MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

64th Annual

AMC 12 B

American Mathematics Contest 12 B

Wednesday, February 20, 2013

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR TELLS YOU.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 12 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators are allowed. No problems on the test will *require* the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form.
8. When your proctor gives the signal, begin working on the problems. You will have **75 minutes** to complete the test.
9. When you finish the exam, *sign your name* in the space provided on the Answer Form.

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

Students who score 100 or above or finish in the top 5% on this AMC 12 will be invited to take the 31st annual American Invitational Mathematics Examination (AIME) on Thursday, March 14, 2013 or Wednesday, April 3, 2013. More details about the AIME and other information are on the back page of this test booklet.

The publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules. After the contest period, permission to make copies of problems in paper or electronic form including posting on web-pages for educational use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear the copyright notice.

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2013

AMC 12 B

DO NOT OPEN UNTIL WEDNESDAY, FEBRUARY 20, 2013

****Administration On An Earlier Date Will Disqualify Your School's Results****

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHERS' MANUAL, which is outside of this package. PLEASE READ THE MANUAL BEFORE FEBRUARY 20, 2013. Nothing is needed from inside this package until February 20.
2. Your PRINCIPAL or VICE-PRINCIPAL must verify on the AMC 12 CERTIFICATION FORM (found in the Teachers' Manual) that you followed all rules associated with the conduct of the exam.
3. The Answer Forms must be mailed by trackable mail to the AMC office no later than 24 hours following the exam.
4. *The publication, reproduction or communication of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, e-mail, internet or media of any type is a violation of the competition rules.*

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1. On a particular January day, the high temperature in Lincoln, Nebraska, was 16 degrees higher than the low temperature, and the average of the high and low temperatures was 3°. In degrees, what was the low temperature in Lincoln that day?

   (A) -13 **(B)** -8 **(C)** -5 **(D)** -3 **(E)** 11

2. Mr. Green measures his rectangular garden by walking two of the sides and finds that it is 15 steps by 20 steps. Each of Mr. Green's steps is 2 feet long. Mr. Green expects a half a pound of potatoes per square foot from his garden. How many pounds of potatoes does Mr. Green expect from his garden?

   (A) 600 **(B)** 800 **(C)** 1000 **(D)** 1200 **(E)** 1400

3. When counting from 3 to 201, 53 is the 51^{st} number counted. When counting backwards from 201 to 3, 53 is the n^{th} number counted. What is n?

   (A) 146 **(B)** 147 **(C)** 148 **(D)** 149 **(E)** 150

4. Ray's car averages 40 miles per gallon of gasoline, and Tom's car averages 10 miles per gallon of gasoline. Ray and Tom each drive the same number of miles. What is the cars' combined rate of miles per gallon of gasoline?

   (A) 10 **(B)** 16 **(C)** 25 **(D)** 30 **(E)** 40

5. The average age of 33 fifth-graders is 11. The average age of 55 of their parents is 33. What is the average age of all of these parents and fifth-graders?

   (A) 22 **(B)** 23.25 **(C)** 24.75 **(D)** 26.25 **(E)** 28

6. Real numbers x and y satisfy the equation $x^2 + y^2 = 10x - 6y - 34$. What is $x + y$?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 6 **(E)** 8

7. Jo and Blair take turns counting from 1 to one more than the last number said by the other person. Jo starts by saying "1", so Blair follows by saying "1, 2". Jo then says "1, 2, 3", and so on. What is the 53^{rd} number said?

   (A) 2 **(B)** 3 **(C)** 5 **(D)** 6 **(E)** 8

8. Line ℓ_1 has equation $3x - 2y = 1$ and goes through $A = (-1, -2)$. Line ℓ_2 has equation $y = 1$ and meets line ℓ_1 at point B. Line ℓ_3 has positive slope, goes through point A, and meets ℓ_2 at point C. The area of $\triangle ABC$ is 3. What is the slope of ℓ_3?

   (A) $\frac{2}{3}$ **(B)** $\frac{3}{4}$ **(C)** 1 **(D)** $\frac{4}{3}$ **(E)** $\frac{3}{2}$

9. What is the sum of the exponents of the prime factors of the square root of the largest perfect square that divides $12!$?

   (A) 5 **(B)** 7 **(C)** 8 **(D)** 10 **(E)** 12

10. Alex has 75 red tokens and 75 blue tokens. There is a booth where Alex can give two red tokens and receive in return a silver token and a blue token, and another booth where Alex can give three blue tokens and receive in return a silver token and a red token. Alex continues to exchange tokens until no more exchanges are possible. How many silver tokens will Alex have at the end?

    (A) 62 **(B)** 82 **(C)** 83 **(D)** 102 **(E)** 103

11. Two bees start at the same spot and fly at the same rate in the following directions. Bee A travels 1 foot north, then 1 foot east, then 1 foot upwards, and then continues to repeat this pattern. Bee B travels 1 foot south, then 1 foot west, and then continues to repeat this pattern. In what directions are the bees traveling when they are exactly 10 feet away from each other?

    (A) A east, B west

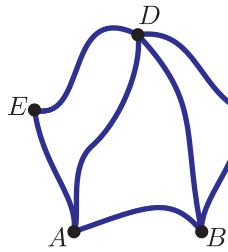
    (B) A north, B south

    (C) A north, B west

    (D) A up, B south

    (E) A up, B west

12. Cities A, B, C, D, and E are connected by roads $\widetilde{AB}$, $\widetilde{AD}$, $\widetilde{AE}$, $\widetilde{BC}$, $\widetilde{BD}$, $\widetilde{CD}$, and $\widetilde{DE}$. How many different routes are there from A to B that use each road exactly once? (Such a route will necessarily visit some cities more than once.)



(A) 7 **(B)** 9 **(C)** 12 **(D)** 16 **(E)** 18

13. The internal angles of quadrilateral $ABCD$ form an arithmetic progression. Triangles ABD and DCB are similar with $\angle DBA = \angle DCB$ and $\angle ADB = \angle CBD$. Moreover, the angles in each of these two triangles also form an arithmetic progression. In degrees, what is the largest possible sum of the two largest angles of $ABCD$?

(A) 210 **(B)** 220 **(C)** 230 **(D)** 240 **(E)** 250

14. Two non-decreasing sequences of nonnegative integers have different first terms. Each sequence has the property that each term beginning with the third is the sum of the previous two terms, and the seventh term of each sequence is N. What is the smallest possible value of N?

(A) 55 **(B)** 89 **(C)** 104 **(D)** 144 **(E)** 273

15. The number 2013 is expressed in the form

$$2013 = \frac{a_1!a_2!\cdots a_m!}{b_1!b_2!\cdots b_n!},$$

where $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ are positive integers and $a_1 + b_1$ is as small as possible. What is $|a_1 - b_1|$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

16. Let $ABCDE$ be an equiangular convex pentagon of perimeter 1. The pairwise intersections of the lines that extend the sides of the pentagon determine a five-pointed star polygon. Let s be the perimeter of this star. What is the difference between the maximum and the minimum possible values of s?

(A) 0 **(B)** $\frac{1}{2}$ **(C)** $\frac{\sqrt{5}-1}{2}$ **(D)** $\frac{\sqrt{5}+1}{2}$ **(E)** $\sqrt{5}$

17. Let a, b, and c be real numbers such that

$$\begin{cases} a+b+c=2, \text{ and} \\ a^2+b^2+c^2=12. \end{cases}$$

What is the difference between the maximum and minimum possible values of c?

(A) 2 **(B)** $\frac{10}{3}$ **(C)** 4 **(D)** $\frac{16}{3}$ **(E)** $\frac{20}{3}$

18. Barbara and Jenna play the following game, in which they take turns. A number of coins lie on a table. When it is Barbara's turn, she must remove 2 or 4 coins, unless only one coin remains, in which case she loses her turn. When it is Jenna's turn, she must remove 1 or 3 coins. A coin flip determines who goes first. Whoever removes the last coin wins the game. Assume both players use their best strategy. Who will win when the game starts with 2013 coins and when the game starts with 2014 coins?

(A) Barbara will win with 2013 coins, and Jenna will win with 2014 coins.

(B) Jenna will win with 2013 coins, and whoever goes first will win with 2014 coins.

(C) Barbara will win with 2013 coins, and whoever goes second will win with 2014 coins.

(D) Jenna will win with 2013 coins, and Barbara will win with 2014 coins.

(E) Whoever goes first will win with 2013 coins, and whoever goes second will win with 2014 coins.

19. In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

(A) 18 **(B)** 21 **(C)** 24 **(D)** 27 **(E)** 30

20. For $135^\circ < x < 180^\circ$, points $P = (\cos x, \cos^2 x)$, $Q = (\cot x, \cot^2 x)$, $R = (\sin x, \sin^2 x)$, and $S = (\tan x, \tan^2 x)$ are the vertices of a trapezoid. What is $\sin(2x)$?

(A) $2 - 2\sqrt{2}$ **(B)** $3\sqrt{3} - 6$ **(C)** $3\sqrt{2} - 5$ **(D)** $-\dfrac{3}{4}$ **(E)** $1 - \sqrt{3}$

21. Consider the set of 30 parabolas defined as follows: all parabolas have as focus the point $(0,0)$ and the directrix lines have the form $y = ax + b$ with a and b integers such that $a \in \{-2,-1,0,1,2\}$ and $b \in \{-3,-2,-1,1,2,3\}$. No three of these parabolas have a common point. How many points in the plane are on two of these parabolas?

(A) 720 **(B)** 760 **(C)** 810 **(D)** 840 **(E)** 870

22. Let $m > 1$ and $n > 1$ be integers. Suppose that the product of the solutions for x of the equation

$$8(\log_n x)(\log_m x) - 7\log_n x - 6\log_m x - 2013 = 0$$

is the smallest possible integer. What is $m + n$?

(A) 12 **(B)** 20 **(C)** 24 **(D)** 48 **(E)** 272

23. Bernardo chooses a three-digit positive integer N and writes both its base-5 and base-6 representations on a blackboard. Later LeRoy sees the two numbers Bernardo has written. Treating the two numbers as base-10 integers, he adds them to obtain an integer S. For example, if $N = 749$, Bernardo writes the numbers 10,444 and 3,245, and LeRoy obtains the sum $S = 13{,}689$. For how many choices of N are the two rightmost digits of S, in order, the same as those of $2N$?

(A) 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 25

24. Let ABC be a triangle where M is the midpoint of $\overline{AC}$, and $\overline{CN}$ is the angle bisector of $\angle ACB$ with N on $\overline{AB}$. Let X be the intersection of the median $\overline{BM}$ and the bisector $\overline{CN}$. In addition $\triangle BXN$ is equilateral and $AC = 2$. What is BN^2?

(A) $\dfrac{10 - 6\sqrt{2}}{7}$ **(B)** $\dfrac{2}{9}$ **(C)** $\dfrac{5\sqrt{2} - 3\sqrt{3}}{8}$ **(D)** $\dfrac{\sqrt{2}}{6}$ **(E)** $\dfrac{3\sqrt{3} - 4}{5}$

25. Let G be the set of polynomials of the form

$$P(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_2z^2 + c_1z + 50,$$

where $c_1, c_2, \ldots, c_{n-1}$ are integers and $P(z)$ has n distinct roots of the form $a + ib$ with a and b integers. How many polynomials are in G?

(A) 288 **(B)** 528 **(C)** 576 **(D)** 992 **(E)** 1056

MATHEMATICAL ASSOCIATION OF AMERICA

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WRITE TO US!

Correspondence about the problems and solutions for this AMC 12 and orders for publications should be addressed to:

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The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. Bernardo M. Abrego

2013 AIME

The 31st annual AIME will be held on Thursday, March 14, with the alternate on Wednesday, April 3. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate only if you score 120 or above or finish in the top 2.5% of the AMC 10, or if you score 100 or above or finish in the top 5% of the AMC 12. Top-scoring students on the AMC 10/12/AIME will be selected to take the 42nd Annual USA Mathematical Olympiad (USAMO) on April 30 - May 1, 2013. The best way to prepare for the AIME and USAMO is to study previous exams. Copies may be ordered as indicated below.

PUBLICATIONS

A complete listing of current publications, with ordering instructions, is at our web site: amc.maa.org