MATHEMATICAL ASSOCIATION OF AMERICA
CELEBRATING A CENTURY OF ADVANCING MATHEMATICS

Solutions Pamphlet

American Mathematics Competitions

66th Annual

AMC 12 B

American Mathematics Contest 12B

Wednesday, February 25, 2015

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

MAA American Mathematics Competitions
Attn: Publications, PO Box 471, Annapolis Junction, MD 20701
Phone 800.527.3690 | Fax 240.396.5647 | amcinfo@maa.org

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 11 Subcommittee Chair:

Jerrold W. Grossman

1. **Answer (C):**

$$2-(-2)^{-2}=2-\frac{1}{(-2)^2}=2-\frac{1}{4}=\frac{7}{4}$$

2. **Answer (B):** The first two tasks together took 100 minutes—from 1:00 to 2:40. Therefore each task took 50 minutes. Marie began the third task at 2:40 and finished 50 minutes later, at 3:30 PM.

3. **Answer (A):** Let x be the integer Isaac wrote two times, and let y be the integer Isaac wrote three times. Then $2x+3y=100$. If $x=28$, then $3y=100-2\cdot 28=44$, and y cannot be an integer. Therefore $y=28$ and $2x=100-3\cdot 28=16$, so $x=8$.

4. **Answer (B):** Marta finished 6th, so Jack finished 5th. Therefore Todd finished 3rd and Rand finished 2nd. Because Hikmet was 6 places behind Rand, it was Hikmet who finished 8th. (David finished 10th.)

5. **Answer (B):** If the Sharks win the next N games, then they win $\frac{1+N}{3+N}\cdot 100\%$ of the games. Therefore $\frac{1+N}{3+N}\geq\frac{95}{100}=\frac{19}{20}$, so $20+20N\geq 57+19N$. Therefore $N\geq 37$.

OR

If the Tigers win no more games, then their 2 wins should be no more than 5%, or $\frac{1}{20}$, of the games played. So the minimum number of games played must be at least 40, and $N\geq 37$.

6. **Answer (A):** There are $13\cdot 13=169$ entries in the body of the table. An entry is odd if and only if both its row factor and its column factor are odd. There are 6 odd whole numbers between 0 and 12, so there are $6\cdot 6=36$ odd entries in the body of the table. The required fraction is $\frac{36}{169}=0.213\ldots\approx 0.21$.

7. **Answer (D):** The lines of symmetry are the 15 lines joining a vertex to the midpoint of the opposite side, so $L=15$. There is rotational symmetry around the center of the 15-gon, and the smallest positive angle of rotation that will transform the 15-gon onto itself is $\frac{360}{15}=24$ degrees; therefore $R=24$. The sum is $15+24=39$.

8. **Answer (D):**

$$\left(625^{\log_5 2015}\right)^{\frac{1}{4}} = \left(\left(5^4\right)^{\log_5 2015}\right)^{\frac{1}{4}} = \left(5^{4\log_5 2015}\right)^{\frac{1}{4}} = \left(5^{\log_5 2015}\right)^{4\cdot\frac{1}{4}} = 2015$$

9. **Answer (C):** Let x be the probability that Larry wins the game. Then $x = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot x$. To see this, note that Larry can win by knocking the bottle off the ledge on his first throw; if he and Julius both miss, then it is as if they started the game all over. Thus $x = \frac{1}{2} + \frac{1}{4}x$, so $\frac{3}{4}x = \frac{1}{2}$ or $x = \frac{2}{3}$.

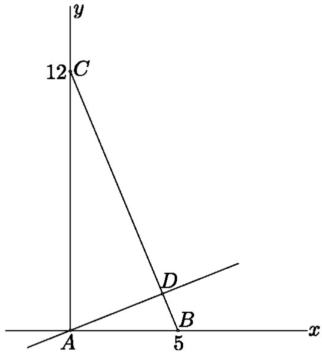
OR

For Larry to win on his nth throw, there must be $2n-2$ misses—$n-1$ by Larry and $n-1$ by Julius—followed by a hit by Larry. Because the probability of each of these independent events is $\frac{1}{2}$, the probability that Larry wins on his nth throw is $\left(\frac{1}{2}\right)^{2n-1}$. Therefore the probability that Larry wins the game is given by a geometric series:

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{2n-1} &= \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \cdots\\ &= \frac{1}{2}\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right)\\ &= \frac{1}{2}\cdot\frac{1}{1-\frac{1}{4}} = \frac{1}{2}\cdot\frac{4}{3} = \frac{2}{3}.\end{aligned}$$

10. **Answer (C):** Let the side lengths be $a < b < c$. By the Triangle Inequality $a + b > c$; it follows that perimeter $P = a + b + c > 2c$. Then $2c < P < 15$, $2c < 14$, and $c < 7$. The only triangles (denoted by three-digit numbers with decreasing digits) that are not equilateral or isosceles are 653, 652, 643, 543, 542, and 432. Of these, only 543 is a right triangle, so the answer is 5.

11. **Answer (E):** Label the vertices of the triangle $A = (0,0)$, $B = (5,0)$, and $C = (0,12)$. By the Pythagorean Theorem $BC = 13$. Two altitudes are 5 and 12. Let $\overline{AD}$ be the third altitude. The area of this triangle is 30, so $\frac{1}{2}\cdot AD\cdot BC = 30$. Therefore $AD = \frac{2\cdot 30}{BC} = \frac{60}{13}$. The sum of the lengths of the altitudes is $5 + 12 + \frac{60}{13} = \frac{281}{13}$.



12. **Answer (D):** If $(x-a)(x-b)+(x-b)(x-c)=0$, then $(x-b)(2x-(a+c))=0$, so the two roots are b and $\frac{a+c}{2}$. The maximum value of their sum is $9+\frac{8+7}{2}=16.5$.

13. **Answer (B):** Because $\angle BAC$ and $\angle BDC$ intercept the same arc, $\angle BDC = 70^\circ$. Then $\angle ADC = 110^\circ$ and $\angle ABC = 180^\circ - \angle ADC = 70^\circ$. Thus $\triangle ABC$ is isosceles, and therefore $AC = BC = 6$.

14. **Answer (D):** Let x equal the area of the circle, y the area of the triangle, and z the area of the overlapped sector. The answer is $(x-z)-(y-z)=x-y$. The area of the circle is 4π and the area of the triangle is $\frac{\sqrt{3}}{4}\cdot 4^2 = 4\sqrt{3}$, so the result is $4(\pi-\sqrt{3})$.

15. **Answer (D):** Rachelle needs a total of at least 14 points to get a 3.5 or higher GPA, so she needs a total of at least 6 points in English and History. The probability of a C in English is $1-\frac{1}{6}-\frac{1}{4}=\frac{7}{12}$, and the probability of a C in History is $1-\frac{1}{4}-\frac{1}{3}=\frac{5}{12}$. The probability that Rachelle earns exactly 6, 7, or 8 total points is computed as follows:
6 points: $\frac{1}{6}\cdot\frac{5}{12}+\frac{1}{4}\cdot\frac{1}{3}+\frac{7}{12}\cdot\frac{1}{4}=\frac{43}{144}$
7 points: $\frac{1}{6}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{4}=\frac{17}{144}$
8 points: $\frac{1}{6}\cdot\frac{1}{4}=\frac{6}{144}$
The probability that Rachelle will get at least a 3.5 GPA is

$$\frac{43}{144}+\frac{17}{144}+\frac{6}{144}=\frac{66}{144}=\frac{11}{24}.$$

16. **Answer (C):** The distance from a vertex of the hexagon to its center is 6. The height of the pyramid can be calculated by the Pythagorean Theorem using the right triangle with other leg 6 and hypotenuse 8; it is $\sqrt{8^2 - 6^2} = 2\sqrt{7}$. The volume is then

$$\frac{1}{3}Bh = \frac{1}{3} \cdot 6\left(6^2 \cdot \frac{\sqrt{3}}{4}\right) \cdot 2\sqrt{7} = 36\sqrt{21}.$$

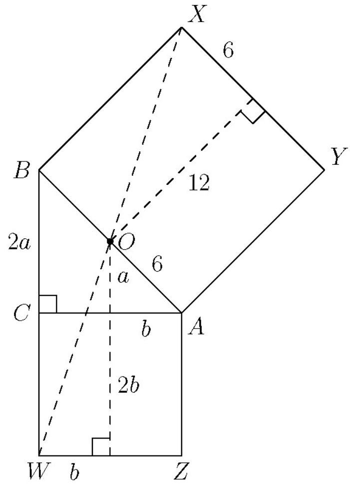
17. **Answer (D):** The probability of exactly two heads is $\binom{n}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{n-2}$, and this must equal the probability of three heads, $\binom{n}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{n-3}$. This results in the equation

$$\frac{n(n-1)}{2} \cdot \frac{3}{4} = \frac{n(n-1)(n-2)}{6} \cdot \frac{1}{4} \qquad \text{or} \qquad \frac{3}{8} = \frac{n-2}{24}.$$

Therefore $n = 11$.

18. **Answer (D):** To be composite, a number must have at least two prime factors, and the smallest prime number is 2. Therefore the smallest element in the range of r is $2 + 2 = 4$. To see that all integers greater than 3 are in the range, note that $r(2^n) = 2n$ for all $n \geq 2$, and $r(2^n \cdot 3) = 2n + 3$ for all $n \geq 1$.

19. **Answer (C):** Let O be the center of the circle on which X, Y, Z, and W lie. Then O lies on the perpendicular bisectors of segments $\overline{XY}$ and $\overline{ZW}$, and $OX = OW$. Note that segments $\overline{XY}$ and $\overline{AB}$ have the same perpendicular bisector and segments $\overline{ZW}$ and $\overline{AC}$ have the same perpendicular bisector, from which it follows that O lies on the perpendicular bisectors of segments $\overline{AB}$ and $\overline{AC}$; that is, O is the circumcenter of $\triangle ABC$. Because $\angle C = 90^\circ$, O is the midpoint of hypotenuse $\overline{AB}$. Let $a = \frac{1}{2}BC$ and $b = \frac{1}{2}CA$. Then $a^2 + b^2 = 6^2$ and $12^2 + 6^2 = OX^2 = OW^2 = b^2 + (a + 2b)^2$. Solving these two equations simultaneously gives $a = b = 3\sqrt{2}$. Thus the perimeter of $\triangle ABC$ is $12 + 2a + 2b = 12 + 12\sqrt{2}$.



20. **Answer (B):** Computing from the definition leads to the following values of $f(i, j)$ for $i = 0, 1, 2, 3, 4, 5, 6$ (the horizontal coordinate in the table) and $j = 0, 1, 2, 3, 4$ (the vertical coordinate).

4	0	1	1	0	3	1	1
3	4	0	4	1	1	1	1
2	3	4	2	4	3	1	1
1	2	3	0	3	1	1	1
0	1	2	3	0	3	1	1
	0	1	2	3	4	5	6

If follows that $f(i, 2) = 1$ for all $i \geq 5$.

21. **Answer (D):** Assume that there are t steps in this staircase and it took Dash $d+1$ jumps. Then the possible values of t are $5d+1, 5d+2, 5d+3, 5d+4, 5d+5$. On the other hand, it took Cozy $d + 20$ jumps, and $t = 2d + 39$ or $t = 2d + 40$. There are 10 possible combinations but only 3 of them lead to integer values of d: $t = 5d + 3 = 2d + 39$, or $t = 5d + 1 = 2d + 40$, or $t = 5d + 4 = 2d + 40$. The possible values of t are 63, 66, and 64, and $s = 63 + 66 + 64 = 193$. The answer is $1 + 9 + 3 = 13$.

22. **Answer (D):** To make the analysis easier, suppose first that everyone gets up and moves to the chair directly across the table. The reseating rule now is that each person must sit in the same chair or in an adjacent chair. There must be either 0, 2, 4, or 6 people who choose the same chair; otherwise there would be an odd-sized gap, which would not permit all the people in that gap to sit in an adjacent chair. If no people choose the same chair, then either everyone moves left, which can be done in 1 way, or everyone moves right, which can be done in 1 way, or people swap with a neighbor, which can be done in 2 ways, for a total of 4 possibilities. If two people choose the same chair, then they must be either directly opposite each other or next to each other; there are $3 + 6 = 9$ such pairs. The remaining four people must swap in pairs, and that can be done in just 1 way in each case. If four people choose the same chair, there are 6 ways to choose those people and the other two people swap. Finally, there is 1 way for everyone to choose the same chair. Therefore there are $4 + 9 + 6 + 1 = 20$ ways in which the reseating can be done.

23. **Answer (B):** Because the volume and surface area are numerically equal, $abc = 2(ab+ac+bc)$. Rewriting the equation as $ab(c-6)+ac(b-6)+bc(a-6) = 0$ shows that $a \leq 6$. The original equation can also be written as $(a-2)bc-2ab-2ac = 0$. Note that if $a = 2$, this becomes $b + c = 0$, and there are no solutions. Otherwise, multiplying both sides by $a - 2$ and adding $4a^2$ to both sides gives $[(a-2)b-2a][(a-2)c-2a] = 4a^2$. Consider the possible values of a.

$a = 1$: $(b + 2)(c + 2) = 4$
There are no solutions in positive integers.

$a = 3$: $(b - 6)(c - 6) = 36$
The 5 solutions for (b, c) are $(7, 42)$, $(8, 24)$, $(9, 18)$, $(10, 15)$, and $(12, 12)$.

$a = 4$: $(b - 4)(c - 4) = 16$
The 3 solutions for (b, c) are $(5, 20)$, $(6, 12)$, and $(8, 8)$.

$a = 5$: $(3b - 10)(3c - 10) = 100$
Each factor must be congruent to 2 modulo 3, so the possible pairs of factors are $(2, 50)$ and $(5, 20)$. The solutions for (b, c) are $(4, 20)$ and $(5, 10)$, but only $(5, 10)$ has $a \leq b$.

$a = 6$: $(b - 3)(c - 3) = 9$
The solutions for (b, c) are $(4, 12)$ and $(6, 6)$, but only $(6, 6)$ has $a \leq b$.

Thus in all there are 10 ordered triples (a, b, c): $(3, 7, 42)$, $(3, 8, 24)$, $(3, 9, 18)$, $(3, 10, 15)$, $(3, 12, 12)$, $(4, 5, 20)$, $(4, 6, 12)$, $(4, 8, 8)$, $(5, 5, 10)$, and $(6, 6, 6)$.

24. **Answer (D):** Points A, B, C, D, and R all lie on the perpendicular bisector of $\overline{PQ}$. Assume R lies between A and B. Let $y = AR$ and $x = \frac{AP}{5}$. Then $BR = 39 - y$ and $BP = 8x$, so $y^2 + 24^2 = 25x^2$ and $(39 - y)^2 + 24^2 = 64x^2$. Subtracting the two equations gives $x^2 = 39-2y$, from which $y^2+50y-399 = 0$, and the only positive solution is $y = 7$. Thus $AR = 7$, and $BR = 32$.

Note that circles A and B are determined by the assumption that R lies between A and B. Thus because the four circles are noncongruent, R does not lie between C and D. Let $w = CR$ and $z = \frac{CP}{5}$. Then $DR = 39 + w$ and $DP = 8z$, so $w^2 + 24^2 = 25z^2$ and $(39+w)^2 + 24^2 = 64z^2$. Subtracting the two equations gives $z^2 = 39 + 2w$, from which $w^2 - 50w - 399 = 0$, and the only positive solution is $w = 57$. Thus $CR = 57$ and $DR = 96$. Again, the uniqueness of the solution implies that R must indeed lie between A and B.

The requested sum is $7 + 32 + 57 + 96 = 192$.

25. **Answer (B):** Modeling the bee's path with complex numbers, set $P_0 = 0$ and $z = e^{\pi i/6}$. It follows that for $j \geq 1$,

$$P_j = \sum_{k=1}^{j} kz^{k-1}.$$

Thus

$$P_{2015} = \sum_{k=0}^{2015} kz^{k-1} = \sum_{k=0}^{2014} (k+1)\, z^k = \sum_{k=0}^{2014} \sum_{j=0}^{k} z^k.$$

Interchanging the order of summation and summing the geometric series gives

$$\begin{aligned}
P_{2015} &= \sum_{j=0}^{2014} \sum_{k=j}^{2014} z^k = \sum_{j=0}^{2014} z^j \sum_{k=0}^{2014-j} z^k \\
&= \sum_{j=0}^{2014} \frac{z^j\left(z^{2015-j}-1\right)}{z-1} = \sum_{j=0}^{2014} \frac{z^{2015}-z^j}{z-1} = \frac{1}{z-1} \sum_{j=0}^{2014} \left(z^{2015}-z^j\right) \\
&= \frac{1}{z-1}\left(2015z^{2015} - \sum_{j=0}^{2014} z^j\right) = \frac{1}{z-1}\left(2015z^{2015} - \frac{z^{2015}-1}{z-1}\right) \\
&= \frac{1}{(z-1)^2}\left(2015z^{2015}(z-1) - z^{2015} + 1\right) \\
&= \frac{1}{(z-1)^2}\left(2015z^{2016} - 2016z^{2015} + 1\right).
\end{aligned}$$

Note that $z^{12} = 1$ and thus $z^{2016} = (z^{12})^{168} = 1$ and $z^{2015} = \frac{1}{z}$. It follows that

$$P_{2015} = \frac{2016}{(z-1)^2}\left(1 - \frac{1}{z}\right) = \frac{2016}{z(z-1)}.$$

Finally,

$$|z-1|^2 = \left|\cos\left(\frac{\pi}{6}\right) - 1 + i\sin\left(\frac{\pi}{6}\right)\right|^2 = \left|\frac{\sqrt{3}}{2} - 1 + \frac{i}{2}\right|^2 = 2 - \sqrt{3} = \frac{(\sqrt{3}-1)^2}{2},$$

and thus

$$|P_{2015}| = \left|\frac{2016}{z(z-1)}\right| = \frac{2016}{|z-1|} = \frac{2016\sqrt{2}}{\sqrt{3}-1} = 1008\sqrt{2}(\sqrt{3}+1)$$
$$= 1008\sqrt{6} + 1008\sqrt{2}.$$

The requested sum is $1008 + 6 + 1008 + 2 = 2024$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Steve Blasberg, Tom Butts, Barbara Currier, Steven Davis, Steve Dunbar, Zuming Feng, Silvia Fernandez, Charles Garner, Richard Gibbs, Jerry Grossman, Joe Kennedy, Cap Khoury, Steve Miller, David Wells, and Carl Yerger.

The

MAA American Mathematics Competitions

are supported by

Academy of Applied Science
Akamai Foundation
American Mathematical Society
American Statistical Association
Art of Problem Solving, Inc.
Casualty Actuarial Society
Conference Board of the Mathematical Sciences
The D.E. Shaw Group
Google
IDEA MATH, LLC
Jane Street Capital
Math for America Inc.
Mu Alpha Theta
National Council of Teachers of Mathematics
Simons Foundation
Society for Industrial & Applied Mathematics
Star League, Inc.