):** Let  $d$  be the requested distance in miles, and suppose that Sharon usually drives at speed  $r$  in miles per hour. Then  $\frac{d}{r} = 3$ . The total time in hours for Sharon's trip with the snowstorm is then  $\frac{\frac{1}{3}d}{r} + \frac{\frac{2}{3}d}{r-20} = \frac{23}{5}$ . Because  $\frac{d}{r} = 3$ , this reduces to

$$1 + \frac{\frac{2}{3}d}{r - 20} = 1 + \frac{\frac{2}{3} \cdot 3r}{\frac{1}{3}r - 20} = \frac{23}{5}.$$

Solving for  $d$  gives  $d = 135$ .

**OR**

The last  $\frac{2}{3}$  of the drive takes  $276 - \frac{1}{3} \cdot 180 = 216$  minutes, which is  $\frac{216}{60}$  hours. If  $r$  is the original speed in miles per hour, then  $\frac{2}{3}$  of the distance is both  $2r$  and  $\frac{216}{60} \cdot (r - 20)$ . Setting these expressions equal and solving yields  $r = 45$ . Therefore the original speed is 45 miles per hour, and the requested distance is  $3 \cdot 45 = 135$  miles.

14. **Answer (C):** Let  $X$  be the set of ways to seat the five people in which Alice sits next to Bob. Let  $Y$  be the set of ways to seat the five people in which Alice sits next to Carla. Let  $Z$  be the set of ways to seat the five people in which Derek sits next to Eric. The required answer is  $5! - |X \cup Y \cup Z|$ . The Inclusion–Exclusion Principle gives

$$|X \cup Y \cup Z| = (|X| + |Y| + |Z|) - (|X \cap Y| + |X \cap Z| + |Y \cap Z|) + |X \cap Y \cap Z|.$$

Viewing Alice and Bob as a unit in which either can sit on the other's left side shows that there are  $2 \cdot 4! = 48$  elements of  $X$ . Similarly there are 48 elements of  $Y$  and 48 elements of  $Z$ . Viewing Alice, Bob, and Carla as a unit with Alice in the middle shows that  $|X \cap Y| = 2 \cdot 3! = 12$ . Viewing Alice and Bob as a unit and Derek and Eric as a unit shows that  $|X \cap Z| = 2 \cdot 2 \cdot 3! = 24$ . Similarly  $|Y \cap Z| = 24$ . Finally, there are  $2 \cdot 2 \cdot 2! = 8$  elements of  $X \cap Y \cap Z$ . Therefore  $|X \cup Y \cup Z| = (48 + 48 + 48) - (12 + 24 + 24) + 8 = 92$ , and the answer is  $120 - 92 = 28$ .

### OR

There are three cases based on where Alice is seated.

- If Alice takes the first or last chair, then Derek or Eric must be seated next to her, Bob or Carla must then take the middle chair, and either of the remaining two individuals can be seated in either of the other two chairs. This gives a total of  $2^4 = 16$  arrangements.
- If Alice is seated in the second or fourth chair, then Derek and Eric will take the seats on her two sides, and this can be done in two ways. Bob and Carla can be seated in the two remaining chairs in two ways, which yields a total of  $2^3 = 8$  arrangements.
- If Alice sits in the middle chair, then Derek and Eric will be seated on her two sides, with Bob and Carla seated in the first and last chairs. This results in  $2^2 = 4$  arrangements.

Thus there are  $16 + 8 + 4 = 28$  possible arrangements in total.

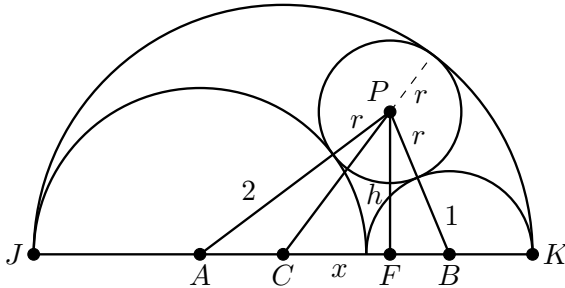
15. **Answer (D):** For  $0 < x < \frac{\pi}{2}$  all three terms are positive, and  $f(x)$  is undefined when  $x = \frac{\pi}{2}$ . For  $\frac{\pi}{2} < x < \frac{3\pi}{4}$ , the term  $3 \tan x$  is less than  $-3$  and dominates the other two terms, so  $f(x) < 0$  there. For  $\frac{3\pi}{4} \leq x < \pi$ ,  $|\cos(x)| \geq |\sin(x)|$  and  $\cos x$  and  $\tan x$  are negative, so  $\sin x + 2 \cos x + 3 \tan x < 0$ . Therefore there is no positive solution of  $f(x) = 0$  for  $x < \pi$ . Because the range of  $f$  includes all values between  $f(\pi) = -2 < 0$  and  $f(\frac{5\pi}{4}) = -\frac{3}{2}\sqrt{2} + 3 > -1.5 \cdot 1.5 + 3 > 0$  on the interval  $[\pi, \frac{5\pi}{4}]$ , the smallest positive solution of  $f(x) = 0$  lies

between  $\pi$  and  $\frac{5\pi}{4}$ . Because  $\pi > 3$  and  $\frac{5\pi}{4} < 4$ , the requested interval is  $(3, 4)$ .

16. **Answer (B):** Let  $C$  be the center of the largest semicircle, and let  $r$  denote the radius of the circle centered at  $P$ . Note that  $PA = 2 + r$ ,  $PC = 3 - r$ ,  $PB = 1 + r$ ,  $AC = 1$ ,  $BC = 2$ , and  $AB = 3$ . Let  $F$  be the foot of the perpendicular from  $P$  to  $\overline{AB}$ , let  $h = PF$ , and let  $x = CF$ . The Pythagorean Theorem in  $\triangle PAF$ ,  $\triangle PCF$ , and  $\triangle PBF$  gives

$$h^2 = (2 + r)^2 - (1 + x)^2 = (3 - r)^2 - x^2 = (1 + r)^2 - (2 - x)^2.$$

This reduces to two linear equations in  $r$  and  $x$ , whose solution is  $r = \frac{6}{7}$ ,  $x = \frac{9}{7}$ .



OR

With the notation and observations above, apply Heron's Formula to  $\triangle PCB$  and  $\triangle PAC$ , noting that the former has twice the area of the latter and each has semiperimeter 3. Thus

$$\sqrt{3 \cdot 1 \cdot r \cdot (r - 2)} = 2\sqrt{3 \cdot 2 \cdot r \cdot (r - 1)},$$

from which it follows that  $r = \frac{6}{7}$ .

OR

With the notation and observations above, let  $\theta = \angle PAB$ . The Law of Cosines applied to  $\triangle PAC$  gives

$$(3 - r)^2 = (2 + r)^2 + 1^2 - 2 \cdot (2 + r) \cdot 1 \cdot \cos \theta,$$

and simplifying yields  $(2 + r) \cos \theta = 5r - 2$ . Applying the Law of Cosines to  $\triangle PAB$  gives

$$(1 + r)^2 = (2 + r)^2 + 3^2 - 2 \cdot (2 + r) \cdot 3 \cdot \cos \theta,$$

and simplifying yields  $3(2 + r) \cos \theta = r + 6$ . Hence  $r + 6 = 3(5r - 2)$ , so  $r = \frac{6}{7}$ .

17. **Answer (D):** The complex numbers  $z$  such that  $z^{24} = 1$  are the roots of  $z^{24} - 1 = (z^6 - 1)(z^6 + 1)((z^6)^2 + 1)$ . The factors can have at most 6, 6, and 12, roots, respectively. Because  $z^{24} - 1$  has 24 distinct roots, the factors do actually have 6, 6, and 12 distinct roots, respectively. The six roots of the first factor satisfy  $z^6 = 1$ , and the six roots of the second factor satisfy  $z^6 = -1$ . The twelve roots of the third factor satisfy  $(z^6)^2 = -1$ , so  $z^6$  is never real in this case. There are  $6 + 6 = 12$  roots such that  $z^6$  is real.

**OR**

The complex values of  $z$  such that  $z^{24} = 1$  are the 24th roots of unity. These values can be written in the form  $e^{\frac{1}{24}k\pi i}$ , where  $k$  is an integer between 0 and 23, inclusive. By Euler's Theorem,

$$z^6 = e^{\frac{1}{2}k\pi i} = \cos\left(\frac{1}{2}k\pi\right) + i \sin\left(\frac{1}{2}k\pi\right).$$

This quantity is a real number if and only if  $\sin\left(\frac{1}{2}k\pi\right) = 0$ , which occurs if and only if  $k$  is even. There are therefore 12 complex values of  $z$  such that  $z^6$  is real.

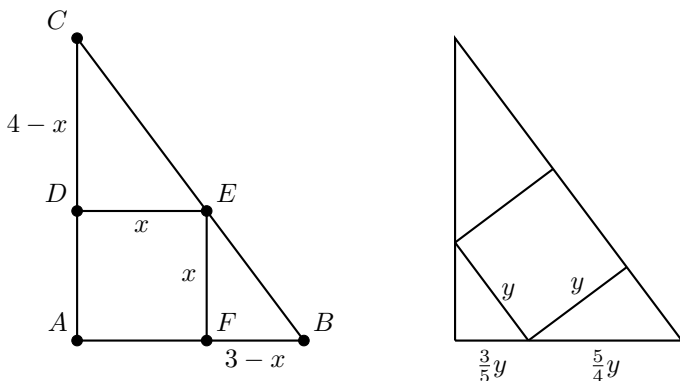
18. **Answer (D):** Note that  $S(n + 1) = S(n) + 1$  unless the numeral for  $n$  ends with a 9. Moreover, if the numeral for  $n$  ends with exactly  $k$  9s, then  $S(n + 1) = S(n) + 1 - 9k$ . Thus the possible values of  $S(n + 1)$  when  $S(n) = 1274$  are all of the form  $1275 - 9k$ , where  $k \in \{0, 1, 2, 3, \dots, 141\}$ . Of the choices, only 1239 can be formed in this manner, and  $S(n + 1)$  will equal 1239 if, for example,  $n$  consists of 4 consecutive 9s preceded by 1238 1s.

**OR**

The value of a positive integer is congruent to the sum of its digits modulo 9. Therefore  $n \equiv S(n) = 1274 \equiv 5 \pmod{9}$ , so  $S(n + 1) \equiv n + 1 \equiv 6 \pmod{9}$ . Of the given choices, only 1239 meets this requirement.



19. **Answer (D):** In the first figure  $\triangle FEB \sim \triangle DCE$ , so  $\frac{x}{3-x} = \frac{4-x}{x}$  and  $x = \frac{12}{7}$ . In the second figure, the small triangles are similar to the large one, so the lengths of the portions of the side of length 3 are as shown. Solving  $\frac{3}{5}y + \frac{5}{4}y = 3$  yields  $y = \frac{60}{37}$ . Thus  $\frac{x}{y} = \frac{12}{7} \cdot \frac{37}{60} = \frac{37}{35}$ .



20. **Answer (E):** Let  $u = \log_b a$ . Because  $u^{2017} = 2017u$ , either  $u = 0$  or  $u = \pm \sqrt[2016]{2017}$ . If  $u = 0$ , then  $a = 1$  and  $b$  can be any integer from 2 to 200. If  $u = \pm \sqrt[2016]{2017}$ , then  $a = b^{\pm \sqrt[2016]{2017}}$ , where again  $b$  can be any integer from 2 to 200. Therefore there are  $3 \cdot 199 = 597$  such ordered pairs.

21. **Answer (D):** Because  $-1$  is a root of  $10x + 10$ ,  $-1$  is added to  $S$ . Then  $1$  is also added to  $S$ , because it is a root of  $(-1)x^{10} + (-1)x^9 + \cdots + (-1)x + 10$ . At this point  $-10$ , a root of  $1 \cdot x + 10$ , can be added to  $S$ . Because  $2$  is a root of  $1 \cdot x^3 + 0 \cdot x^2 + 1 \cdot x + (-10)$ , and  $-2$  is a root of  $1 \cdot x + 2$ , both  $2$  and  $-2$  can be added to  $S$ . The polynomials  $2x + (-10)$  and  $2x + 10$  allow  $5$  and  $-5$  into  $S$ . At this point  $S = \{0, \pm 1, \pm 2, \pm 5, \pm 10\}$ . No more elements can be added to  $S$ , because by the Rational Root Theorem, any integer root of a polynomial with integer coefficients whose constant term is a factor of 10 must be a factor of 10. Therefore  $S$  contains 9 elements.

**Note:** It is not true that in general if  $S$  starts with  $\{0, c\}$  then all factors of  $c$  can be added to  $S$ . For example, applying the procedure to  $\{0, 35\}$  gives only  $\{0, \pm 1, \pm 35\}$ , although of course it takes some argument to rule out  $\pm 5$  and  $\pm 7$ .

22. **Answer (E):** Let  $A = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$ , let  $C = \{(0, 0)\}$ , and let  $I = \{(1, 1), (-1, 1), (-1, -1), (1, -1)\}$ . A particle

in  $A$  will move to  $A$  with probability  $\frac{2}{8}$ , to  $C$  with probability  $\frac{1}{8}$ , to  $I$  with probability  $\frac{2}{8}$ , and to an interior point of a side of the square with probability  $\frac{3}{8}$ . Similarly, a particle in  $C$  will move to  $A$  with probability  $\frac{4}{8}$  and to  $I$  with probability  $\frac{4}{8}$ ; and a particle in  $I$  will move to  $A$  with probability  $\frac{2}{8}$ , to  $C$  with probability  $\frac{1}{8}$ , to a corner of the square with probability  $\frac{1}{8}$ , and to an interior point of a side of the square with probability  $\frac{4}{8}$ . Let  $a$ ,  $c$ , and  $i$  be the probabilities that the particle will first hit the square at a corner, given that it is currently in  $A$ ,  $C$ , and  $I$ , respectively. The transition probabilities noted above lead to the following system of equations.

$$\begin{aligned} a &= \frac{2}{8}a + \frac{1}{8}c + \frac{2}{8}i \\ c &= \frac{4}{8}a + \frac{4}{8}i \\ i &= \frac{2}{8}a + \frac{1}{8}c + \frac{1}{8} \end{aligned}$$

This system can be solved by elimination to yield  $a = \frac{1}{14}$ ,  $c = \frac{4}{35}$ , and  $i = \frac{11}{70}$ . The required fraction is  $c$ , whose numerator and denominator sum to 39.

23. **Answer (C):** Let  $q$  be the additional root of  $f(x)$ . Then

$$\begin{aligned} f(x) &= (x - q)(x^3 + ax^2 + x + 10) \\ &= x^4 + (a - q)x^3 + (1 - qa)x^2 + (10 - q)x - 10q. \end{aligned}$$

Thus  $100 = 10 - q$ , so  $q = -90$  and  $c = -10q = 900$ . Also  $1 = a - q = a + 90$ , so  $a = -89$ . It follows, using the factored form of  $f$  shown above, that  $f(1) = (1 - (-90)) \cdot (1 - 89 + 1 + 10) = 91 \cdot (-77) = -7007$ .

24. **Answer (A):** Because  $\overline{YE}$  and  $\overline{EF}$  are parallel to  $\overline{AD}$  and  $\overline{AC}$ , respectively,  $\triangle XEY \sim \triangle XAD$  and  $\triangle XEF \sim \triangle XAC$ . Therefore

$$\frac{XY}{XE} = \frac{XD}{XA} \quad \text{and} \quad \frac{XF}{XE} = \frac{XC}{XA}.$$

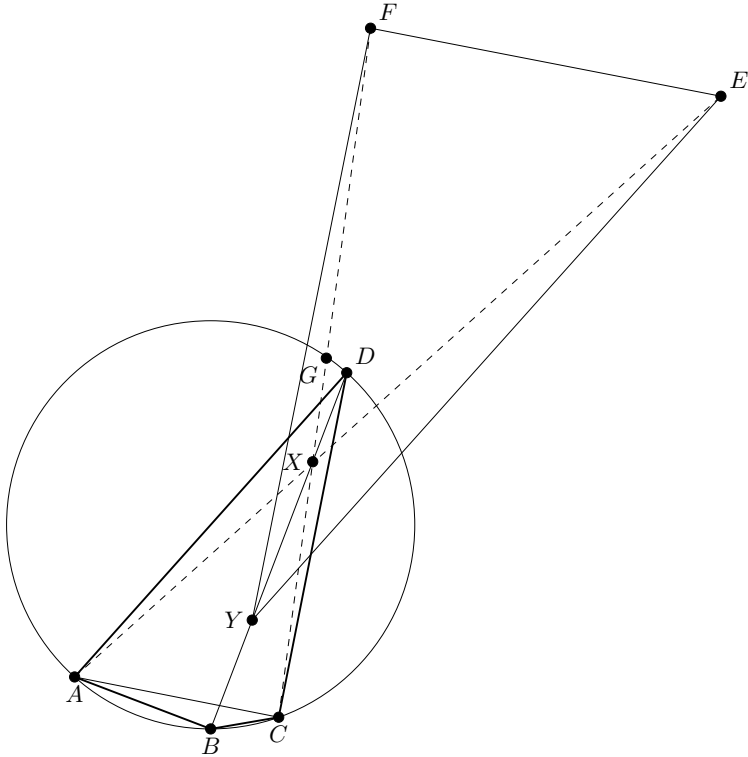
It follows that

$$\frac{XC}{XD} = \frac{XF}{XY}.$$

The Power of a Point Theorem applied to circle  $O$  and point  $X$  implies that  $XC \cdot XG = XD \cdot XB$ . Together with the previous equation this

implies that  $XF \cdot XG = XB \cdot XY$ . Let  $d = BD$ ; then  $DX = \frac{1}{4}d$  and  $BY = \frac{11}{36}d$ . It follows that

$$\begin{aligned} XF \cdot XG &= XB \cdot XY = (BD - DX) \cdot (BD - DX - BY) \\ &= \left(d - \frac{1}{4}d\right) \left(d - \frac{1}{4}d - \frac{11}{36}d\right) \\ &= \frac{3}{4} \cdot \frac{4}{9}d^2 = \frac{d^2}{3}. \end{aligned}$$



To determine  $d$ , note that because  $ABCD$  is a cyclic quadrilateral it follows that  $\alpha = \angle BAD = \pi - \angle DCB$ . Applying the Law of Cosines to  $\triangle ABD$  and  $\triangle BCD$  yields

$$\cos \alpha = \frac{AB^2 + AD^2 - BD^2}{2 \cdot AB \cdot AD} = \frac{3^2 + 8^2 - d^2}{2 \cdot 3 \cdot 8} = \frac{73 - d^2}{48},$$

and

$$-\cos \alpha = \cos(\pi - \alpha) = \frac{CB^2 + CD^2 - BD^2}{2 \cdot CB \cdot CD} = \frac{2^2 + 6^2 - d^2}{2 \cdot 2 \cdot 6} = \frac{40 - d^2}{24}.$$

Therefore

$$\frac{73 - d^2}{48} = \frac{d^2 - 40}{24},$$

and solving for  $d^2$  gives  $d^2 = 51$ . Hence  $XF \cdot XG = \frac{1}{3}d^2 = 17$ .

25. **Answer (E):** If  $z_j$  is an element of the set  $A = \{\sqrt{2}i, -\sqrt{2}i\}$ , then  $|z_j| = \sqrt{2}$ . Otherwise  $z_j$  is an element of

$$B = V \setminus A = \left\{ \frac{1}{\sqrt{8}}(1 + i), \frac{1}{\sqrt{8}}(-1 + i), \frac{1}{\sqrt{8}}(1 - i), \frac{1}{\sqrt{8}}(-1 - i) \right\}$$

and  $|z_j| = \frac{1}{2}$ . It follows that  $|P| = \prod_{j=1}^{12} |z_j| = 1$  exactly when 8 of the 12 factors  $z_j$  are in  $A$  and 4 of the factors are in  $B$ . The product of 8 complex numbers each of which is in  $A$  is a real number, either 16 or  $-16$ . The product of 4 numbers each of which is in  $B$  is one of  $\frac{1}{16}$ ,  $\frac{1}{16}i$ ,  $-\frac{1}{16}$ , or  $-\frac{1}{16}i$ . Thus a product  $P = \prod_{j=1}^{12} z_j$  is  $-1$  exactly when 8 of the  $z_j$  are from  $A$ , 4 of the  $z_j$  are from  $B$ , and the last of the 4 elements from  $B$  is chosen so that the product is  $-1$  rather than  $i$ ,  $-i$ , or 1. Because the probability is  $\frac{1}{3}$  that a particular factor  $z_j$  is from  $A$ , the probability is  $\frac{2}{3}$  that a particular factor  $z_j$  is from  $B$ , and the probability is  $\frac{1}{6}$  that a particular factor  $z_j$  is a specific element of  $V$ , the probability that the product  $P$  will be  $-1$  is given by

$$\binom{12}{4} \binom{1}{3}^8 \binom{2}{3}^3 \binom{1}{6} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} \cdot \frac{1}{3^8} \cdot \frac{2^3}{3^3} \cdot \frac{1}{6} = \frac{2^2 \cdot 5 \cdot 11}{3^{10}}.$$

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