February 2nd , 2013 Mock AMC 10

Welcome!

2013 has 3 distinct prime factors. 2013 = 3 * 11 * 61

2014 is also a year with three prime factors, because 2014 = 2 *19 * 53. And get this: The same is true about 2015, which equals 5 * 13 * 31.

There won't be another three consecutive years that are the products of three prime numbers until 2665 - more than 500 years from now!

Factorization of numbers around 2013:

2016 = 2 * 2 * 2 * 2 * 2 * 3 * 3 * 7
2015 = 5 * 13 * 31
2014 = 2 * 19 * 53
2013 = 3 * 11 * 61
2012 = 2 * 2 * 503
2011 = prime
2010 = 2 * 3* 5 * 67
2009 = 7 * 7 * 41
2008 = 2 * 2 * 2 * 251
2007 = 3 * 3 * 223
2006 = 2 * 17 * 59

2013 is composed of the four sequential digits 0, 1, 2, and 3. This hasn't happened since 1432, over 500 years ago!

The last prime-numbered years before 2013 were 2003 and 2011. The next prime numbers after 2013 will be 2017 and 2027.

Prime numbers around 2013:

1993 1997 1999 2003 2011 2017 2027 2029 2039 2053

2013 is one of only 45 "alliterative" numbers, which, when spelled out ("two thousand, thirteen") has all of its multiple words starting with the same letter (in this case, T). The first 20 alliterative numbers are: 22, 23, 32, 33, 44, 45, 54, 55, 66, 67, 76, 77, 88, 99, 2000, 2002, 2003, 2010, 2012, 2013. What are the next ones?

2013 is the first year since 1987 to have 4 different numbers as digits.

Have Fun!

Dan Plotnick

1. Find $1! \cdot 3 - 2! \cdot 4 + 3! \cdot 5 - 4! \cdot 6 + \cdots - 2012! \cdot 2014 + 2013!$.

A) 0 B) 2014! C) 1 D) 2013! E) 2013! + 1

2. How many times does the prime factor 7 appear in the prime factorization of

$$1001 \cdot 1002 \cdot 1003 \cdot \ldots \cdot 2012 \cdot 2013 \text{ ?}$$

A) 168 B) 169 C) 170 D) 171 E) 172

3. Half of the books on a teacher's book shelf are mathematics books, a third of them are physics books, and 1/15-th of them are history books. The remainder of the books are romance novels. If 2 of the mathematics books, and 4 of the physics books are replaced by romance novels, then romance novels will comprise 15% of the books on the bookshelf. How many books, total, are there on the bookshelf?

A) 60 B) 90 C) 120 D) 150 E) 180

4. If the letters a, A, b, B, c, and C are arranged at random in a row, what is the probability that the lower case letters appear in increasing alphabetical order?

A) $\frac{1}{6}$ B) $\frac{1}{2}$ C) $\frac{1}{720}$ D) $\frac{1}{36}$ E) $\frac{1}{30}$

5. In a regular hexagon, the longest diagonals are increased by 30 percent in length. By what percentage will the area of the hexagon increase as a result?

A) 30 B) 40 C) 49 D) 60 E) 69

6. The arithmetic mean of N real numbers is N. The arithmetic mean of a subset of M of the given numbers is M, where M < N. What is the arithmetic mean of the remaining N – M numbers?

A) M B) N C) N – M D) N + M E) Cannot be determined

7. A fair coin is tossed repeatedly. What is the probability that we obtain a total of two tails before we obtain a total of three heads?

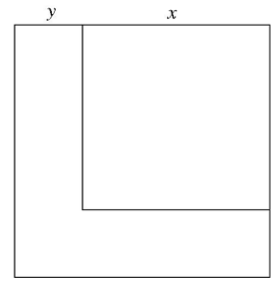
A) $\frac{1}{2}$ B) $\frac{9}{16}$ C) $\frac{5}{8}$ D) $\frac{11}{16}$ E) $\frac{3}{4}$

8. Esther rides her bike around a course in the shape of an equilateral triangle. Her speed is 10 miles per hour on the first side of the course, 15 miles per hour on the second side of the course, and 20 miles per hour on the third and final side of the course. If S is Esther's speed for the entire trip, in miles per hour, which of these statements about S is true?

A) S < 13 miles per hour B) 13 < S < 14 miles per hour
C) 14 < S < 15 miles per hour D) S = 15 miles per hour
E) S > 15 miles per hour

9. It usually takes Daniel 8 hours to mow the yard. Andy can do it in 6 hours. Daniel got Andy to help one Saturday morning, but when one-half of the lawn was mowed, Andy quit and Daniel had to finish by himself. How long did Daniel mow altogether including the time he worked with Andy?

A) $5\frac{2}{5}$ hrs. B) $5\frac{1}{3}$ hrs. C) $4\frac{4}{5}$ hrs. D) $5\frac{5}{7}$ hrs. E) None of these



10. A large square has a smaller square cut from its corner in such a way that the area of the square removed equals the area of the remaining region. If *x* represents the length of a side of the removed square, and *y* represents the remaining length, find the ratio $\frac{x}{y}$.

A) $\frac{5}{12}$ B) $\frac{12}{5}$ C) $\frac{\sqrt{2}+1}{1}$ D) $\frac{8}{3}$ E) $\frac{6(\sqrt{2}-1)}{1}$

11. Let x, y, and z be positive real numbers such that x + y + z = 1 and xy + yz + xz = $\frac{1}{3}$. The number of possible values of the expression $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$ is

A) 1 B) 2 C) 3 D) more than 3, but finite many E) infinitely many

12. Through a point on the hypotenuse of a right triangle, lines are drawn parallel to the legs of the triangle so that the triangle is divided into a square and two smaller right triangles. The area of one of the two small right triangles is *m* times the area of the square. The ratio of the area of the other small right triangle to the area of the square is

A) $\frac{1}{2m+1}$ B) m C) $1-m$ D) $\frac{1}{4m}$ E) $\frac{1}{8m^2}$

13. The solutions of the equation $x^2+px+q=0$ are the cubes of the solutions of the equation $x^2+mx+n=0$. Which of the following is true?

A) $p=m^3+3mn$ B) $p=m^3-3mn$ C) $p=3mn-m^3$
D) $p+q=m^3$ E) $p+q=2m^3$

14. How many composite (i.e. not prime) integers from 1 to 2013 have a prime number of positive integral divisors. Note: 1 is not a prime.

A) 17 B) 18 C) 19 D) 20 E) 21

15. Let $f(x)=|3x-6|-|x+1|+|2x+4|$ Find the minimum value of f(x).

A) 11 B) 0 C) -5 D) 7 E) 5

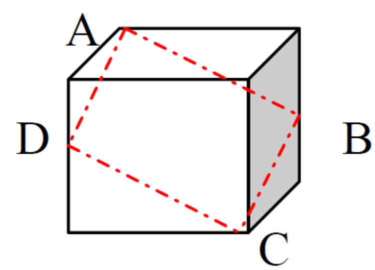
16. How many positive integers less than or equal to 2013 can be written in the form m^2-n^2 where $1 \le n < m$.

A) 503 B) 1006 C) 1508 D) 1510 E) 2013

17. Determine all values of p such that the equation $|x^2+2x+p|$ = 2 has exactly four solutions.

A) p > 3 B) p < 3 C) p < 3 and p > -1
D) p > 3 or p < -1 E) p < -1

18. A pair of opposite vertices and the midpoints of a pair of opposite edges of a cube are connected to form a quadrilateral. If each edge of the cube has length 6, find the area of the quadrilateral.

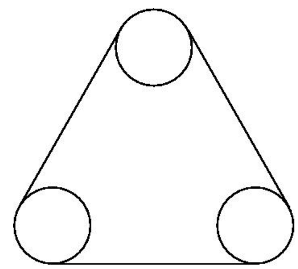


A) $18\sqrt{3}$ B) $18\sqrt{6}$ C) $12\sqrt{2}$ D) 18 E) $12\sqrt{6}$

19. Let $a_1, a_2, a_3, \ldots$ be a sequence of integers satisfying $a_{n-1} + a_n = 3n$ for all n ≥ 2. If $a_1 = 1$ find a_{2013} .

A) 987 B) 2013 C) 3019 D) 3022 E) 6039

20. Three wheels, each of radius 1, have their centers at respective vertices of an equilateral triangle of side length 4. A belt is wrapped continuously around the wheels. Find the length of the belt.



A) $3(\pi + 4)$ B) $\frac{2\pi}{3} + 4$ C) $2(\pi + 6)$ D) $\frac{3\pi}{2} + 12$

E) None of these.

21. If 7 distinct fair 6-sided dice are rolled at the same time, what is the probability that the sum will be 10?

A) $\frac{7}{279936}$ B) $\frac{7}{23328}$ C) $\frac{1}{139968}$ D) $\frac{1}{11664}$ E) $\frac{10}{343}$

22. Suppose that 7! Is written as the product abcd where a, b, c and d are positive integers such that each of a, b, c and d has the same number of positive integral divisors. For example, 6 has four positive integral divisors 1, 2, 3, and 6. What is the largest possible value of a + b + c + d?

A) 51 B) 49 C) 37 D) 41 E) 36

23. How many ordered pairs of positive integer numbers (x,y) satisfy the equation $x + y + xy = 2013$?

A) 2 B) 3 C) 4 D) 6 E) 8

24. The set of positive integer multiples is partitioned in the following way:

{1}, {2,3}, {4,5,6}, {7,8,9,10}, …

Let S_k be the sum of the elements in the k-th set. Find S_{25}.

A) 6924 B) 7825 C) 13825 D) 15625 E) 17576

25. Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on top of them. What is the distance from the plane to the top of the larger sphere?

A) $3 + \frac{\sqrt{30}}{2}$ B) $3 + \frac{\sqrt{69}}{3}$ C) $3 + \frac{\sqrt{123}}{4}$ D) $3 + 2\sqrt{2}$ E) $5 + \sqrt{3}$