Mock AMC 10 2016

1. C
2. C
3. B
4. D
5. A
6. C
7. C
8. B
9. B
10. E
11. B
12. D
13. C
14. D
15. E
16. D
17. B
18. A
19. A
20. C
21. B
22. A
23. D
24. E
25. E

Q.1

$$\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{5}\cdot\frac{1}{6}\cdot\frac{1}{7}\cdot\frac{1}{8}\cdot\frac{1}{9}\cdot 2016N = 1$$

$$\frac{1}{2^1}\cdot\frac{1}{3^1}\cdot\frac{1}{2^2}\cdot\frac{1}{5^1}\cdot\frac{1}{2^1\cdot 3^1}\cdot\frac{1}{7}\cdot\frac{1}{2^3}\cdot\frac{1}{3^2}\cdot 2^5\cdot 3^2\cdot 7^1\cdot N = 1$$

$$\frac{1}{2^7\cdot 3^4\cdot 5^1\cdot 7^1}\cdot 2^5\cdot 3^2\cdot 7^1 N = 1$$

$$\frac{N}{2^2\cdot 3^2\cdot 5} = 1$$

$$N = 2^2\cdot 3^2\cdot 5 = 4\cdot 9\cdot 5 = \boxed{180}$$

Q2.

The next triangular number after $T_{63} = 2016$

is $T_{64} = \frac{64(65)}{2} = 32 \cdot 65 = 2080$

$2025 = 45^2$

2017 is the next prime

$2197 = 13^3$

$$
\begin{aligned}
2080 - 2016 &= 64 \\
2025 - 2016 &= 9 \\
2017 - 2016 &= 1 \\
2197 - 2016 &= 181 \\
\hline
&\;\; 255 = 2^8 - 1 \qquad n = 8
\end{aligned}
$$

Q.3 Notice that 2016 is a triangular number $T_{63} = 2016$

$$T_n = 2016 = 32 \cdot 63 = \frac{1}{2} 63 \cdot 64 = \frac{n(n+1)}{2}, \quad n = 63$$

The sum is $1 \cdot 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 4 + \dots + 63 \cdot 63$

$$S_{63} = 1^2 + 2^2 + 3^2 + \dots + 63^2$$

The formula for the sum of the first n squares is

$$S_n = \frac{n(n+1)(2n+1)}{6}$$

$$S_{63} = \frac{63 \cdot 64}{2} \cdot \frac{127}{3} = 85344$$

The sum of the digits is 85344

Q.4

$a + d = 100$

$bc = 99$

Represent $a, b, c,$ and d by

$m-3n, \; m-n, \; m+n, \; m+3n$ with common difference n.

$\quad a \qquad b \qquad c \qquad d$

$(m+3n)+(m-3n) = 2m = 100$ so $m = 50$

$bc = 99$ so $(m-n)(m+n) = 99$

$m^2 - n^2 = 99$

$2500 - n^2 = 99$

$2500 - 99 = n^2$

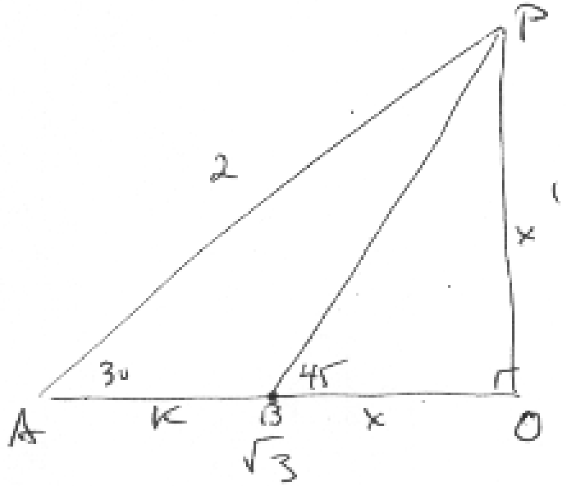
$2401 = n^2$

$49 = n$

The smallest number is

$50 - 3 \cdot 49 = \boxed{-97}$

Q.5



Δ AOP is 30-60-90
so the sides are in proportion $2:1:\sqrt{3}$

$$\frac{x}{x+k} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$3x = (x+k)\sqrt{3}$$

$$3x = x\sqrt{3} + k\sqrt{3}$$

$$x(3-\sqrt{3}) = k\sqrt{3}$$

$$x = \frac{k\sqrt{3}}{3-\sqrt{3}} = \frac{k\sqrt{3}(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})}$$

$$x = \frac{k(3\sqrt{3}+3)}{6} = \frac{1}{2}k(\sqrt{3}+1)$$

In this problem $k = 20$

so $x = 10(\sqrt{3}+1) = 10 + 10\sqrt{3}$

Q.6

$$P(K) = \frac{1}{2}$$

$$P(R) = \frac{1}{3}$$

$$P(A) = \frac{1}{4}$$

$$P(K \cup R \cup A) = 1 - P(K \cup R \cup A)'$$

$$= 1 - P(K' \cap R' \cap A')$$

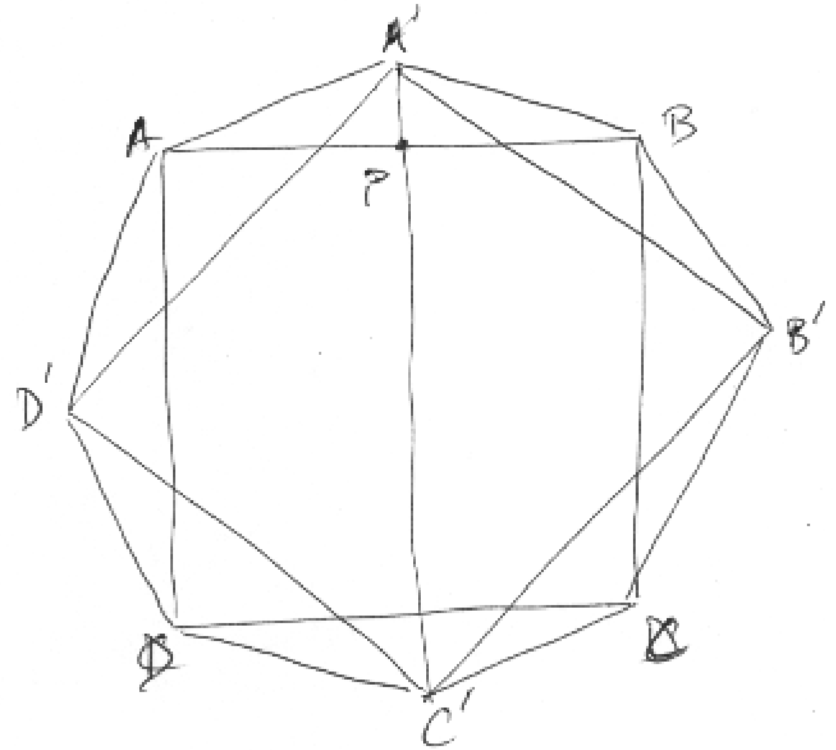
$$= 1 - P(K')P(R')P(A')$$

$$= 1 - \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4}$$

$$= 1 - \frac{1}{4} =$$

$$= \frac{3}{4}$$

Q.7



$A'C' = 2\sqrt{2}$

$AP = \frac{1}{2}\left(2\sqrt{2}-2\right) = \sqrt{2}-1$

$AB = 2$

AREA $K_{AA'B} = \frac{1}{2} \cdot 2\left(\sqrt{2}-1\right) = \sqrt{2}-1$

AREA OCTAGON = AREA SQUARE ABCD + 4 AREA OF AA'B (By Symmetry)

$= 4 + 4(\sqrt{2}-1)$

$= 4 + 4\sqrt{2} - 4$

$= 4\sqrt{2}$

Q.8

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{x} \qquad x=2016$$

$$\frac{a+b}{ab}=\frac{1}{x}$$

$$xa+xb=ab$$

$$ab-xa-xb=0$$

$$ab-xa-xb+x^2=x^2$$

$$(a-x)(b-x)=x^2$$

HOW MANY WAYS ARE THERE TO FACTOR x^2?

$x=2016=2^5\cdot3^2\cdot7^1$, so 2016 has $6\cdot3\cdot2=36$ divisors

$x^2=2016^2=2^{10}\cdot3^4\cdot7^2$ has $11\cdot5\cdot3=165$ divisors

Each pair of divisors produces a solution, but since $a<b$ the one corresponding to $2016\cdot2016$ is not counted ($a=b$) and we only count $\frac{1}{2}$.

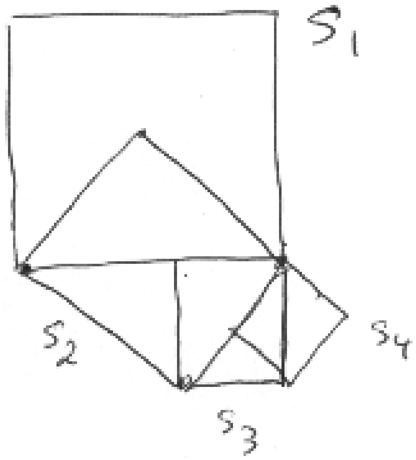
The number of solutions to

$$(a-2016)(b-2016)=2016^2$$

is then $\frac{1}{2}(165-1)=82$

Q.9

Geometric Sequen



The side of $S_1 = \sqrt{100} = 10$

The side of $S_2 = \frac{1}{2} 10\sqrt{2} = 5\sqrt{2}$

The side of $S_3 = \frac{1}{2}(5\sqrt{2}) \cdot \sqrt{2} = 5$

$\vdots$

$S_{n+1} = \frac{1}{2} S_n \sqrt{2}$

The sequence is

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
10,	$5\sqrt{2}$,	5,	$\frac{1}{2}5\sqrt{2}$,	$\frac{5}{2}$,	$\frac{5}{4}\sqrt{2}$,	$\frac{5}{4}$,	$\frac{5}{8}\sqrt{2}$,	$\frac{5}{8}$
				2.5		1.25		.625

So certainly $S_9^2 = (.625)^2 < 1$

What about $S_8^2 = \left(\frac{5}{8}\sqrt{2}\right)^2 = \frac{50}{64} < 1$ also, so $n = 8$

Q.10 Vivian has to get

3 Heads HHH

or 2 Heads, 2 tails HHTT

or 1 Head, 4 tails HTTTT

or 6 Tails TTTTTT

P(3 heads) is $\left(\frac{1}{2}\right)^3 = \frac{1}{2^3}$

P(2 heads, 2 tails) is $\binom{4}{2} \cdot \frac{1}{2^4}$

P(1 head, 4 tails) is $\binom{5}{1} \cdot \frac{1}{2^5}$

P(6 tails) is $\frac{1}{2^6}$

P(6 points) is $\frac{1}{2^3} + \binom{4}{2} \cdot \frac{1}{2^4} + \binom{5}{1} \cdot \frac{1}{2^5} + \frac{1}{2^6}$

$$P(6\text{ point}) = \frac{1}{8} + \frac{6}{16} + \frac{5}{32} + \frac{1}{64}$$

$$= \frac{8}{64} + \frac{24}{64} + \frac{10}{64} + \frac{1}{64}$$

$$= \frac{43}{64}$$

Q.11

$$4^x - 3(2^{x+3}) + 128 = 0$$

Make the substitution $t = 2^x$, then

$$4^x = (2\cdot 2)^x = 2^x 2^x = t^2$$
$$2^{x+3} = 2^x 2^3 = 8\cdot 2^x = 8t$$

$$t^2 - 3(8t) + 128 = 0$$
$$t^2 - 24t + 128 = 0$$
$$(t-16)(t-8) = 0$$

$t = 16$, $2^x = 16$, $x = 4$.

$t = 8$, $2^x = 8$, $x = 3$

roots r and s are 3 and 4

$$\frac{r+s}{|r-s|} = \frac{3+4}{|3-4|} = \frac{7}{1} = 7$$

Q.12 The number of ways of permuting the letters of the word ASSASSIN is $\dfrac{8!}{4!2!} = \dfrac{8\times7\times6\times5}{2} = 840$ ways

The non S letters are AAIN

these can be arranged in $\dfrac{4!}{2!} = 12$ ways

Suppose we have

$\square \times \square \times \square \times \square \times \square$

where X is one of the arrangements of AAIN we put the Ss in the boxes. Since we can't have 2 Ss next to each other, we have 5 boxes and 4 Ss, so there are $\binom{5}{4} = 5$ ways of putting the Ss for each arrangement of AAIN for a total of $5\times12=60$ orderings that meet the condition of the problem.

$$\frac{60}{840} = \frac{6}{84} = \frac{1}{14}$$

$840 = 2^3 \cdot 3 \cdot 5 \cdot 7$

Q.13

$x^2 - 3x + 5 = 0$ roots $\alpha + \beta$

$\alpha + \beta = 3$

$\alpha\beta = 5$

$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$$
$$= 9 - 10$$
$$= -1$$

$x^2 + 5x - 3 = 0$ roots r, s

$r + s = -5$

$rs = -3$

$$r^2 + s^2 = (r+s)^2 - 2rs$$
$$= 25 + 6$$
$$= 31$$

$\alpha r + \beta s$, $\alpha s + \beta r$ are the roots of $x^2 + px + q = 0$

$$(\alpha r + \beta s)(\alpha s + \beta r) = q$$
$$\alpha^2 rs + \alpha\beta r^2 + \alpha\beta s^2 + \beta^2 rs = q$$
$$(\alpha^2 + \beta^2) rs + \alpha\beta(r^2 + s^2) = q$$
$$-1 \cdot -3 + 5 \cdot 31 = 158$$

$$\alpha r + \beta s + \alpha s + \beta r = -p$$
$$(\alpha+\beta)(r+s) = -p$$
$$3 \cdot -5 = -p$$
$$15 = p$$

$p + q = 158 + 15 = 173$

Q.14 There are a total of $3n+3$ points. The vertices are excluded.

Any three form a triangle there are $\binom{3n+3}{3}$ such triangles

- except - if the 3 points are colinear (from the same side).

This can be done in $\binom{n}{3}+\binom{n+1}{3}+\binom{n+2}{3}$ ways

So the expression is

$$\binom{3n+3}{3} - \left(\binom{n}{3}+\binom{n+1}{3}+\binom{n+2}{3}\right)$$

and we want this equal to 205.

$$\frac{(3n+3)(3n+2)(3n+1)}{3!} - \frac{1}{6}\left[n(n-1)(n-2)+(n+1)n(n-1)+(n+2)(n+1)n\right]$$

$$\frac{1}{2}(n+1)(3n+2)(3n+1) - \frac{1}{6}\left(n(n^2-3n+2)+n(n^2-1)+n(n^2+3n+2)\right)$$

$$\frac{1}{2}(n+1)(9n^2+9n+2) - \frac{1}{6}\left(n^3-3n^2+2n+n^3-n+n^3+3n^2+2n\right)$$

$$\frac{1}{2}(9n^3+18n^2+11n+2) - \frac{1}{6}(3n^3+3n)$$

$$\frac{1}{2}\left(9n^3+18n^2+11n+2-n^3-n\right)$$

$$\frac{1}{2}\left(8n^3+18n^2+10n+2\right) = 4n^3+9n^2+5n+1$$

$n=1$, value is 19
$n=2$, value is 79
$n=3$, value is 205
$n=4$, value is 421
$n=5$, value is 751
$n=6$, value is 1219

NOTE CONNECTION TO PASCAL'S TRIANGLE

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126
1 10 45 120 210 252
1 11 55 165 330 462
1 12 66 220
1 13 78 286
1 14 91 364
1 15 105 455

$220-(1+4+10)=205$
$455-(4+10+20)=421$

Q.15 $x^4-3x^3-2x^2-3x+1=0$

This is a symmetric equation, divide by x^2

$$x^2-3x-2-\frac{3}{x}+\frac{1}{x^2}=0$$

NOW USE THE SUBSTITUTION $x+\frac{1}{x}=t$.

$x+\frac{1}{x}=t$, so $\left(x+\frac{1}{x}\right)^2=t^2$

$x^2+2+\frac{1}{x^2}=t^2$, so $x^2+\frac{1}{x^2}=t^2-2$

$$x^2-3x-2-\frac{3}{x}+\frac{1}{x^2}=0$$

$$\left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)-2=0$$

$$t^2-2-3t-2=0$$

$$t^2-3t-4=0$$

$$(t-4)(t+1)=0$$

$x+\frac{1}{x}=4$

$x^2+1=4x$

$x^2-4x+1=0$ $\quad x=\frac{4\pm\sqrt{12}}{2}=2\pm\sqrt{3}$, sum of real roots is 4

$x+\frac{1}{x}=-1$

$x^2+1=-x$

$x^2+x+1=0$ $\quad x=\frac{-1\pm\sqrt{-3}}{2}=$ complex

Q.16 Suppose the numbers are x and y.

$x, y \equiv 0 \mod 5$	5	10	15	20	25
$\equiv 1 \mod 5$	6	11	16	21	26
$\equiv 2 \mod 5$	7	12	17	22	27
$\equiv 3 \mod 5$	8	13	18	23	28
$\equiv 4 \mod 5$	9	14	19	24	29

If both x and y are congruent mod 5, then $x^2 - y^2$ is divisible by 5

these can be chosen in $5 \cdot \binom{5}{2} = 5 \cdot 10 = 50$ ways

If $x \equiv 1 \mod 5$ and $y \equiv 4 \mod 5$, then $x^2 - y^2$ is divisible by 5

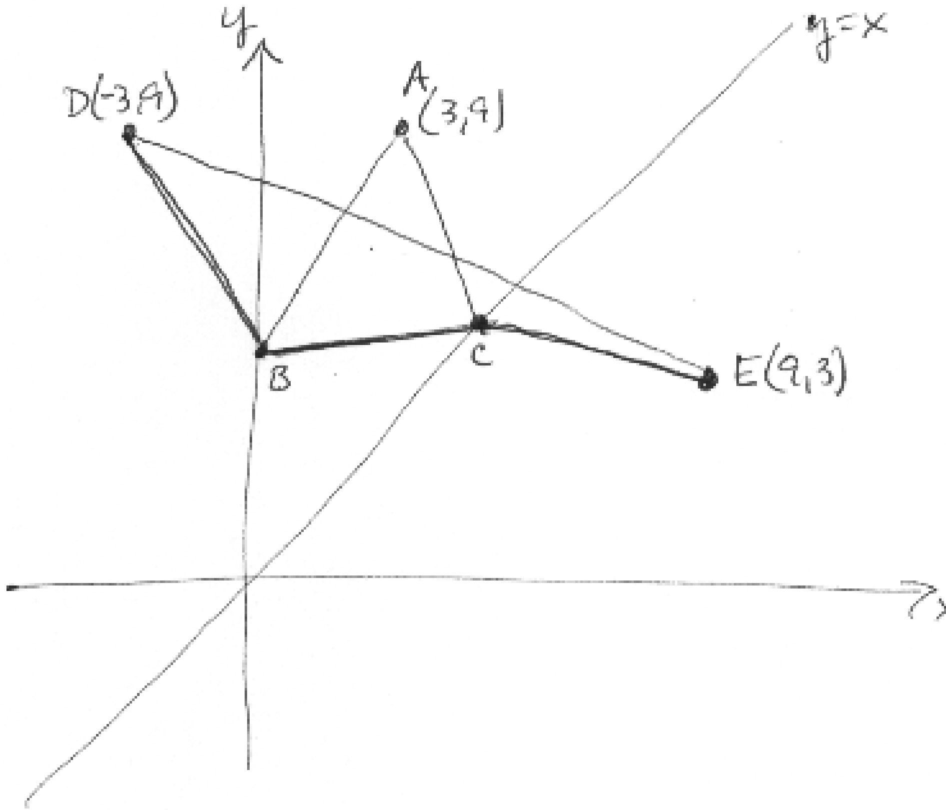
these can be chosen in $\binom{5}{1}\binom{5}{1} = 5 \cdot 5 = 25$ ways

If $x \equiv 2 \mod 5$ and $y \equiv 3 \mod 5$, then $x^2 - y^2$ is divisible by 5

these can be chosen in $\binom{5}{1}\binom{5}{1} = 5 \cdot 5 = 25$ ways

$50 + 25 + 25 = 100$ ways.

Q.17



The perimeter of $\triangle ABC$ is $AB+BC+CA$.

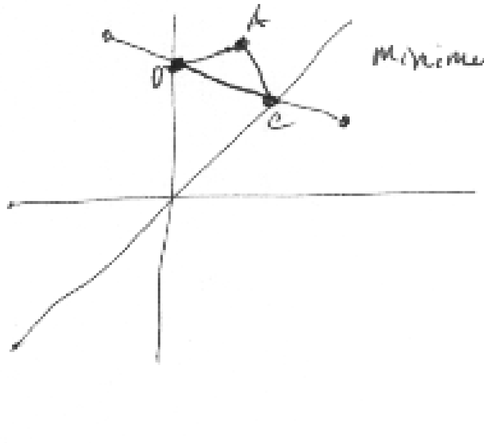
Reflect $A(3,9)$ in the y-axis to $D(-3,9)$ $AB=BD$.

Reflect $A(3,9)$ in the line $y=x$ to $E(9,3)$ $AC=CE$

$$AB+BC+CA = DB+BC+CE$$

which is minimized when $\overline{DE}$ is a straight line

$$DB+BC+CE \geq DE = \sqrt{(9-(-3))^2+(3-9)^2} = \sqrt{180} = 6\sqrt{5}$$



minimum perimeter

Q.18

Let the n^{th} term of arithmetic progression 1 be $a_1+(n-1)d_1$
Let the n^{th} term of arithmetic progression 2 be $a_2+(n-1)d_2$

then $$\frac{a_1+(n-1)d_1}{a_2+(n-1)d_2} = \frac{14n-6}{8n+23}$$

$$= \frac{2a_1+2(n-1)d_1}{2a_2+2(n-1)d_2} = \frac{2a_1+(2n-2)d_1}{2a_2+(2n-2)d_2}$$

$$= \frac{\frac{m}{2}\left(2a_1+(2n-2)d_1\right)}{\frac{m}{2}\left(2a_2+(2n-2)d_2\right)} = \frac{\frac{m}{2}\left(2a_1+(m-1)d_1\right)}{\frac{m}{2}\left(2a_2+(m-1)d_2\right)}$$

$$= \frac{\sum m \text{ terms of arithmetic progression } 1}{\sum m \text{ terms of arithmetic progression } 2}$$

IF $m-1=2n-2$ OR $2n=m+1$

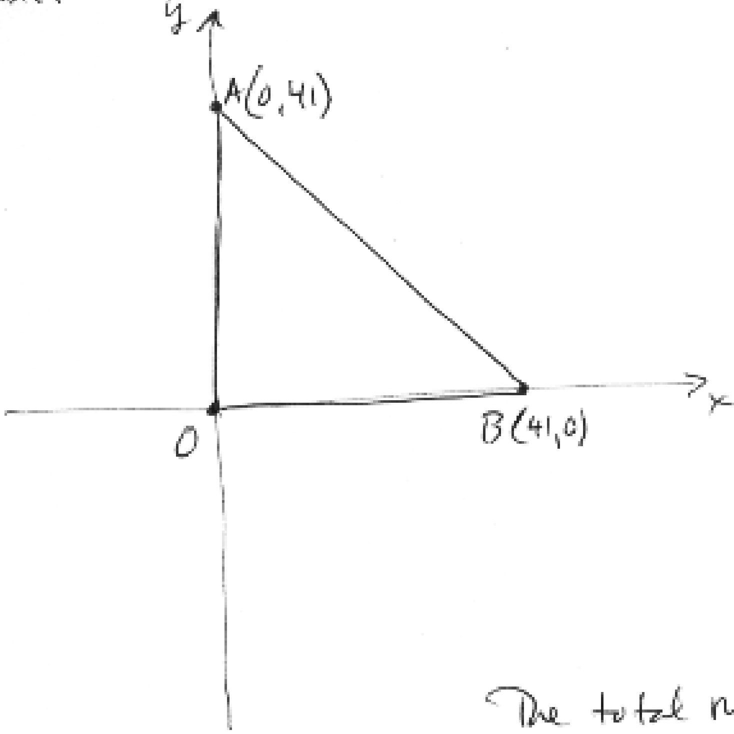
So
$$\frac{14n-6}{8n+23} = \frac{7(2n-2)+8}{4(2n-2)+31} = \frac{7(m-1)+8}{4(m-1)+31} = \boxed{\frac{7m+1}{4m+27}}$$

NOTE: IF $n=1$, the ratio is $\frac{8}{31}$. Theoretically, we could use a 1 term arithmetic progression to find the answer, but all choices have value $\frac{8}{31}$ for $m=1$.

How about $n=2$, the ratio is $22:39$, So appropriate arithmetic progressions would be $8, 22$ and $31, 39$ the ratio of the sums is $\frac{30}{70} = \frac{3}{7}$, for $m=2$.

Which choice has value $\frac{3}{7}$ for $m=2$?

Q. #19



The area of AOB is $\frac{1}{2} \cdot 41 \cdot 41$

$\overline{AO}$ has 42 lattice points

$(0,0), (0,1), \ldots, (0,41)$

$\overline{OB}$ has 42 lattice points

$(0,0), (1,0), \ldots, (41,0)$

$\overline{AB}$ is the line segment $x+y=41$ with 42 lattice points

$(0,41), (1,40), \ldots, (41,0)$

The total number of points on the boundary of AOB is then $42+42+42-3=123$.

By Pick's Theorem, the area is

$$A = \frac{B}{2} + I - 1$$

Where B is the number of points on the boundary and I is the number of points on the interior. So

$$\frac{41^2}{2} = \frac{123}{2} + I - 1$$

$$\frac{1681-123}{2} = I - 1$$

$$\frac{1558}{2} + 1 = I = 779 + 1 = 780$$

NOTE THAT 780 IS THE 39^{th} TRIANGULAR NUMBER, WHICH PROVIDES AN ALTERNATE METHOD OF SOLUTION.

If the points are not easily counted on the boundary or there is no symmetry, Pick's is the only good method.

Q.20 Heron's formula for area is

$$K_{ABC} = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where } s = \frac{a+b+c}{2}$$

$$K_{ABC} = \sqrt{\frac{a+b+c}{2} \cdot \frac{b+c-a}{2} \cdot \frac{c+a-b}{2} \cdot \frac{a+b-c}{2}}$$

$$K_{ABC} = \frac{1}{4}\sqrt{(a+b+c)(b+c-a)\cdot(c+a-b)(a+b-c)}$$

$$K_{ABC} = \frac{1}{4}\sqrt{((b+c)+a)((b+c)-a)(a-(b-c))(a+(b-c))}$$

$$K_{ABC} = \frac{1}{4}\sqrt{((b+c)^2-a^2)(a^2-(b-c)^2}$$

Now $c = 2b$, substituting, and $a = 3\sqrt{3}$

$$K_{ABC} = \frac{1}{4}\sqrt{(9b^2-27)(27-b^2)}$$

$$K_{ABC} = \frac{3}{4}\sqrt{(b^2-3)(27-b^2)}$$

$$K_{ABC} = \frac{3}{4}\sqrt{-b^4+30b^2-81}$$

The maximum value of the "quadratic" in b^2 occurs at $b^2 = \frac{-30}{-2\cdot 1} = 15$

$$-b^4+30b^2-81\Big|_{b^2=15} = -225+30\cdot 15-81 = 144$$

$$K_{ABC} \le \frac{3}{4}\sqrt{144}$$

$$= \frac{3}{4}\cdot 12$$

$$= 9$$

So the maximum area is 9.

Q.21

$$\prod_{n=3}^{\infty} \frac{n^3-8}{n^3+8}$$

NOTE THAT $X^3-Y^3 = (X-Y)(X^2+XY+Y^2)$

and $X^3+Y^3 = (X+Y)(X^2-XY+Y^2)$

$$\prod_{n=3}^{\infty} \frac{n^3-8}{n^3+8} = \frac{3^3-8}{3^3+8}\cdot\frac{4^3-8}{4^3+8}\cdot\frac{5^3-8}{5^3+8}\cdots\cdots\frac{n^3-8}{n^3+8}\cdots\cdots$$

$$= \frac{(3-2)(3^2+2\cdot3+4)}{(3+2)(3^2-2\cdot3+4)}\cdot\frac{(4-2)(4^2+2\cdot4+4)}{(4+2)(4^2-2\cdot4+4)}\cdot\frac{(5-2)(5^2+2\cdot5+4)}{(5+2)(5^2-2\cdot5+4)}\cdots\frac{(n-2)(n^2+2n+4)}{(n+2)(n^2-2n+4)}\cdots$$

$$= \frac{(3-2)}{(3+2)}\cdot\frac{(4-2)}{(4+2)}\cdot\frac{(5-2)}{(5+2)}\cdots\frac{(n-2)}{(n+2)}\cdots\frac{(3^2+2\cdot3+4)}{(3^2-2\cdot3+4)}\cdot\frac{(8^2+2\cdot8+4)}{(8^2-2\cdot8+4)}\cdots\frac{(n^2+2n+4)}{(n^2-2n+4)}\cdots$$

$$= \frac{1}{5}\cdot\frac{2}{6}\cdot\frac{3}{7}\cdot\frac{4}{8}\cdot\frac{5}{9}\cdot\frac{6}{10}\cdots\cdots\frac{19}{7}\cdot\frac{28}{12}\cdot\frac{39}{19}\cdot\frac{52}{28}\cdot\frac{63}{39}\cdots$$

$$= \frac{1}{\cancel{5}}\cdot\frac{2}{\cancel{6}}\cdot\frac{3}{\cancel{7}}\cdot\frac{4}{\cancel{8}}\cdot\frac{\cancel{5}}{\cancel{9}}\cdot\frac{\cancel{6}}{\cancel{10}}\cdots\frac{\cancel{19}}{7}\cdot\frac{\cancel{28}}{12}\cdot\frac{\cancel{39}}{\cancel{19}}\cdot\frac{\cancel{52}}{\cancel{28}}\cdot\frac{\cancel{63}}{\cancel{39}}\cdots$$

$$= \frac{1\cdot2\cdot3\cdot4}{1}\cdot\frac{1}{7\cdot12} = \frac{24}{7\cdot12} = \frac{2}{7}$$

TELESCOPING PRODUCT

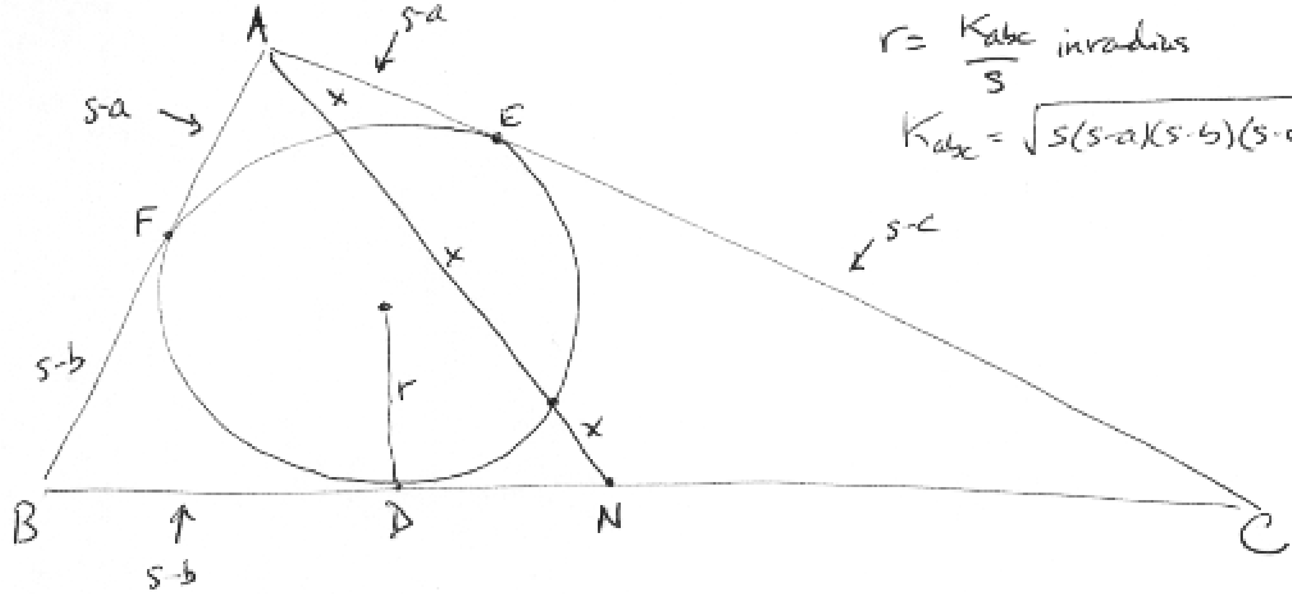
Generalize the problem $\prod \frac{n^3-k^3}{n^3+k^3}$

Q.22

K_{ABC} = area of ABC $\quad s = \dfrac{a+b+c}{2}$ semi-perimeter

$r = \dfrac{K_{abc}}{s}$ inradius

$K_{abc} = \sqrt{s(s-a)(s-b)(s-c)}$



By Power of a Point $\overline{AE}^2 = x \cdot 2x$, so $2x^2 = (s-a)^2 = \left(\frac{a+b+c}{2} - a\right)^2 = \left(\frac{b+c-a}{2}\right)^2 =$

$$2x^2 = \frac{1}{4}(b+c-10)^2$$

Also by Power of a Point $\overline{DN}^2 = x \cdot 2x$, so since $\overline{DN} = \overline{BN} - \overline{BD} = \frac{a}{2} - (s-b)$

$$2x^2 = \overline{DN}^2 = \left(\frac{a}{2} - (s-b)\right)^2 = \left(\frac{a}{2} - \left(\frac{a+b+c}{2} - b\right)\right)^2 = \left(\frac{a}{2} - \left(\frac{a+c-b}{2}\right)\right)^2$$

$$2x^2 = \left(\frac{b-c}{2}\right)^2 = \left(\frac{(b-c)^2}{4}\right) = \frac{(b-c)^2}{4}$$

so $\quad \dfrac{(b-c)^2}{4} = \dfrac{(b+c-10)^2}{4}$ and $(b-c)^2 = (b+c-10)^2$

so $\quad b^2 - 2bc + c^2 = b^2 + c^2 + 100 + 2bc - 20b - 20c$

$$0 = 100 + 4bc - 20b - 20c$$

$$0 = (2b-10)(2c-10)$$

so $c = 5$.

By Stewarts Theorem

$$b^2 \cdot \frac{a}{2} + c^2 \cdot \frac{a}{2} = (3x)^2 \cdot a + \frac{a}{2} \cdot \frac{a}{2} \cdot a$$

$$b^2 + c^2 = 18x^2 + \frac{a^2}{2} \quad \text{or} \quad b^2 + 25 = 18x^2 + 50, \text{ so } b^2 - 25 = 18x^2$$

we also know that $2x^2 = (s-a)^2 = \left(\frac{b+c-a}{2}\right)^2 = \left(\frac{b-5}{2}\right)^2$, so $(b-5)^2 = 8x^2$

$b^2 - 25 = 18x^2 \Rightarrow 4b^2 - 100 = 72x^2 \qquad b^2 - 10b + 25 = 8x^2 \Rightarrow 9b^2 - 90b + 225 = 72x^2$

$$4b^2 - 100 = 9b^2 - 90b + 225 \quad \text{or} \quad 5b^2 - 90b + 325 = 0$$

or $\quad b^2 - 18b + 65 = 0 = (b-5)(b-13)$, so $b = 13$.

$s = \dfrac{a+b+c}{2} = \dfrac{10+5+13}{2} = 14$, $K_{abc} = \sqrt{14 \cdot 4 \cdot 9 \cdot 1} = 6\sqrt{14}$, $r = \dfrac{6\sqrt{14}}{14} = \dfrac{3\sqrt{14}}{7}$

Q23.

$$(10^{2016}+5)^2 = 225N$$

$$10^{4032} + 2\cdot 5\cdot 10^{2016} + 25 = 225N$$

$$25\cdot 4\cdot 10^{4030} + 4\cdot 25\cdot 10^{2015} + 25 = 225N$$

$$4\cdot 10^{4030} + 4\cdot 10^{2015} + 1 = 9N$$

$$N = \frac{4\cdot 10^{4030} + 4\cdot 10^{2015} + 1}{9}$$

$$N = \frac{4\cdot(10^{4030}-1) + 4\cdot(10^{2015}-1) + 8 + 1}{9}$$

$$N = \frac{4\cdot(10^{4030}-1)}{9} + \frac{4\cdot(10^{2015}-1)}{9} + \frac{9}{9}$$

$$N = \frac{4\cdot(10^{4030}-1)}{10-1} + \frac{4(10^{2015}-1)}{10-1} + 1$$

$$N = \underbrace{4444\ldots488888\ldots89}_{4030 \text{ DIGITS}}$$

2015 4's, 2014 8's, 1 9

sum of digits is $4\cdot 2015 = 8060$

$8\cdot 2014 = 16112$

$9 = 9$

$8060 + 16112 + 9 = 24181$

$$\begin{array}{r} {}^{1} \\ 16112 \\ 8060 \\ 9 \\ \hline 24181 \end{array}$$

Q. 24

$2 < n^2 < 3$

$\sqrt{2} < n < \sqrt{3}$

$\frac{1}{\sqrt{3}} < n^{-1} < \frac{1}{\sqrt{2}}$, so n^{-1} is between 0 and 1.

Also

$0 < n^2 - 2 < 1$, so $n^2 - 2$ is between 0 and 1.

But $\{n^{-1}\} = \{n^2\} = \{n^2 - 2\}$

so $n^2 - 2 = n^{-1}$

$n^3 - 2n = 1$

$n^3 - 2n - 1 = 0$

$(n+1)(n^2 - n - 1) = 0$

$n = -1$ or $n = \frac{1 \pm \sqrt{5}}{2}$

so $n = \frac{1+\sqrt{5}}{2}$ since n is not negative

Now $n^{-1} = \frac{2}{1+\sqrt{5}} = \frac{2 - 2\sqrt{5}}{1-5} = \frac{2\sqrt{5} - 2}{4} = \frac{\sqrt{5}-1}{2}$

$n^2 = \frac{6 + 2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$

$n^3 = \left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) = \frac{3 + 4\sqrt{5} + 5}{4} = 2 + \sqrt{5}$

$n^6 = (2+\sqrt{5})^2 = 9 + 4\sqrt{5}$

$n^{12} = (9 + 4\sqrt{5})^2 = 81 + 72\sqrt{5} + 80 = 161 + 72\sqrt{5}$

$144n^{-1} = 144\left(\frac{\sqrt{5}-1}{2}\right) = 72\sqrt{5} - 72$

$n^{12} - 144n^{-1} = 161 + 72\sqrt{5} - (72\sqrt{5} - 72)$

$= 161 + 72$

$= 233$

Q.25

$(*)$ $f(1)+2f(2)+3f(3)+\cdots+nf(n)=n(n+1)f(n)$

put $n=n+1$

$(**)$ $f(1)+2f(2)+3f(3)+\cdots+nf(n)+(n+1)f(n+1)=(n+1)(n+2)f(n+1)$

Subtracting $(*)$ from $(**)$

$$(n+1)f(n+1)=(n+1)(n+2)f(n+1)-n(n+1)f(n)$$

$$f(n+1)=(n+2)f(n+1)-nf(n)$$

$$nf(n)=(n+1)f(n+1)$$

$$2f(2)=3f(3)=4f(4)=\cdots=nf(n)$$

Substitute back into $(*)$

$$f(1)+\underbrace{nf(n)+nf(n)+\cdots+nf(n)}_{n-1\text{ times}}=n(n+1)f(n)$$

or $\quad 1+(n-1)\cdot nf(n)=n(n+1)f(n)$

or $\quad 2nf(n)=1$

so $\quad f(n)=\frac{1}{2n}$ and $f(2016)=\frac{1}{4032}$

ALTERNATE: Given the choices, trying $f(n)=kn$, $f(n)=\frac{k}{n}$ is a good idea. The second with $k=\frac{1}{2}$ works:

$$f(1)+2\cdot\frac{k}{2}+3\cdot\frac{k}{3}+\cdots+n\cdot\frac{k}{n}=n(n+1)\cdot\frac{k}{n}$$

$$1+(n-1)k=(n+1)k$$

$$1=2k$$

$$\frac{1}{2}=k$$

$f(n)=kn$ doesn't work.