MATHEMATICAL ASSOCIATION OF AMERICA
MAA 100
CELEBRATING A CENTURY OF ADVANCING MATHEMATICS

Solutions Pamphlet

MAA American Mathematics Competitions

67th Annual

AMC 12B

American Mathematics Contest 12B

Wednesday, February 17, 2016

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 12 were prepared by
MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the
co-chairs Jerrold W. Grossman and Silvia Fernandez.

1. **Answer (D):**

$$\frac{2(\frac{1}{2})^{-1} + \frac{(\frac{1}{2})^{-1}}{2}}{\frac{1}{2}} = \left(2 \cdot 2 + \frac{2}{2}\right) \cdot 2 = 10$$

2. **Answer (A):** The harmonic mean of 1 and 2016 is

$$\frac{2 \cdot 1 \cdot 2016}{1 + 2016} = 2 \cdot \frac{2016}{2017} \approx 2 \cdot 1 = 2.$$

3. **Answer (D):**

$$\Bigg|\,\Big|\,|-2016|-(-2016)\,\Big| - |-2016|\,\Bigg| - (-2016)$$
$$= \Bigg|\,\Big|\,2016 + 2016\,\Big| - 2016\,\Bigg| + 2016 = 2016 + 2016 = 4032$$

4. **Answer (C):** Let α and β be the measures of the angles, with $\alpha < \beta$. Then $\frac{\beta}{\alpha} = \frac{5}{4}$. Because $\alpha < \beta$, it follows that $90^\circ - \beta < 90^\circ - \alpha$, so $90^\circ - \alpha = 2(90^\circ - \beta)$. This leads to the system of linear equations $4\beta - 5\alpha = 0$ and $2\beta - \alpha = 90^\circ$. Solving the system gives $\alpha = 60^\circ$, $\beta = 75^\circ$. The requested sum is $\alpha + \beta = 135^\circ$.

5. **Answer (B):** Because $919 = 7 \cdot 131 + 2$, the war lasted 131 full weeks plus 2 days. Therefore it ended 2 days beyond Thursday, which is Saturday.

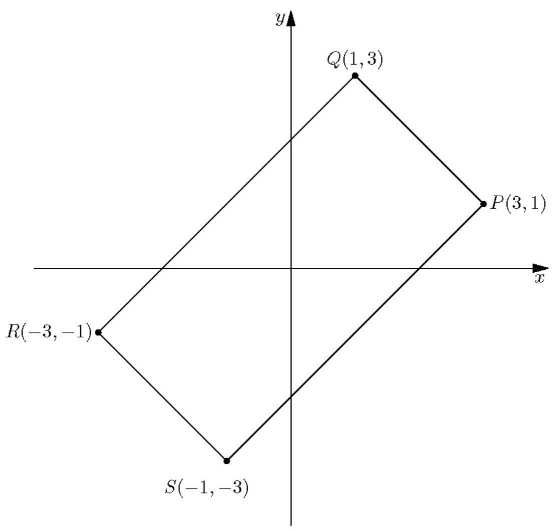
6. **Answer (C):** Let the vertex of the triangle that lies in the first quadrant be (x, x^2). Then the base of the triangle is $2x$ and the height is x^2, so $\frac{1}{2} \cdot 2x \cdot x^2 = 64$. Thus $x^3 = 64$, $x = 4$, and $BC = 2x = 8$.

7. **Answer (D):** In the first pass Josh marks out the odd numbers $1, 3, 5, 7, \ldots, 99$, leaving the multiples of 2: $2, 4, 6, 8, \ldots, 100$. In the second pass Josh marks out $2, 6, 10, \ldots, 98$, leaving the multiples of 4: $4, 8, 12, \ldots, 100$. Similarly, in the n^{th} pass Josh marks out the numbers that are not multiples of 2^n, leaving the numbers that are multiples of 2^n. It follows that in the 6^{th} pass Josh marks out the numbers that are multiples of 2^5 but not multiples of 2^6, namely 32 and 92. This leaves 64, the only number in his original list that is a multiple of 2^6. Thus the last number remaining is 64.

8. **Answer (D):** The weight of an object of uniform density is proportional to its volume. The volume of the triangular piece of wood of uniform thickness is proportional to the area of the triangle. The side length of the second piece is $\frac{5}{3}$ times the side length of the first piece, so the area of the second piece is $\left(\frac{5}{3}\right)^2$ times the area of the first piece. Therefore the weight is $12 \cdot \left(\frac{5}{3}\right)^2 = \frac{100}{3} \approx 33.3$ ounces.

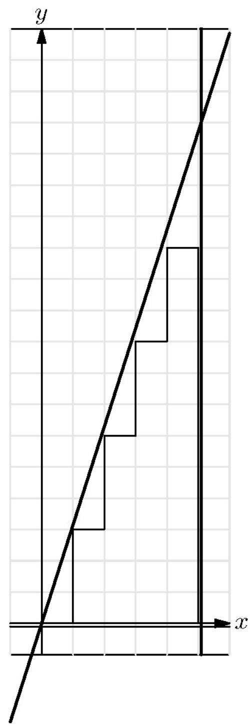
9. **Answer (B):** Let x be the number of posts along the shorter side; then there are $2x$ posts along the longer side. When counting the number of posts on all the sides of the garden, each corner post is counted twice, so $2x + 2(2x) = 20 + 4$. Solving this equation gives $x = 4$. Thus the dimensions of the rectangle are $(4 - 1) \cdot 4 = 12$ yards by $(8 - 1) \cdot 4 = 28$ yards. The requested area is given by the product of these dimensions, $12 \cdot 28 = 336$ square yards.

10. **Answer (A):** The slopes of $\overline{PQ}$ and $\overline{RS}$ are -1, and the slopes of $\overline{QR}$ and $\overline{PS}$ are 1, so the figure is a rectangle. The side lengths are $PQ = (a-b)\sqrt{2}$ and $PS = (a + b)\sqrt{2}$, so the area is $2(a - b)(a + b) = 2(a^2 - b^2) = 16$. Therefore $a^2 - b^2 = 8$. The only perfect squares whose difference is 8 are 9 and 1, so $a = 3$, $b = 1$, and $a + b = 4$.

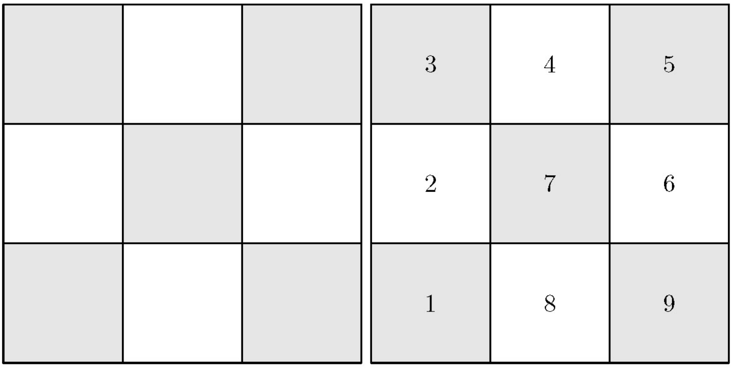


11. **Answer (D):** Note that $3 < \pi < 4$, $6 < 2\pi < 7$, $9 < 3\pi < 10$, and $12 < 4\pi < 13$. Therefore there are 3 1-by-1 squares of the desired type in the strip $1 \le x \le 2$, 6 1-by-1 squares in the strip $2 \le x \le 3$, 9 1-by-1 squares in

the strip $3 \le x \le 4$, and 12 1-by-1 squares in the strip $4 \le x \le 5$. Furthermore there are 2 2-by-2 squares in the strip $1 \le x \le 3$, 5 2-by-2 squares in the strip $2 \le x \le 4$, and 8 2-by-2 squares in the strip $3 \le x \le 5$. There is 1 3-by-3 square in the strip $1 \le x \le 4$, and there are 4 3-by-3 squares in the strip $2 \le x \le 5$. There are no 4-by-4 or larger squares. Thus in all there are $3+6+9+12+2+5+8+1+4 = 50$ squares of the desired type within the given region.



12. **Answer (C):** Shade the squares in a checkerboard pattern as shown in the first figure. Because consecutive numbers must be in adjacent squares, the shaded squares will contain either five odd numbers or five even numbers. Because there are only four even numbers available, the shaded squares contain the five odd numbers. Thus the sum of the numbers in all five shaded squares is $1+3+5+7+9 = 25$. Because all but the center add up to $18 = 25 - 7$, the center number must be 7. The situation described is actually possible, as the second figure demonstrates.



13. **Answer (E):** Let Alice, Bob, and the airplane be located at points A, B, and C, respectively. Let D be the point on the ground directly beneath the airplane, and let h be the airplane's altitude, in miles. Then $\triangle ACD$ and $\triangle BCD$ are $30-60-90^\circ$ right triangles with right angles at D, so $AD = \sqrt{3}h$ and $BD = \frac{h}{\sqrt{3}}$. Then by the Pythagorean Theorem applied to the right triangle on the ground,

$$100 = AB^2 = AD^2 + BD^2 = \left(\sqrt{3}h\right)^2 + \left(\frac{h}{\sqrt{3}}\right)^2 = \frac{10h^2}{3}.$$

Thus $h = \sqrt{30}$, and the closest of the given choices is 5.5.

14. **Answer (E):** Let r be the common ratio of the geometric series; then

$$S = \frac{1}{r} + 1 + r + r^2 + \cdots = \frac{\frac{1}{r}}{1-r} = \frac{1}{r-r^2}.$$

Because $S > 0$, the smallest value of S occurs when the value of $r - r^2$ is maximized. The graph of $f(r) = r - r^2$ is a downward-opening parabola with vertex $(\frac{1}{2}, \frac{1}{4})$, so the smallest possible value of S is $\frac{1}{(\frac{1}{4})} = 4$. The optimal series is $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$.

15. **Answer (D):** Suppose that one pair of opposite faces of the cube are assigned the numbers a and b, a second pair of opposite faces are assigned the numbers c and d, and the remaining pair of opposite faces are assigned the numbers e and f. Then the needed sum of products is $ace + acf + ade + adf + bce + bcf + bde + bdf = (a+b)(c+d)(e+f)$. The sum of these three factors is $2+3+4+5+6+7 = 27$. A product of positive numbers whose sum is fixed is maximized when the factors are all equal. Thus the greatest possible value occurs when $a+b = c+d = e+f = 9$, as in $(a,b,c,d,e,f) = (2,7,3,6,4,5)$. This results in the value $9^3 = 729$.

16. **Answer (E):** A sum of consecutive integers is equal to the number of integers in the sum multiplied by their median. Note that $345 = 3 \cdot 5 \cdot 23$. If there are an odd number of integers in the sum, then the median and the number of integers must be complementary factors of 345. The only possibilities are 3 integers with median $5 \cdot 23 = 115$, 5 integers with median $3 \cdot 23 = 69$, $3 \cdot 5 = 15$ integers with median 23, and 23 integers with median $3 \cdot 5 = 15$. Having more integers in the sum would force some of the integers to be negative. If there are an even number of integers in the sum, say $2k$, then the median will be $\frac{j}{2}$, where k and j are complementary factors of 345. The possibilities are 2 integers with median $\frac{345}{2}$, 6 integers with median $\frac{115}{2}$, and 10 integers with median $\frac{69}{2}$. Again, having more integers in the sum would force some of the integers to be negative. This gives a total of 7 solutions.

17. **Answer (D):** Let $x = BH$. Then $CH = 8 - x$ and $AH^2 = 7^2 - x^2 = 9^2 - (8-x)^2$, so $x = 2$ and $AH = \sqrt{45}$. By the Angle Bisector Theorem in $\triangle ACH$, $\frac{AP}{PH} = \frac{CA}{CH} = \frac{9}{6}$, so $AP = \frac{3}{5}AH$. Similarly, by the Angle Bisector Theorem in $\triangle ABH$, $\frac{AQ}{QH} = \frac{BA}{BH} = \frac{7}{2}$, so $AQ = \frac{7}{9}AH$. Then $PQ = AQ - AP = (\frac{7}{9} - \frac{3}{5})AH = \frac{8}{45}\sqrt{45} = \frac{8}{15}\sqrt{5}$.

18. **Answer (B):** The graph of the equation is symmetric about both axes. In the first quadrant, the equation is equivalent to $x^2 + y^2 - x - y = 0$. Completing the square gives $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2}$, so the graph in the first quadrant is an arc of the circle that is centered at $C(\frac{1}{2}, \frac{1}{2})$ and contains the points $A(1,0)$ and $B(0,1)$. Because C is the midpoint of $\overline{AB}$, the arc is a semicircle. The region enclosed by the graph in the first quadrant is the union of isosceles right triangle AOB, where $O(0,0)$ is the origin, and a semicircle with diameter $\overline{AB}$. The triangle and the semicircle have areas $\frac{1}{2}$ and $\frac{1}{2} \cdot \pi \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{4}$, respectively, so the area of the region enclosed by the graph in all quadrants is $4(\frac{1}{2} + \frac{\pi}{4}) = \pi + 2$.

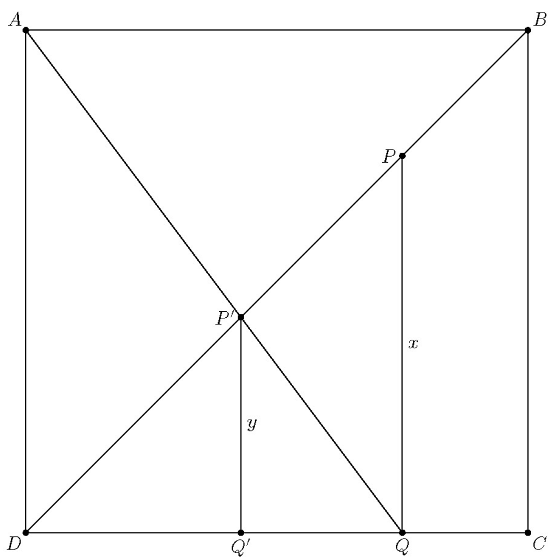
19. **Answer (B):** The probability that a flipper obtains his first head on the n^{th} flip is $(\frac{1}{2})^n$, because the sequence of outcomes must be exactly TT … TH, with $n-1$ Ts. Therefore the probability that all of them obtain their first heads on the n^{th} flip is $\left((\frac{1}{2})^n\right)^3 = (\frac{1}{8})^n$. The probability that all three flip their coins the same number of times is computed by summing an infinite geometric series:

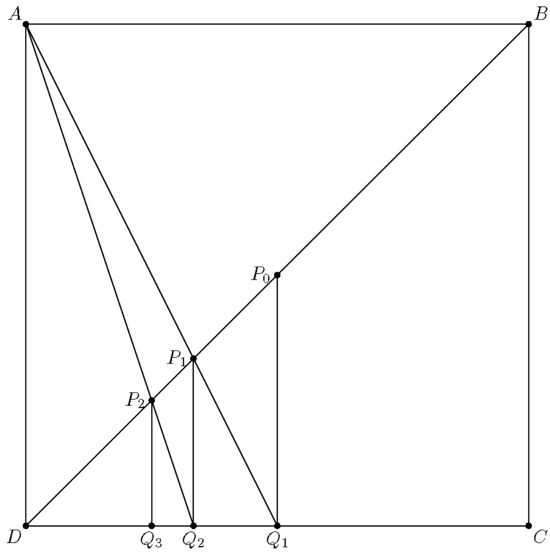
$$\left(\frac{1}{8}\right)^1 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \cdots = \frac{\frac{1}{8}}{1 - \frac{1}{8}} = \frac{1}{7}.$$

20. **Answer (A):** There must have been $10 + 10 + 1 = 21$ teams, and therefore there were $\binom{21}{3} = \frac{21 \cdot 20 \cdot 19}{6} = 1330$ subsets $\{A, B, C\}$ of three teams. If such a

subset does not satisfy the stated condition, then it consists of a team that beat both of the others. To count such subsets, note that there are 21 choices for the winning team and $\binom{10}{2} = 45$ choices for the other two teams in the subset. This gives $21 \cdot 45 = 945$ such subsets. The required answer is $1330 - 945 = 385$. To see that such a scenario is possible, arrange the teams in a circle, and let each team beat the 10 teams that follow it in clockwise order around the circle.

21. **Answer (B):** For any point P between B and D, let Q be the foot of the perpendicular from P to $\overline{CD}$, let P' be the intersection of $\overline{AQ}$ and $\overline{BD}$, and let Q' be the foot of the perpendicular from P' to $\overline{CD}$. Let $x = PQ$ and $y = P'Q'$. Because $\triangle PQD$ and $\triangle P'Q'D$ are isosceles right triangles, $DQ = x$ and $DQ' = y$. Because $\triangle ADQ$ is similar to $\triangle P'Q'Q$, $\frac{1}{x} = \frac{y}{x-y}$. Solving for y gives $y = \frac{x}{1+x}$.





Now let P_0 be the midpoint of $\overline{BD}$. Then $P_0Q_1 = DQ_1 = \frac{1}{2}$. It follows from the analysis above that $P_1Q_2 = DQ_2 = \frac{1}{3}$, $P_2Q_3 = DQ_3 = \frac{1}{4}$, and in general $P_iQ_{i+1} = DQ_{i+1} = \frac{1}{i+2}$. The area of $\triangle DQ_iP_i$ is

$$\frac{1}{2} \cdot DQ_i \cdot P_iQ_{i+1} = \frac{1}{2} \cdot \frac{1}{i+1} \cdot \frac{1}{i+2} = \frac{1}{2}\left(\frac{1}{i+1} - \frac{1}{i+2}\right).$$

The requested infinite sum telescopes:

$$\sum_{i=1}^{\infty} \text{Area of } \triangle DQ_iP_i = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots\right).$$

Its value is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

22. **Answer (B):** Because $\frac{1}{n} = \frac{abcdef}{999999}$, it follows that n is a divisor of $10^6 - 1 = (10^3-1)(10^3+1) = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$. Because $\frac{1}{n+6} = \frac{wxyz}{9999}$, it follows that $n+6$ divides $10^4 - 1 = 3^2 \cdot 11 \cdot 101$. However, $n+6$ does not divide $10^2 - 1 = 3^2 \cdot 11$, because otherwise the decimal representation of $\frac{1}{n+6}$ would have period 1 or 2. Thus $n = 101k - 6$, where $k = 1$, 3, 9, 11, 33, or 99. Because $n < 1000$, the only possible values of k are 1, 3, and 9, and the corresponding values of n are 95, 297, and 903. Of these, only $297 = 3^3 \cdot 11$ divides $10^6 - 1$. Thus $n \in [201, 400]$. It may be checked that $\frac{1}{297} = 0.\overline{003367}$ and $\frac{1}{303} = 0.\overline{0033}$.

23. **Answer (A):** In the first octant, the first inequality reduces to $x + y + z \le 1$, and the inequality defines the region under a plane that intersects the coordinate

axes at $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. By symmetry, the first inequality defines the region inside a regular octahedron centered at the origin and having internal diagonals of length 2. The upper half of this octahedron is a pyramid with altitude 1 and a square base of side length $\sqrt{2}$, so the volume of the octahedron is $2 \cdot \frac{1}{3} \cdot \left(\sqrt{2}\right)^2 \cdot 1 = \frac{4}{3}$. The second inequality defines the region obtained by translating the first region up 1 unit. The intersection of the two regions is bounded by another regular octahedron with internal diagonals of length 1. Because the linear dimensions of the third octahedron are half those of the first, its volume is $\frac{1}{8}$ that of the first, or $\frac{1}{6}$.

24. **Answer (D):** Note that $\gcd(a,b,c,d) = 77$ and $\operatorname{lcm}(a,b,c,d) = n$ if and only if $\gcd(\frac{a}{77}, \frac{b}{77}, \frac{c}{77}, \frac{d}{77}) = 1$ and $\operatorname{lcm}(\frac{a}{77}, \frac{b}{77}, \frac{c}{77}, \frac{d}{77}) = \frac{n}{77}$. Thus there are 77,000 ordered quadruples (a,b,c,d) such that $\gcd(a,b,c,d) = 1$ and $\operatorname{lcm}(a,b,c,d) = \frac{n}{77}$. Let $m = \frac{n}{77}$ and suppose that p is a prime that divides m. Let $A = A(p)$, $B = B(p)$, $C = C(p)$, $D = D(p)$, and $M = M(p) \geq 1$ be the exponents of p such that p^A, p^B, p^C, p^D, and p^M are the largest powers of p that divide a, b, c, d, and m, respectively. The gcd and lcm requirements are equivalent to $\min(A,B,C,D) = 0$ and $\max(A,B,C,D) = M$. For a fixed value of M, there are $(M+1)^4$ quadruples (A,B,C,D) with each entry in $\{0,1,\ldots,M\}$. There are M^4 of them for which $\min(A,B,C,D) \geq 1$, and also M^4 of them such that $\max(A,B,C,D) \leq M-1$. Finally, there are $(M-1)^4$ quadruples (A,B,C,D) such that $\min(A,B,C,D) \geq 1$ and $\max(A,B,C,D) \leq M-1$. Thus the number of quadruples such that $\min(A,B,C,D) = 0$ and $\max(A,B,C,D) = M$ is equal to $(M+1)^4 - 2M^4 + (M-1)^4 = 12M^2 + 2 = 2(6M^2+1)$. Multiplying these quantities over all primes that divide m yields the total number of quadruples (a,b,c,d) with the required properties. Thus

$$77{,}000 = 2^3 \cdot 5^3 \cdot 7 \cdot 11 = \prod_{p|m} 2(6(M(p))^2 + 1).$$

Note that $6(M(p))^2 + 1$ is odd and this product must contain three factors of 2, so there must be exactly three primes that divide m. Let p_1, p_2, and p_3 be these primes. Note that $6 \cdot 1^2 + 1 = 7$, $6 \cdot 2^2 + 1 = 5^2$, and $6 \cdot 3^2 + 1 = 5 \cdot 11$. None of these could appear as a factor more than once because 77,000 is not divisible by 7^2, 5^4, or 11^2. Moreover, the product of these three is equal to $5^3 \cdot 7 \cdot 11$. All other factors of the form $6M^2+1$ are greater than these three, so without loss of generality the only solution is $M(p_1) = 1$, $M(p_2) = 2$, and $M(p_3) = 3$. It follows that $m = p_1^1 p_2^2 p_3^3$, and the smallest value of m occurs when $p_1 = 5$, $p_2 = 3$, and $p_3 = 2$. Therefore the smallest possible values of m and n are $5 \cdot 3^2 \cdot 2^3 = 360$ and $77(5 \cdot 3^2 \cdot 2^3) = 27{,}720$, respectively.

25. **Answer (A):** Express each term of the sequence (a_n) as $2^{\frac{b_n}{19}}$. (Equivalently, let b_n be the logarithm of a_n to the base $\sqrt[19]{2}$.) The recursive definition of the

sequence (a_n) translates into $b_0 = 0$, $b_1 = 1$, and $b_n = b_{n-1} + 2b_{n-2}$ for $n \geq 2$. Then the product $a_1 a_2 \cdots a_k$ is an integer if and only if $\sum_{i=1}^{k} b_i$ is divisible by 19. Let $c_n = b_n \bmod 19$. It follows that $a_1 a_2 \cdots a_k$ is an integer if and only if $p_k = \sum_{i=1}^{k} c_i$ is divisible by 19. Let $q_k = p_k \bmod 19$. Because the largest answer choice is 21, it suffices to compute c_k and q_k successively for k from 1 up to at most 21, until q_k first equals 0. The modular computations are straightforward from the definitions.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
c_k	1	1	3	5	11	2	5	9	0	18	18	16	14	8	17	14	10
q_k	1	2	5	10	2	4	9	18	18	17	16	13	8	16	14	9	0

Thus the requested answer is 17.

OR

Using standard techniques, the recurrence relation for b_n can be solved to get $b_n = \frac{1}{3}(2^n - (-1)^n)$. Let $S_k = b_1 + b_2 + \cdots + b_k$. Then it is straightforward to show that $S_k = \frac{1}{3}(2^{k+1} - 1)$ for k odd, and $S_k = \frac{2}{3}(2^k - 1)$ for k even. Let $P_k = a_1 a_2 \cdots a_k$. It follows that, for k odd, P_k is an integer if and only if 19 divides $2^{k+1} - 1$; and, for k even, P_k is an integer if and only if 19 divides $2^k - 1$. A little computation shows that this first occurs at $k = 17$, when $2^{18} - 1 = 2^{18} - 1 = (2^9 - 1)(2^9 + 1) = 511 \cdot 513 = 511 \cdot 19 \cdot 27$. (In fact, one can show that P_k is an integer if and only if k is congruent to 0 or -1 mod 18.)

Problems and solutions were contributed by Bernardo Abrego, Sam Baethge, Barb Currier, Marta Eso, Chuck Garner, Jerry Grossman, Joe Kennedy, Michael Khoury, Matthew McMullen, Harold Reiter, and David Wells.

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