MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

65th Annual

AMC 12 A

American Mathematics Contest 12 A

Tuesday, February 4, 2014

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

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1. **Answer (C):** Note that
$$10 \cdot \left(\frac{1}{2}+\frac{1}{5}+\frac{1}{10}\right)^{-1} = 10 \cdot \left(\frac{8}{10}\right)^{-1} = \frac{25}{2}.$$

2. **Answer (B):** Because child tickets are half the price of adult tickets, the price of 5 adult tickets and 4 child tickets is the same as the price of $5+\frac{1}{2}\cdot 4 = 7$ adult tickets. In the same way, the price of 8 adult tickets and 6 child tickets is the same as the price of $8+\frac{1}{2}\cdot 6 = 11$ adult tickets, which is equal to $11 \cdot \frac{1}{7} \cdot 24.50 = 38.50$ dollars.

3. **Answer (B):** If Ralph passed the orange house first, then because the blue and yellow houses are not neighbors, the house color ordering must be orange, blue, red, yellow. If Ralph passed the blue house first, then there are 2 possible placements for the yellow house, and each choice determines the placement of the orange and red houses. These 2 house color orderings are blue, orange, yellow, red, and blue, orange, red, yellow. There are 3 possible orderings for the colored houses.

4. **Answer (A):** One cow gives $\frac{b}{a}$ gallons in c days, so one cow gives $\frac{b}{ac}$ gallons in 1 day. Thus d cows will give $\frac{bd}{ac}$ gallons in 1 day. In e days d cows will give $\frac{bde}{ac}$ gallons of milk.

5. **Answer (C):** Because over 50% of the students scored 90 or lower, and over 50% of the students scored 90 or higher, the median score is 90. The mean score is
$$\frac{10}{100}\cdot 70 + \frac{35}{100}\cdot 80 + \frac{30}{100}\cdot 90 + \frac{25}{100}\cdot 100 = 87,$$
for a difference of $90-87=3$.

6. **Answer (D):** Let $10a+b$ be the larger number. Then $10a+b-(10b+a) = 5(a+b)$, which simplifies to $2a=7b$. The only nonzero digits that satisfy this equation are $a=7$ and $b=2$. Therefore the larger number is 72, and the required sum is $72+27=99$.

7. **Answer (A):** Each term in a geometric progression is r times the preceding term. The ratio is
$$r = \frac{3^{\frac{1}{3}}}{3^{\frac{1}{2}}} = 3^{\frac{1}{3}-\frac{1}{2}} = 3^{-\frac{1}{6}}.$$

Thus the third term is correctly given as $r \cdot 3^{\frac{1}{3}} = 3^{-\frac{1}{6}} \cdot 3^{\frac{1}{3}} = 3^{\frac{1}{6}}$, and the fourth term is $r \cdot 3^{\frac{1}{6}} = 3^{-\frac{1}{6}} \cdot 3^{\frac{1}{6}} = 3^0 = 1$.

8. **Answer (C):** Let $P > 100$ be the listed price. Then the price reductions in dollars are as follows:

   Coupon 1: $\frac{P}{10}$
   Coupon 2: 20
   Coupon 3: $\frac{18}{100}(P - 100)$

   Coupon 1 gives a greater price reduction than coupon 2 when $\frac{P}{10} > 20$, that is, $P > 200$. Coupon 1 gives a greater price reduction than coupon 3 when $\frac{P}{10} > \frac{18}{100}(P - 100)$, that is, $P < 225$. The only choice that satisfies these inequalities is \$219.95.

9. **Answer (B):** The five consecutive integers starting with a are a, $a+1$, $a+2$, $a+3$, and $a+4$. Their average is $a+2 = b$. The average of five consecutive integers starting with b is $b + 2 = a + 4$.

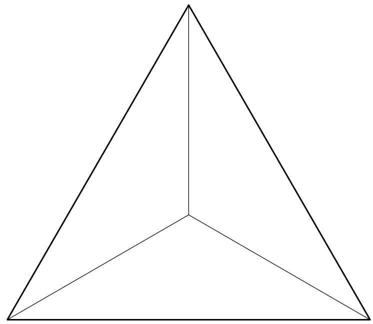
10. **Answer (B):**

    Let h represent the altitude of each of the isosceles triangles from the base on the equilateral triangle. Then the area of each of the congruent isosceles triangles is $\frac{1}{2} \cdot 1 \cdot h = \frac{1}{2}h$. The sum of the areas of the three isosceles triangles is the same as the area of the equilateral triangle, so $3 \cdot \frac{1}{2}h = \frac{1}{4}\sqrt{3}$ and $h = \frac{1}{6}\sqrt{3}$. As a consequence, the Pythagorean Theorem implies that the side length of the isosceles triangles is

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{12}} = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3}.$$

OR

Suppose that the isosceles triangles are constructed internally with respect to the equilateral triangle. Because the sum of their areas is equal to the area of the equilateral triangle, it follows that the center of the equilateral triangle is a vertex common to all three isosceles triangles. The distance from the center of the equilateral triangle to any of its vertices is two thirds of its height. Thus the required side length is equal to $\frac{2}{3} \cdot \frac{1}{2}\sqrt{3} = \frac{1}{3}\sqrt{3}$.



11. **Answer (C):** Let d be the remaining distance after one hour of driving, and let t be the remaining time until his flight. Then $d = 35(t+1)$, and $d = 50(t-0.5)$. Solving gives $t = 4$ and $d = 175$. The total distance from home to the airport is $175 + 35 = 210$ miles.

OR

Let d be the distance between David's home and the airport. The time required to drive the entire distance at 35 MPH is $\frac{d}{35}$ hours. The time required to drive at 35 MPH for the first 35 miles and 50 MPH for the remaining $d - 35$ miles is $1 + \frac{d-35}{50}$. The second trip is 1.5 hours quicker than the first, so

$$\frac{d}{35} - \left(1 + \frac{d-35}{50}\right) = 1.5.$$

Solving yields $d = 210$ miles.

12. **Answer (D):**

Let the larger and smaller circles have radii R and r, respectively. Then the length of chord $\overline{AB}$ can be expressed as both r and $2R\sin 15°$. The ratio of the areas of the circles is

$$\frac{\pi R^2}{\pi r^2} = \frac{1}{4\sin^2 15°} = \frac{1}{2(1-\cos 30°)} = \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}.$$

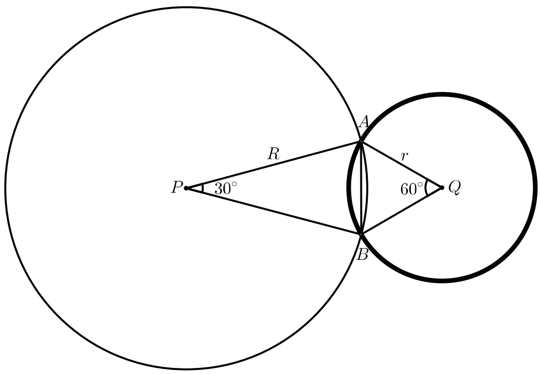
OR

Let the larger and smaller circles have radii R and r, and centers P and Q, respectively. Because $\triangle QAB$ is equilateral, it follows that $r = AB$. The Law of Cosines applied to $\triangle PBA$ gives

$$\begin{aligned} r^2 = AB^2 &= PA^2 + PB^2 - 2PA \cdot PB \cos 30^\circ \\ &= 2R^2 - 2R^2 \cos 30^\circ = R^2(2-\sqrt{3}). \end{aligned}$$

Thus

$$\frac{\pi R^2}{\pi r^2} = \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}.$$



13. **Answer (B):** If each friend rooms alone, then there are $5! = 120$ ways to assign the guests to the rooms. If one pair of friends room together and the others room alone, then there are $\binom{5}{2} = 10$ ways to choose the roommates and then $5 \cdot 4 \cdot 3 \cdot 2 = 120$ ways to assign the rooms to the 4 sets of occupants, for a total of $10 \cdot 120 = 1200$ possible arrangements. The only other possibility is to have two sets of roommates. In this case the roommates can be chosen in $5 \cdot \frac{1}{2}\binom{4}{2} = 15$ ways (choose the solo lodger first), and then there are $5 \cdot 4 \cdot 3 = 60$ ways to assign the rooms, for a total of $15 \cdot 60 = 900$ possibilities. Therefore the answer is $120 + 1200 + 900 = 2220$.

14. **Answer (C):** Let $d = b - a$ be the common difference of the arithmetic progression. Then $b = a + d$, $c = a + 2d$, and because a, c, b is a geometric progression,

$$\frac{a+2d}{a} = \frac{a+d}{a+2d}.$$

Thus $(a+2d)(a+2d) = a(a+d)$, which simplifies to $3ad + 4d^2 = 0$. Because $d > 0$, it follows that $3a + 4d = 0$ and therefore $a = -4k$ and $d = 3k$ for some positive integer k. Thus $c = (-4k) + 2(3k) = 2k$, and the smallest value of c is $2 \cdot 1 = 2$.

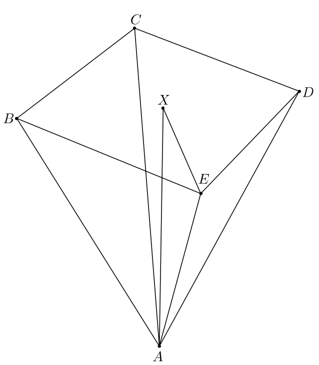
15. **Answer (B):** Note that $abcba = a000a + b0b0 + c00$. Because $1+2+\cdots+9 = \frac{1}{2}(9\cdot 10) = 45$, the sum of all integers of the form $a000a$ is $450\,045$. For each value of a there are $10 \cdot 10 = 100$ choices for b and c. Similarly, the sum of all integers of the form $b0b0$ is $45\,450$. For each value of b there are $9 \cdot 10 = 90$ choices for a and c. The sum of all integers of the form $c00$ is 4500, and for each c there are $9 \cdot 10 = 90$ choices for a and b. Thus $S = 100 \cdot 450\,045 + 90 \cdot 45\,450 + 90 \cdot 4500 = 45(1\,000\,100 + 90\,900 + 9000) = 45(1\,100\,000) = 49\,500\,000$. The sum of the digits is 18.

16. **Answer (D):** By direct multiplication, $8 \cdot 888\ldots 8 = 7111\ldots 104$, where the product has 2 fewer ones than the number of digits in $888\ldots 8$. Because $7 + 4 = 11$, the product must have $1000 - 11 = 989$ ones, so $k - 2 = 989$ and $k = 991$.

17. **Answer (A):** Connect the centers of the large sphere and the four small spheres at the top to form an inverted square pyramid as shown. Since $BCDE$ is a square of side 2, $EX = \sqrt{2}$. Also, $AE = 3$ and $\triangle AXE$ is a right triangle, so

$$AX = \sqrt{3^2 - \left(\sqrt{2}\right)^2} = \sqrt{7}.$$

The distance from the plane containing $BCDE$ to the top of the box is 1. Therefore the total height of the box is $2(1 + AX) = 2 + 2\sqrt{7}$.



18. **Answer (C):** For every $a > 0$, $a \neq 1$, the domain of $\log_a x$ is the set $\{x : x > 0\}$. Moreover, for $0 < a < 1$, $\log_a x$ is a decreasing function on its domain, and for $a > 1$, $\log_a x$ is an increasing function on its domain. Thus the function $f(x)$ is defined if and only if $\log_4(\log_{\frac{1}{4}}(\log_{16}(\log_{\frac{1}{16}}(x)))) > 0$, and this inequality is

equivalent to each of the following:

$$\log_{\frac{1}{4}}(\log_{16}(\log_{\frac{1}{16}}(x))) > 1, \quad 0 < \log_{16}(\log_{\frac{1}{16}} x) < \frac{1}{4},$$
$$1 < \log_{\frac{1}{16}} x < 2, \text{ and } \quad \frac{1}{256} < x < \frac{1}{16}.$$

Thus $\frac{m}{n} = \frac{1}{16} - \frac{1}{256} = \frac{15}{256}$, and $m + n = 271$.

19. **Answer (E):** Solve the equation for k to obtain $k = -\frac{12}{x} - 5x$. For each integer value of x except $x = 0$, there is a corresponding rational value for k. As a function of x, $|k| = \frac{12}{x} + 5x$ is increasing for $x \geq 2$. Thus by inspection, the integer values of x that ensure $|k| < 200$ satisfy the inequality $-39 \leq x \leq 39$. There are 78 such values. Assume that a and b are two different integer values of x that produce the same k. Then $k = -\frac{12}{a} - 5a = -\frac{12}{b} - 5b$, which simplifies to $(5ab - 12)(a - b) = 0$. Because $a \neq b$, it follows that $5ab = 12$, but there are no integers satisfying this equation. Thus the values of k corresponding to the 78 values of x are all distinct, and the answer is therefore 78.

20. **Answer (D):** Let B' be the reflection of point B across $\overline{AC}$, and let C' be the reflection of point C across $\overline{AB}$. Then $AB' = AB = 10$, $AC' = AC = 6$, $BE = B'E$, $CD = C'D$, and $\angle B'AC' = 120°$. By the Law of Cosines, $B'C'^2 = 10^2 + 6^2 - 2 \cdot 10 \cdot 6 \cos 120° = 196$; thus $B'C' = 14$. Furthermore, $B'C' \leq B'E + DE + C'D = BE + DE + CD$. Therefore the answer is 14.

21. **Answer (A):** If $x = n + r$, where n is an integer, $1 \leq n \leq 2013$, and $0 \leq r < 1$, then $f(x) = n(2014^r - 1)$. The condition $f(x) \leq 1$ is equivalent to $2014^r \leq 1 + \frac{1}{n}$, or $0 \leq r \leq \log_{2014}\left(\frac{n+1}{n}\right)$. Thus the required sum is

$$\log_{2014}\frac{2}{1} + \log_{2014}\frac{3}{2} + \log_{2014}\frac{4}{3} + \cdots + \log_{2014}\frac{2014}{2013}$$
$$= \log_{2014}\left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{2014}{2013}\right) = \log_{2014}(2014) = 1.$$

22. **Answer (B):** Because $2^2 < 5$ and $2^3 > 5$, there are either two or three integer powers of 2 strictly between any two consecutive integer powers of 5. Thus for each n there is at most one m satisfying the given inequalities, and the question asks for the number of cases in which there are three powers rather than two. Let d (respectively, t) be the number of nonnegative integers n less than 867 such that there are exactly two (respectively, three) powers of 2 strictly between

5^n and 5^{n+1}. Because $2^{2013} < 5^{867} < 2^{2014}$, it follows that $d + t = 867$ and $2d + 3t = 2013$. Solving the system yields $t = 279$.

23. **Answer (B):** Note that

$$\frac{10^n}{99^2} = \frac{10^n}{9801} = b_{n-1}b_{n-2}\dots b_2b_1b_0.\overline{b_{n-1}b_{n-2}\dots b_2b_1b_0}.$$

Subtracting the original equation gives

$$\frac{10^n - 1}{99^2} = b_{n-1}b_{n-2}\dots b_2b_1b_0.$$

Thus $10^n - 1 = 99^2 \cdot b_{n-1}b_{n-2}\dots b_2b_1b_0$. It follows that $10^n - 1$ is divisible by 11 and thus n is even, say $n = 2N$. For $0 \le j \le N - 1$, let $a_j = 10b_{2j+1} + b_{2j}$. Note that $0 \le a_j \le 99$, and because

$$\frac{10^{2N} - 1}{10^2 - 1} = 1 + 10^2 + 10^4 + \cdots + 10^{2(N-1)},$$

it follows that

$$\sum_{k=0}^{N-1} 10^{2k} = (10^2 - 1)\sum_{k=0}^{N-1} a_k 10^{2k},$$

and so

$$\sum_{k=0}^{N-1} 10^{2k} + \sum_{k=0}^{N-1} a_k 10^{2k} = \sum_{k=1}^{N} a_{k-1}10^{2k}.$$

Considering each side of the equation as numbers written in base 100, it follows that $1 + a_0 \equiv 0 \pmod{100}$, so $a_0 = 99$ and there is a carry for the 10^2 digit in the sum on the left side. Thus $1 + (1 + a_1) \equiv a_0 = 99 \pmod{100}$ and so $a_1 = 97$, and there is no carry for the 10^4 digit. Next, $1 + a_2 \equiv a_1 = 97 \pmod{100}$, and so $a_2 = 96$ with no carry for the 10^6 digit. In the same way $a_j = 98 - j$ for $1 \le j \le 98$. Then $1 + a_{99} \equiv a_{98} = 0 \pmod{100}$ would yield $a_{99} = 99$, and then the period would start again. Therefore $N = 99$ and $b_{n-1}b_{n-2}\dots b_2b_1b_0 = 0001020304\dots969799$. By momentarily including 9 and 8 as two extra digits, the sum would be $(0 + 1 + 2 + \cdots + 9)\cdot 20 = 900$, so the required sum is $900 - 9 - 8 = 883$.

24. **Answer (C):** For integers $n \ge 1$ and $k \ge 0$, if $f_{n-1}(x) = \pm k$, then $f_n(x) = k - 1$. Thus if $f_0(x) = \pm k$, then $f_k(x) = 0$. Furthermore, if $f_n(x) = 0$, then $f_{n+1}(x) = -1$ and $f_{n+2}(x) = 0$. It follows that the zeros of f_{100} are the solutions of $f_0(x) = 2k$ for $-50 \le k \le 50$. To count these solutions, note that

$$f_0(x) = \begin{cases} x + 200 & \text{if } x < -100, \\ -x & \text{if } -100 \le x < 100, \text{ and} \\ x - 200 & \text{if } x \ge 100. \end{cases}$$

The graph of $f_0(x)$ is piecewise linear with turning points at $(-100, 100)$ and $(100, -100)$. The line $y = 2k$ crosses the graph three times for $-49 \le k \le 49$ and twice for $k = \pm 50$. Therefore the number of zeros of $f_{100}(x)$ is $99 \cdot 3 + 2 \cdot 2 = 301$.

25. **Answer (B):** Let $O = (0, 0)$, $A = (4, 3)$, and $B = (-4, -3)$. Because $A, B \in P$ and O is the midpoint of $\overline{AB}$, it follows that $\overline{AB}$ is the latus rectum of the parabola P. Thus the directrix is parallel to $\overline{AB}$. Let T be the foot of the perpendicular from O to the directrix of P. Because $OT = OA = OB = 5$ and $\overline{OT}$ is perpendicular to $\overline{AB}$, it follows that $T = (3, -4)$. Thus the equation of the directrix is $y+4 = \frac{3}{4}(x-3)$, and in general form the equation is $4y-3x+25 = 0$.

Using the formula for the distance from a point to a line, as well as the definition of P as the locus of points equidistant from O and the directrix, the equation of P is

$$\sqrt{x^2 + y^2} = \frac{|4y - 3x + 25|}{\sqrt{4^2 + 3^2}}.$$

After squaring and rearranging, this is equivalent to

$$\begin{aligned} 25x^2 + 25y^2 &= 25(x^2 + y^2) = (4y - 3x + 25)^2 \\ &= 16y^2 + 9x^2 - 24xy + 25^2 + 50(4y - 3x), \end{aligned}$$

and

$$(4x + 3y)^2 = 25(25 + 2(4y - 3x)). \tag{1}$$

Assume x and y are integers. Then $4x + 3y$ is divisible by 5. If $4x + 3y = 5s$ for $s \in \mathbb{Z}$, then $2s^2 = 50+16y-12x = 50+16y-3(5s-3y) = 50+25y-15s$. Thus s is divisible by 5. If $s = 5t$ for $t \in \mathbb{Z}$, then $2t^2 = 2+y-3t$, and so $y = 2t^2+3t-2$. In addition $4x = 5s - 3y = 25t - 3y = 25t - 3(2t^2 + 3t - 2) = -6t^2 + 16t + 6$, and thus t is odd. If $t = 2u + 1$ for $u \in \mathbb{Z}$, then

$$x = -6u^2 + 2u + 4 \text{ and } y = 8u^2 + 14u + 3. \tag{2}$$

Conversely, if x and y are defined as in (2) for $u \in \mathbb{Z}$, then x and y are integers and they satisfy (1), which is the equation of P. Lastly, with $u \in \mathbb{Z}$,

$$\begin{aligned} |4x + 3y| &= \left|-24u^2 + 8u + 16 + 24u^2 + 42u + 9\right| \\ &= |50u + 25| \le 1000 \end{aligned}$$

if and only if u is an integer such that $|2u + 1| \le 39$. That is, $-20 \le u \le 19$, and so the required answer is $19 - (-21) = 40$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Tom Butts, Steven Davis, Peter Gilchrist, Jerry Grossman, Joe Kennedy, Gerald Kraus, Roger Waggoner, Kevin Wang, David Wells, LeRoy Wenstrom, and Ronald Yannone.

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