MATHEMATICAL ASSOCIATION OF AMERICA
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Solutions Pamphlet

American Mathematics Competitions

66th Annual

AMC 12 A

American Mathematics Contest 12A

Tuesday, February 3, 2015

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

MAA American Mathematics Competitions
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The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 11 Subcommittee Chair:

Jerrold W. Grossman

1. **Answer (C):**
$$(1-1+25+0)^{-1}\times 5=\frac{1}{25}\times 5=\frac{1}{5}$$

2. **Answer (E):** By the Triangle Inequality the third side must be greater than $20-15=5$ and less than $20+15=35$. Therefore the perimeter must be greater than $5+20+15=40$ and less than $35+20+15=70$. Among the choices only 72 cannot be the perimeter.

3. **Answer (E):** The sum of the 14 test scores was $14\cdot 80=1120$. The sum of all 15 test scores was $15\cdot 81=1215$. Therefore Payton's score was $1215-1120=95$.

OR

To bring the average up to 81, the total must include 1 more point for each of the 14 students, in addition to 81 points for Payton. Therefore Payton's score was $81+14=95$.

4. **Answer (B):** Let x and y be the two positive numbers, with $x>y$. Then $x+y=5(x-y)$. Thus $4x=6y$, so $\frac{x}{y}=\frac{3}{2}$.

5. **Answer (D):** As long as x and y and their rounded values are positive, rounding the dividend x up in a division problem $\frac{x}{y}$ makes the answer larger, and rounding x down makes the answer smaller. Similarly, rounding the divisor y up makes the answer smaller, and rounding y down makes the answer larger. In a subtraction problem $x-y$, rounding x up or rounding y down increases the answer, and rounding x down or rounding y up decreases it. Only in choice (D) do all the roundings contribute to increasing the answer. In the other situations, the estimate may be larger or smaller than the exact value, depending on the the amount by which each number is rounded and their values. In particular, the rounding may make the answer smaller.
For (A), $\frac{999\,999}{900}-490>\frac{1\,000\,000}{1\,000}-500$.
For (B), $\frac{999\,999}{900}-510>\frac{1\,000\,000}{1\,000}-500$.
For (C), $\frac{999\,999}{1\,001}-490>\frac{1\,000\,000}{1\,000}-500$.
For (E), $\frac{1\,000\,001}{1\,001}-490>\frac{1\,000\,000}{1\,000}-500$.

6. **Answer (B):** Let p be Pete's present age, and let c be Claire's age. Then $p-2=3(c-2)$ and $p-4=4(c-4)$. Solving these equations gives $p=20$ and

$c = 8$. Thus Pete is 12 years older than Claire, so the ratio of their ages will be $2:1$ when Claire is 12 years old. That will occur $12 - 8 = 4$ years from now.

7. **Answer (D):** Let r, h, R, H be the radii and heights of the first and second cylinders, respectively. The volumes are equal, so $\pi r^2 h = \pi R^2 H$. Also $R = r + 0.1r = 1.1r$. Thus $\pi r^2 h = \pi(1.1r)^2 H = \pi(1.21r^2)H$. Dividing by πr^2 yields $h = 1.21H = H + 0.21H$. Thus the first height is 21% more than the second height.

8. **Answer (C):** Let the sides of the rectangle have lengths $3a$ and $4a$. By the Pythagorean Theorem, the diagonal has length $5a$. Because $5a = d$, the side lengths are $\frac{3}{5}d$ and $\frac{4}{5}d$. Therefore the area is $\frac{3}{5}d \cdot \frac{4}{5}d = \frac{12}{25}d^2$, so $k = \frac{12}{25}$.

9. **Answer (C):** Because the marbles left for Cheryl are determined at random, the second of Cheryl's marbles is equally likely to be any of the 5 marbles other than her first marble. One of those 5 marbles matches her first marble in color. Therefore the probability is $\frac{1}{5}$.

OR

Because all the choices are made at random, Cheryl is equally likely to take any of the $\binom{6}{2} = 15$ possible pairs of marbles. Exactly 3 of these are pairs of same-colored marbles. Therefore the requested probability is $\frac{3}{15} = \frac{1}{5}$.

10. **Answer (E):** Adding 1 to both sides of the equation and factoring yields $(x+1)(y+1) = 81 = 3^4$. Because x and y are distinct positive integers and $x > y$, the only possibility is that $x+1 = 3^3 = 27$ and $y+1 = 3^1 = 3$. Therefore $x = 26$.

11. **Answer (D):** If the smaller circle is in the interior of the larger circle, there are no common tangent lines. If the smaller circle is internally tangent to the larger circle, there is exactly one common tangent line. If the circles intersect at two points, there are exactly two common tangent lines. If the circles are externally tangent, there are exactly three tangent lines. Finally, if the circles do not intersect, there are exactly four tangent lines. Therefore, k can be any of the numbers 0, 1, 2, 3, or 4, which gives 5 possibilities.

12. **Answer (B):** The y-intercepts of the two parabolas are -2 and 4, respectively,

and in order to intersect the x-axis, the first must open upward and the second downward. Because the area of the kite is 12, the x-intercepts of both parabolas must be -2 and 2. Thus $4a-2=0$ so $a=\frac{1}{2}$, and $4-4b=0$ so $b=1$. Therefore $a+b=1.5$.

13. **Answer (E):** Note that each of the 12 teams plays 11 games, so $\frac{12\cdot 11}{2}=66$ games are played in all. If every game ends in a draw, then each team will have a score of 11, so statement (E) is not true. Each of the other statements is true. Each of the games generates 2 points in the score list, regardless of its outcome, so the sum of the scores must be $66\cdot 2=132$; thus (D) is true. Because the sum of an odd number of odd numbers plus any number of even numbers is odd, and 132 is even, there must be an even number of odd scores; thus (A) is true. Because there are 12 scores in all, there must also be an even number of even scores; thus (B) is true. Two teams cannot both have a score of 0 because the game between them must result in 1 point for each of them or 2 points for one of them; thus (C) is true.

14. **Answer (D):** By the change of base formula, $\dfrac{1}{\log_m n}=\log_n m$. Thus

$$1=\frac{1}{\log_2 a}+\frac{1}{\log_3 a}+\frac{1}{\log_4 a}=\log_a 2+\log_a 3+\log_a 4=\log_a 24.$$

It follows that $a=24$.

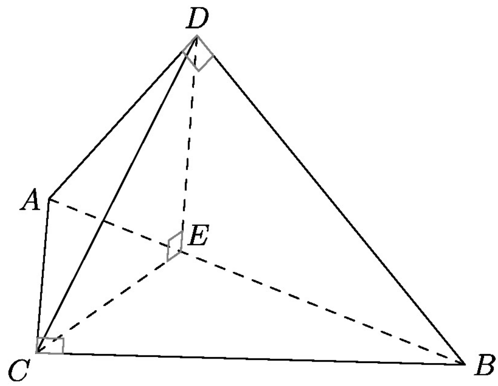
15. **Answer (C):** The numerator and denominator of this fraction have no common factors. To express the fraction as a decimal requires rewriting it with a power of 10 as the denominator. The smallest denominator that permits this is 10^{26}:

$$\frac{123\,456\,789}{2^{26}\cdot 5^4}=\frac{123\,456\,789\cdot 5^{22}}{2^{26}\cdot 5^4\cdot 5^{22}}=\frac{123\,456\,789\cdot 5^{22}}{10^{26}},$$

so the numeral will have 26 places after the decimal point. In fact

$$\frac{123\,456\,789}{2^{26}\cdot 5^4}=0.00294\,34392\,21382\,14111\,32812\,5.$$

16 **Answer (C):** Triangles ABC and ABD are $3-4-5$ right triangles with area 6. Let $\overline{CE}$ be the altitude of $\triangle ABC$. Then $CE=\frac{12}{5}$. Likewise in $\triangle ABD$, $DE=\frac{12}{5}$. Triangle CDE has sides $\frac{12}{5}$, $\frac{12}{5}$, and $\frac{12}{5}\sqrt{2}$, so it is an isosceles right triangle with right angle CED. Therefore $\overline{DE}$ is the altitude of the tetrahedron to base ABC. The tetrahedron's volume is $\frac{1}{3}\cdot 6\cdot\frac{12}{5}=\frac{24}{5}$.



17. **Answer (A):** There are $2^8 = 256$ equally likely outcomes of the coin tosses. Classify the possible arrangements around the table according to the number of heads flipped. There is 1 possibility with no heads, and there are 8 possibilities with exactly one head. There are $\binom{8}{2} = 28$ possibilities with exactly two heads, 8 of which have two adjacent heads. There are $\binom{8}{3} = 56$ possibilities with exactly three heads, of which 8 have three adjacent heads and $8 \cdot 4$ have exactly two adjacent heads (8 possibilities to place the two adjacent heads and 4 possibilities to place the third head). Finally, there are 2 possibilities using exactly four heads where no two of them are adjacent (heads and tails must alternate). There cannot be more than four heads without two of them being adjacent. Therefore there are $1 + 8 + (28 - 8) + (56 - 8 - 32) + 2 = 47$ possibilities with no adjacent heads, and the probability is $\frac{47}{256}$.

18. **Answer (C):** The zeros of f are integers and their sum is a, so a is an integer. If r is an integer zero, then $r^2 - ar + 2a = 0$ or

$$a = \frac{r^2}{r-2} = r + 2 + \frac{4}{r-2}.$$

So $\frac{4}{r-2} = a - r - 2$ must be an integer, and the possible values of r are 6, 4, 3, 1, 0, and -2. The possible values of a are 9, 8, 0, and -1, all of which yield integer zeros of f, and their sum is 16.

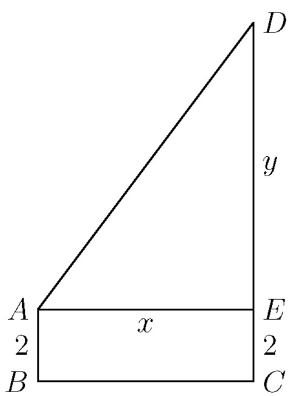
OR

As above, a must be an integer. The function f has zeros at

$$x = \frac{a \pm \sqrt{a^2 - 8a}}{2}.$$

These values are integers only if $a^2 - 8a = w^2$ for some integer w. Solving for a in terms of w gives $a = 4 \pm \sqrt{16 + w^2}$, so $16 + w^2$ must be a perfect square. The only integer solutions for w are 0 and ± 3, from which it follows that the values of a are 0, 8, 9, and -1, all of which yield integer values of x. The requested sum is 16.

19. **Answer (B):** In every such quadrilateral, $CD \geq AB$. Let E be the foot of the perpendicular from A to $\overline{CD}$; then $CE = 2$ and $AE = BC$. Let $x = AE$ and $y = DE$; then $AD = 2 + y$. By the Pythagorean Theorem, $x^2 + y^2 = (2 + y)^2$, or $x^2 = 4 + 4y$. Therefore x is even, say $x = 2z$, and $z^2 = 1 + y$. The perimeter of the quadrilateral is $x + 2y + 6 = 2z^2 + 2z + 4$. Increasing positive integer values of z give the required quadrilaterals, with increasing perimeter. For $z = 31$ the perimeter is 1988, and for $z = 32$ the perimeter is 2116. Therefore there are 31 such quadrilaterals.



20. **Answer (A):** Let g and h be the lengths of the altitudes of T and T' from the sides with lengths 8 and b, respectively. The Pythagorean Theorem implies that $g = \sqrt{5^2 - 4^2} = 3$, and so the area of T is $\frac{1}{2} \cdot 8 \cdot 3 = 12$, and the perimeter is $5 + 5 + 8 = 18$. The Pythagorean Theorem implies that $h = \frac{1}{2}\sqrt{4a^2 - b^2}$. Thus $18 = 2a + b$ and

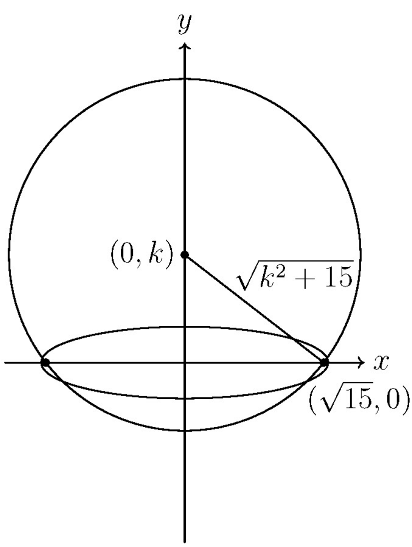
$$12 = \frac{1}{2}b \cdot \frac{1}{2}\sqrt{4a^2 - b^2} = \frac{1}{4}b\sqrt{4a^2 - b^2}.$$

Solving for a and substituting in the square of the second equation yields

$$\begin{aligned} 12^2 &= \frac{b^2}{16}\left(4a^2 - b^2\right) = \frac{b^2}{16}\left((18 - b)^2 - b^2\right) \\ &= \frac{b^2}{16} \cdot 18 \cdot (18 - 2b) = \frac{9}{4}b^2\,(9 - b)\,. \end{aligned}$$

Thus $64 - b^2(9 - b) = b^3 - 9b^2 + 64 = (b - 8)(b^2 - b - 8) = 0$. Because T and T' are not congruent, it follows that $b \neq 8$. Hence $b^2 - b - 8 = 0$ and the positive solution of this equation is $\frac{1}{2}(\sqrt{33} + 1)$. Because $25 < 33 < 36$, the solution is between $\frac{1}{2}(5 + 1) = 3$ and $\frac{1}{2}(6 + 1) = 3.5$, so the closest integer is 3.

21. **Answer (D):** The ellipse with equation $x^2 + 16y^2 = 16$ is centered at the origin, with a major axis of length 8 and a minor axis of length 2. If the foci have coordinates $(\pm c, 0)$, then $c^2 + 1^2 = 4^2$. Thus $c = \pm\sqrt{15}$. Any circle passing through both foci must have its center on the y-axis; thus r is at least as large as the distance from the foci to the y-axis. That is, $r \geq \sqrt{15}$. For any $k \geq 0$, the circle of radius $\sqrt{k^2 + 15}$ and center $(0, k)$ passes through both foci (in the interior of the ellipse) and the points $(0, k \pm \sqrt{k^2 + 15})$. The point $(0, k + \sqrt{k^2 + 15})$ is in the exterior of the ellipse since $k + \sqrt{k^2 + 15} > \sqrt{15} > 1$. The point $(0, k - \sqrt{k^2 + 15})$ is in the exterior of the ellipse if and only if $k - \sqrt{k^2 + 15} < -1$, that is, if and only if $k < 7$. Thus, for $k \geq 0$, the circle with center $(0, k)$ intersects the ellipse in four points if and only if $0 \leq k < 7$. As k increases, the radius $r = \sqrt{k^2 + 15}$ increases as well, so the set of possible radii is the interval $[\sqrt{15}, \sqrt{7^2 + 15}) = [\sqrt{15}, 8)$. The requested answer is $\sqrt{15} + 8$.



22. **Answer (D):** Note that $S(1) = 2$, $S(2) = 4$, and $S(3) = 8$. Call a sequence with A and B entries valid if it does not contain 4 or more consecutive symbols that are the same. For $n \geq 4$, every valid sequence of length $n - 1$ can be extended to a valid sequence of length n by appending a symbol different from its last symbol. Similarly, valid sequences of length $n-2$ or $n-3$ can be extended

to valid sequences of length n by appending either two or three equal symbols different from its last symbol. Note that all of these sequences are pairwise distinct. Conversely, every valid sequence of length n ends with either one, two, or three equal consecutive symbols. Removal of these equal symbols at the end produces every valid sequence of length $n-1$, $n-2$, or $n-3$, respectively. Thus $S(n) = S(n-1) + S(n-2) + S(n-3)$. This recursive formula implies that the remainders modulo 3 of the sequence $S(n)$ for $1 \le n \le 16$ are

$$2, 1, 2, 2, 2, 0, 1, 0, 1, 2, 0, 0, 2, 2, 1, 2.$$

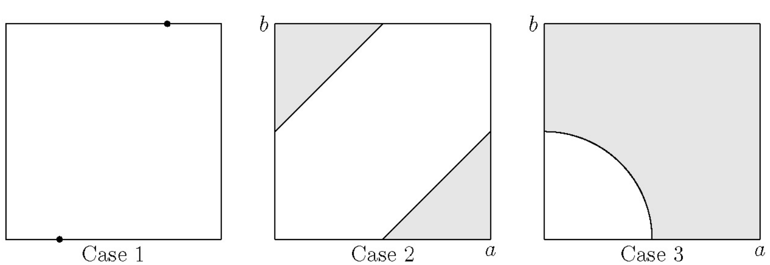
Thus the sequence is periodic with period-length 13. Because $2015 = 13 \cdot 155$, it follows that $S(2015) \equiv S(13) \equiv 2 \pmod 3$. Similarly, the remainders modulo 4 of the sequence $S(n)$ for $1 \le n \le 7$ are $2, 0, 0, 2, 2, 0, 0$. Thus the sequence is periodic with period-length 4. Because $2015 = 4 \cdot 503 + 3$, it follows that $S(2015) \equiv S(3) \equiv 0 \pmod 4$. Therefore $S(2015) = 4k$ for some integer k, and $4k \equiv 2 \pmod 3$. Hence $k \equiv 2 \pmod 3$ and $S(2015) = 4k \equiv 8 \pmod{12}$.

23. **Answer (A):** Let the square have vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, and consider three cases.

Case 1: The chosen points are on opposite sides of the square. In this case the distance between the points is at least $\frac{1}{2}$ with probability 1.

Case 2: The chosen points are on the same side of the square. It may be assumed that the points are $(a, 0)$ and $(b, 0)$. The pairs of points in the ab-plane that meet the requirement are those within the square $0 \le a \le 1$, $0 \le b \le 1$ that satisfy either $b \ge a + \frac{1}{2}$ or $b \le a - \frac{1}{2}$. These inequalities describe the union of two isosceles right triangles with leg length $\frac{1}{2}$, together with their interiors. The area of the region is $\frac{1}{4}$, and the area of the square is 1, so the probability that the pair of points meets the requirement in this case is $\frac{1}{4}$.

Case 3: The chosen points are on adjacent sides of the square. It may be assumed that the points are $(a, 0)$ and $(0, b)$. The pairs of points in the ab-plane that meet the requirement are those within the square $0 \le a \le 1$, $0 \le b \le 1$ that satisfy $\sqrt{a^2 + b^2} \ge \frac{1}{2}$. These inequalities describe the region inside the square and outside a quarter-circle of radius $\frac{1}{2}$. The area of this region is $1 - \frac{1}{4}\pi(\frac{1}{2})^2 = 1 - \frac{\pi}{16}$, which is also the probability that the pair of points meets the requirement in this case.



Cases 1 and 2 each occur with probability $\frac{1}{4}$, and Case 3 occurs with probability $\frac{1}{2}$. The requested probability is

$$\frac{1}{4}\cdot 1+\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\left(1-\frac{\pi}{16}\right)=\frac{26-\pi}{32},$$

and $a+b+c=59$.

24. **Answer (D):** There are 20 possible values for each of a and b, namely those in the set

$$S=\left\{0,1,\frac{1}{2},\frac{3}{2},\frac{1}{3},\frac{2}{3},\frac{4}{3},\frac{5}{3},\frac{1}{4},\frac{3}{4},\frac{5}{4},\frac{7}{4},\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},\frac{6}{5},\frac{7}{5},\frac{8}{5},\frac{9}{5}\right\}.$$

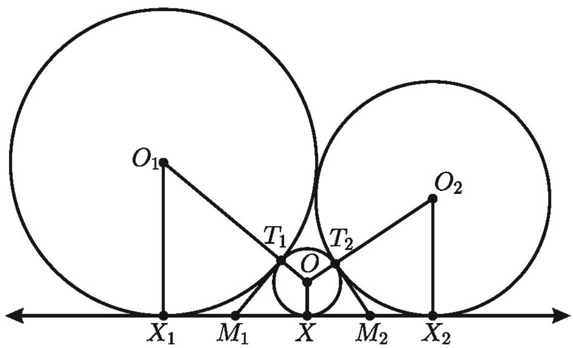
If x and y are real numbers, then $(x+iy)^2=x^2-y^2+i(2xy)$ is real if and only if $xy=0$, that is, $x=0$ or $y=0$. Therefore $(x+iy)^4$ is real if and only if $x^2-y^2=0$ or $xy=0$, that is, $x=0$, $y=0$, or $x=\pm y$. Thus $((\cos(a\pi)+i\sin(b\pi))^4$ is a real number if and only if $\cos(a\pi)=0$, $\sin(b\pi)=0$, or $\cos(a\pi)=\pm\sin(b\pi)$. If $\cos(a\pi)=0$ and $a\in S$, then $a=\frac{1}{2}$ or $a=\frac{3}{2}$ and b has no restrictions, so there are 40 pairs (a,b) that satisfy the condition. If $\sin(b\pi)=0$ and $b\in S$, then $b=0$ or $b=1$ and a has no restrictions, so there are 40 pairs (a,b) that satisfy the condition, but there are 4 pairs that have been counted already, namely $(\frac{1}{2},0)$, $(\frac{1}{2},1)$, $(\frac{3}{2},0)$, and $(\frac{3}{2},1)$. Thus the total so far is $40+40-4=76$.

Note that $\cos(a\pi)=\sin(b\pi)$ implies that $\cos(a\pi)=\cos(\pi(\frac{1}{2}-b))$ and thus $a\equiv\frac{1}{2}-b \pmod 2$ or $a\equiv-\frac{1}{2}+b \pmod 2$. If the denominator of $b\in S$ is 3 or 5, then the denominator of a in simplified form would be 6 or 10, and so $a\notin S$. If $b=\frac{1}{2}$ or $b=\frac{3}{2}$, then there is a unique solution to either of the two congruences, namely $a=0$ and $a=1$, respectively. For every $b\in\left\{\frac{1}{4},\frac{3}{4},\frac{5}{4},\frac{7}{4}\right\}$, there is exactly one solution $a\in S$ to each of the previous congruences. None of the solutions are equal to each other because if $\frac{1}{2}-b\equiv-\frac{1}{2}+b \pmod 2$, then $2b\equiv 1 \pmod 2$; that is, $b=\frac{1}{2}$ or $b=\frac{3}{2}$. Similarly, $\cos(a\pi)=-\sin(b\pi)=\sin(-b\pi)$ implies that $\cos(a\pi)=\cos(\pi(\frac{1}{2}+b))$ and thus $a\equiv\frac{1}{2}+b \pmod 2$ or $a\equiv-\frac{1}{2}-b \pmod 2$. If the denominator of $b\in S$ is 3 or 5, then the denominator of a would be 6 or 10, and so $a\notin S$. If $b=\frac{1}{2}$ or $b=\frac{3}{2}$, then there is a unique solution to either of the two congruences, namely $a=1$ and $a=0$, respectively. For every $b\in\left\{\frac{1}{4},\frac{3}{4},\frac{5}{4},\frac{7}{4}\right\}$, there is exactly one solution $a\in S$ to each of the previous congruences, and, as before, none of these solutions are equal to each other. Thus there are a total of $2+8+2+8=20$ pairs $(a,b)\in S^2$ such that $\cos(a\pi)=\pm\sin(b\pi)$. The requested probability is $\frac{76+20}{400}=\frac{96}{400}=\frac{6}{25}$.

Note: By de Moivre's Theorem the fourth power of the complex number $x+iy$ is real if and only if it lies on one of the four lines $x=0$, $y=0$, $x=y$, or $x=-y$. Then the counting of (a,b) pairs proceeds as above.

25. **Answer (D):** Suppose that circles C_1 and C_2 in the upper half-plane have

centers O_1 and O_2 and radii r_1 and r_2, respectively. Assume that C_1 and C_2 are externally tangent and tangent to the x-axis at X_1 and X_2, respectively. Let C with center O and radius r be the circle externally tangent to C_1 and C_2 and tangent to the x-axis. Let X be the point of tangency of C with the x-axis, and let T_1 and T_2 be the points of tangency of C with C_1 and C_2, respectively. Let M_1 and M_2 be the points on the x-axis such that $\overline{M_1T_1} \perp \overline{O_1T_1}$ and $\overline{M_2T_2} \perp \overline{O_2T_2}$.



Because $\overline{M_1X_1}$ and $\overline{M_1T_1}$ are both tangent to C_1, it follows that $X_1M_1 = M_1T_1$. Similarly, $\overline{M_1T_1}$ and $\overline{M_1X}$ are both tangent to C, and thus $M_1T_1 = M_1X$. Because $\angle OT_1M_1$, $\angle M_1X_1O_1$, $\angle M_1T_1O$, and $\angle OXM_1$ are all right angles and $\angle T_1M_1X = \pi - \angle X_1M_1T_1$, it follows that quadrilaterals $O_1X_1M_1T_1$ and M_1XOT_1 are similar. Thus

$$\frac{r_1}{X_1M_1} = \frac{O_1X_1}{X_1M_1} = \frac{M_1X}{XO} = \frac{X_1M_1}{r}.$$

Therefore $X_1M_1 = \sqrt{rr_1}$, and similarly $M_2X_2 = \sqrt{rr_2}$. By the distance formula,

$$(r_1 + r_2)^2 = (O_1O_2)^2 = (X_1X_2)^2 + (r_1 - r_2)^2.$$

Thus

$$\begin{aligned} 2\sqrt{r_1r_2} = X_1X_2 &= X_1M_1 + M_1X + XM_2 + M_2X_2 \\ &= 2(X_1M_1 + M_2X_2) = 2\sqrt{r}(\sqrt{r_1} + \sqrt{r_2}); \end{aligned}$$

that is,

$$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}. \tag{1}$$

It follows that

$$\sum_{C\in L_k} \frac{1}{\sqrt{r(C)}} = \sum_{C\in L_k} \left(\frac{1}{\sqrt{r(C_1)}} + \frac{1}{\sqrt{r(C_2)}} \right),$$

where C_1 and C_2 are the consecutive circles in $\bigcup_{j=0}^{k-1} L_j$ that are tangent to C. Note that every circle in $\bigcup_{j=0}^{k-1} L_j$ appears twice in the sum on the right-hand side, except for the two circles in L_0, which appear only once. Thus

$$\sum_{C\in L_k} \frac{1}{\sqrt{r(C)}} = 2\sum_{j=1}^{k-1} \sum_{C\in L_j} \frac{1}{\sqrt{r(C)}} + \sum_{C\in L_0} \frac{1}{\sqrt{r(C)}}.$$

In particular, if $k = 1$, then

$$\sum_{C\in L_1} \frac{1}{\sqrt{r(C)}} = \sum_{C\in L_0} \frac{1}{\sqrt{r(C)}} = \frac{1}{70} + \frac{1}{73}.$$

For simplicity let $x = \frac{1}{70} + \frac{1}{73}$. Let $k \geq 2$, and suppose by induction that for $1 \leq j \leq k-1$,

$$\sum_{C\in L_j} \frac{1}{\sqrt{r(C)}} = 3^{j-1}x.$$

It follows that

$$\sum_{C\in L_k} \frac{1}{\sqrt{r(C)}} = 2\left(\sum_{j=1}^{k-1} 3^{j-1}x\right) + x = 2x\left(\frac{3^{k-1}-1}{2}\right) + x = x3^{k-1}.$$

Therefore

$$\begin{aligned}\sum_{C\in S} \frac{1}{\sqrt{r(C)}} &= \sum_{k=0}^{6} \sum_{C\in L_k} \frac{1}{\sqrt{r(C)}} = x + \sum_{k=1}^{6} x3^{k-1} = x\left(1 + \frac{3^6-1}{2}\right)\\ &= x\left(\frac{3^6+1}{2}\right) = \frac{143}{70\cdot 73}\left(\frac{730}{2}\right) = \frac{143}{14}.\end{aligned}$$

Note: Equation (1) is a special case of the Kissing Circles Theorem.

The problems and solutions in this contest were proposed by Bernardo Abrego, Tom Butts, Barb Currier, Steven Davis, Steve Dunbar, Silvia Fernandez, Charles Garner, Peter Gilchrist, Jerry Grossman, Jon Kane, Dan Kennedy, Joe Kennedy, Michael Khoury, Roger Waggoner, Dave Wells, Ronald Yannone, and Carl Yerger.

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