MAA AMC

American Mathematics Competitions

Solutions Pamphlet

MAA American Mathematics Competitions

20th Annual

AMC 10B

American Mathematics Competition 10B

Wednesday, February 13, 2019

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *Copies of the problem booklet and solution pamphlet may be shared with your students for educational purposes. However, the publication, reproduction, or communication of the problems or solutions for this competition with anyone outside of the classroom is a violation of the competition rules. This includes dissemination via copier, telephone, email, internet, or media of any type.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by MAA's Subcommittee on the AMC 10/AMC 12 Exams, under the direction of the co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (D):** Let x be the volume of the first container and y the volume of the second container. Then
$$\frac{5}{6}x = \frac{3}{4}y, \qquad \text{so} \qquad \frac{x}{y} = \frac{3}{4} \cdot \frac{6}{5} = \frac{9}{10}.$$

2. **Answer (E):** An implication is false if and only if the hypothesis is true but the conclusion is false. Choice **(E)**, $n = 27$, is a counterexample to the statement because the hypothesis is true (27 is not prime) but the conclusion is false ($27 - 2 = 25$ is not prime). For answer choices **(A)** and **(C)**, n is prime, so the hypothesis is false and these values of n do not provide a counterexample. For choices **(B)**, **(C)**, and **(D)**, $n-2$ is prime, so the conclusion is true and these values of n do not provide a counterexample.

3. **Answer (B):** Let n be the number of non-seniors. Because there are 500 students in all, there are $500 - n$ seniors. Because 40% of the seniors play a musical instrument, 60% of the seniors do not play a musical instrument. This leads to the equation
$$0.60(500 - n) + 0.30n = 0.468 \cdot 500.$$
The equation simplifies to $0.30n = 66$. Solving this equation for n gives $n = 220$. Because $100\% - 30\% = 70\%$ of the non-seniors play a musical instrument, there are $0.70 \cdot 220 = 154$ non-seniors who play a musical instrument.

4. **Answer (A):** If a, b, c form an arithmetic progression, then $a = b-d$ and $c = b + d$ for some number d. Then the given linear equation becomes $(b - d)x + by = b + d$, which is equivalent to
$$b(x + y - 1) - d(x + 1) = 0.$$
This will hold for all values of b and d if and only if $x + y - 1 = 0$ and $x + 1 = 0$, which means $x = -1$ and $y = 2$. Thus $(-1, 2)$ is the unique point through which all such lines pass.

OR

If a, b, c form an arithmetic progression, then $b - a = c - b$. This equation is equivalent to $a(-1) + b(2) = c$, so $x = -1$ and $y = 2$ give a point through which the line passes. Conversely, if $a = 1$, $b = 2$, and $c = 3$, then the equation of the line is $x + 2y = 3$, and none of the other four choices satisfies this equation.

OR

Letting $(a, b, c) = (0, 1, 2)$ yields $y = 2$, and letting $(a, b, c) = (1, 0, -1)$ yields $x = -1$. In fact, $(-1, 2)$ is on the line $ax + by = c$ if and only if $-a + 2b = c$, which is equivalent to $b - a = c - b$, which is the defining condition for a, b, c to be an arithmetic progression.

5. **Answer (E):** The reflection of the point (a, b) across the line $y = x$ is the point (b, a). Because the coordinates of the points in the original triangle are all positive, it follows that the coordinates of the images will also be all positive. Thus **(A)** is always true. It is a property of reflections that the line segment connecting a point not on the line of reflection and its image is perpendicular to the line of reflection. This fact shows that **(C)** and **(D)** are always true. Reflection is a rigid transformation, and therefore areas are preserved, so **(B)** is always true. The statement **(E)** is not true in general. As an example, consider a triangle ABC such that the side $\overline{AB}$ is parallel to the line $y = x$. Then the side $\overline{A'B'}$ in the image will also be parallel to $y = x$, which shows that lines AB and $A'B'$ are not perpendicular. Thus the statement **(E)** is not always true. In fact, because reflection across the line $y = x$ interchanges the roles of x and y, the slope of a non-vertical/non-horizontal line and the slope of its image are reciprocals, not negative reciprocals. A line will be perpendicular to its reflection across the line $y = x$ if and only if the line is horizontal or vertical, in which case its image will be vertical or horizontal, respectively.

6. **Answer (C):** Dividing the given equation by $n!$, simplifying, and completing the square yields

$$\begin{aligned}(n+1) + (n+2)(n+1) &= 440\\ n^2 + 4n + 3 &= 440\\ n^2 + 4n + 4 &= 441\\ (n+2)^2 &= 21^2.\end{aligned}$$

Thus $n + 2 = 21$ and $n = 19$. The requested sum of the digits of n is $1 + 9 = 10$.

7. **Answer (B):** Because Casper has exactly enough money to buy 12 pieces of red candy, the amount of money he has must be a multiple of 12 cents. Similarly, it must be a multiple of both 14 cents and 15 cents. Furthermore, this amount of money will buy a whole number of purple candies that cost 20 cents each, so the amount of money must also be a multiple of 20. The least common multiple of $12 = 2^2 \cdot 3$, $14 = 2 \cdot 7$, $15 = 3 \cdot 5$, and $20 = 2^2 \cdot 5$ is $2^2 \cdot 3 \cdot 5 \cdot 7 = 420$. Therefore the number of purple candies that Casper can buy, n, must be a multiple of $420 \div 20 = 21$. Thus the least possible value of n is 21. In this case the red candies cost $420 \div 12 = 35$ cents each, the green candies cost $420 \div 14 = 30$ cents each, and the blue candies cost $420 \div 15 = 28$ cents each.

8. **Answer (B):** The height of each equilateral triangle is $\sqrt{3}$, so the side length of the square is $2\sqrt{3}$. The area of the square is then $\left(2\sqrt{3}\right)^2 = 12$, and the area of the shaded region is

$$12 - 4 \cdot \frac{\sqrt{3}}{4} \cdot 2^2 = 12 - 4\sqrt{3}.$$

9. **Answer (A):** If $x \geq 0$, then $|x| = x$, so $\lfloor |x| \rfloor = \lfloor x \rfloor$. Furthermore, if $x \geq 0$, then $\lfloor x \rfloor \geq 0$, so $\left| \lfloor x \rfloor \right| = \lfloor x \rfloor$. Therefore $f(x) = \lfloor x \rfloor - \lfloor x \rfloor = 0$ when $x \geq 0$.

   Otherwise, $x < 0$, so $|x| = -x$.

   If $x < 0$ and x is an integer, then $\lfloor |x| \rfloor = \lfloor -x \rfloor = -x$ and $\left| \lfloor x \rfloor \right| = |x| = -x$. Therefore $f(x) = (-x) - (-x) = 0$ in this case.

   If $x < 0$ and x is not an integer, then $\lfloor |x| \rfloor = \lfloor -x \rfloor = -\lfloor x \rfloor - 1$ and $\left| \lfloor x \rfloor \right| = -\lfloor x \rfloor$. Therefore $f(x) = \left(-\lfloor x \rfloor - 1\right) - \left(-\lfloor x \rfloor\right) = -1$ in this case.

   Thus the range of $f(x)$ is $\{-1, 0\}$.

10. **Answer (A):** In order for the area of $\triangle ABC$ to be 100, the altitude to the base $\overline{AB}$ must be 20. Thus C must lie on one of the two lines parallel to and 20 units from line AB. In order for the perimeter of $\triangle ABC$ to be 50, the sum of the lengths of the other two sides must be 40, which implies that point C lies on an ellipse whose foci are A and B and whose semi-minor axis has length $\frac{1}{2}\sqrt{40^2 - 10^2} = \sqrt{375}$, which is less than 20. Therefore the ellipse does not intersect either of the parallel lines, and there are no such points C.

OR

As above, the altitude to the base $\overline{AB}$ of length 10 is 20. Therefore $CA \geq 20$ and $CB \geq 20$, and at least one of those sides has length greater than 20. This contradicts the fact that the perimeter is 50, so no such points C exist.

11. **Answer (A):** Let x be the number of green marbles in Jar 1. Then there are $95-x$ green marbles in Jar 2. Jar 1 contains $9x$ blue marbles and $10x$ marbles in all, and Jar 2 contains $8(95-x)$ blue marbles and $9(95-x)$ marbles in all. Because the jars contain the same number of marbles, $10x = 9(95-x)$, and this equation has the solution $x = 45$. Therefore Jar 1 contains $9 \cdot 45 = 405$ blue marbles, and Jar 2 contains $8(95-45) = 400$ blue marbles. Jar 1 contains $405-400 = 5$ more blue marbles than does Jar 2.

OR

The number of marbles in each jar must be a multiple of both 10 and 9, so it is a multiple of 90. If there are $90n$ marbles in each jar, then there are $9n$ green marbles in Jar 1 and $10n$ green marbles in Jar 2. Thus $9n + 10n = 95$, so $n = 5$. Therefore there are 5 fewer green marbles in Jar 1 than in Jar 2, so there are also 5 more blue marbles in Jar 1 than Jar 2.

12. **Answer (C):** One can convert 2018 to base seven by repeatedly dividing by 7; the successive remainders are the digits in the base-seven representation, from right to left. Thus $2018 = 5612_{\text{seven}}$. It follows that the base-seven representations of positive integers less than 2019 have at most four digits, each digit is at most 6, and the leftmost digit is at most 5. If the leftmost digit is 4, then the remaining digits can all be 6 for a sum of $4+6+6+6 = 22$. If the leftmost digit is 5, then the remaining digits cannot all be 6. Therefore the required sum of digits cannot exceed $5+5+6+6 = 22$. Because $5566_{\text{seven}} < 5612_{\text{seven}} < 2019$ (and $4666_{\text{seven}} < 5612_{\text{seven}} < 2019$), the requested maximum sum is 22.

13. **Answer (A):** The mean of the given numbers is

$$\frac{4+6+8+17+x}{5} = \frac{x+35}{5} = \frac{x}{5} + 7.$$

The median depends on the value of x.

If $x < 6$, then the median is 6. If the mean and median are equal, then $\frac{x}{5} + 7 = 6$, which is equivalent to $x = -5$.

If $6 \le x \le 8$, then the median is x. If the mean and median are equal, then $\frac{x}{5} + 7 = x$, which is equivalent to $x = \frac{35}{4}$. But this is outside of the given range.

If $x > 8$, then the median is 8. If the mean and median are equal then $\frac{x}{5} + 7 = 8$, which is equivalent to $x = 5$. Again this is outside of the given range.

Therefore the only value of x for which the mean equals the median is -5, so the requested sum is also -5.

14. **Answer (C):** Because 5, 10, and 15 all have a single factor of 5 in their prime factorization, 19! ends with 3 0s. Hence $H = 0$. To determine T and M, divisibility tests for 9 and 11 can be used. Because 19! is divisible by 9, its digit sum, $T + M + 33$, must also be divisible by 9, which implies $T + M = 3$ or $T + M = 12$. Similarly, because 19! is divisible by 11, its alternating digit sum, $(T + 13) - (M + 20) = T - M - 7$, must also be divisible by 11. This implies that $T - M = -4$ or $T - M = 7$. Combining these constraints results in only one solution in which T and M are digits, namely $T = 4$ and $M = 8$. Hence $T + M + H = 4 + 8 + 0 = 12$.

15. **Answer (A):** Let a and b, with $a < b$, be the shared side lengths. Then T_1 has hypotenuse b and legs a and $\sqrt{b^2 - a^2}$, and T_2 has hypotenuse $\sqrt{a^2 + b^2}$ and legs a and b. Thus $\frac{1}{2}a\sqrt{b^2 - a^2} = 1$ and $\frac{1}{2}ab = 2$. Multiplying the first equation by 2 and then squaring gives $a^2b^2 - a^4 = 4$. From the second equation, $a^2b^2 = 16$, so $16 - a^4 = 4$, which means $a^4 = 12$. Then

$$b^4 = \left(\frac{4}{a}\right)^4 = \frac{4^4}{a^4} = \frac{256}{12} = \frac{64}{3}.$$

Therefore the square of the product of the other sides is

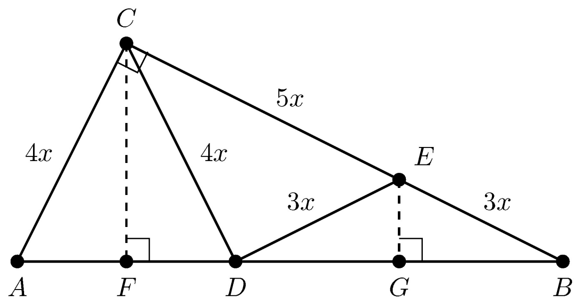
$$\left(\sqrt{b^2 - a^2} \cdot \sqrt{a^2 + b^2}\right)^2 = b^4 - a^4 = \frac{64}{3} - 12 = \frac{28}{3}.$$

16. **Answer (A):** Let $AC = DC = 4x$ and $DE = BE = 3x$. Because $\angle A \cong \angle ADC$, $\angle B \cong \angle EDB$, and $\angle A$ and $\angle B$ are complementary, it follows that $\angle CDE$ is a right angle. Thus $CE = 5x$. Let F and

G lie on $\overline{AB}$ so that $\overline{CF}$ and $\overline{EG}$ are perpendicular to $\overline{AB}$. Then it follows that

$$\frac{3}{8} = \frac{BE}{BC} = \frac{BG}{BF} = \frac{\frac{1}{2}BD}{BD + \frac{1}{2}AD} = \frac{BD}{2BD + AD},$$

so $8BD = 6BD + 3AD$. It follows that $AD : DB = 2 : 3$.



17. **Answer (C):** The probability that the two balls are tossed into the same bin is

$$\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{8} \cdot \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}.$$

Therefore the probability that the balls are tossed into different bins is $\frac{2}{3}$. By symmetry the probability that the red ball is tossed into a higher-numbered bin than the green ball is $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$.

18. **Answer (C):** By symmetry Henry's walks will converge toward walking between two points, one at a distance x from the gym and the other at the same distance x from his home. Because Henry would be $2 - x$ kilometers from home when he is closest to the gym and also because his trip toward home would take him to $\frac{1}{4}$ this distance from home, $x = \frac{1}{4}(2 - x)$. Solving this yields $x = \frac{2}{5}$. Therefore, Henry's walks will approach $2 - 2 \cdot \frac{2}{5} = 1\frac{1}{5}$ kilometers in length.

OR

If Henry is a_k kilometers from home after his kth walk toward the gym and b_k kilometers from home after his kth walk toward home, then $a_0 = b_0 = 0$, and for $k \geq 1$,

$$a_k = b_{k-1} + \frac{3}{4}(2 - b_{k-1}) = \frac{3}{2} + \frac{1}{4}b_{k-1}$$

and

$$b_k = \frac{1}{4}a_k = \frac{3}{8} + \frac{1}{16}b_{k-1}.$$

Iterating shows that the sequence (b_k) converges to

$$B = \frac{3}{8} + \frac{3}{8}\cdot\frac{1}{16} + \frac{3}{8}\cdot\left(\frac{1}{16}\right)^2 + \cdots = \frac{3}{8}\cdot\frac{1}{1-\frac{1}{16}} = \frac{2}{5},$$

from which it then follows that (a_k) converges to $A = \frac{3}{2} + \frac{1}{4}\cdot\frac{2}{5} = \frac{8}{5}$. The requested absolute difference is $\left|\frac{8}{5} - \frac{2}{5}\right| = 1\frac{1}{5}$.

OR

Let x_k denote Henry's distance from home after his kth walk. The following formulas give the value of x_k:

$$x_k = \frac{2}{5} - \frac{2}{5\cdot 4^k} \quad \text{when } k \text{ is even}$$

and

$$x_k = \frac{8}{5} - \frac{2}{5\cdot 4^k} \quad \text{when } k \text{ is odd.}$$

To prove this by mathematical induction, first note that indeed $x_0 = \frac{2}{5} - \frac{2}{5\cdot 4^0} = 0$ and $x_1 = \frac{8}{5} - \frac{2}{5\cdot 4^1} = \frac{3}{2} = \frac{3}{4}\cdot 2$. Then for even values of $k \geq 2$, Henry was heading home, so

$$x_k = \frac{1}{4}x_{k-1} = \frac{1}{4}\left(\frac{8}{5} - \frac{2}{5\cdot 4^{k-1}}\right) = \frac{2}{5} - \frac{2}{5\cdot 4^k};$$

and for odd values of $k \geq 3$, Henry was heading toward the gym, so

$$\begin{aligned} x_k &= x_{k-1} + \frac{3}{4}(2 - x_{k-1}) \\ &= \frac{3}{2} + \frac{1}{4}x_{k-1} \\ &= \frac{3}{2} + \frac{1}{4}\left(\frac{2}{5} - \frac{2}{5\cdot 4^{k-1}}\right) \\ &= \frac{8}{5} - \frac{2}{5\cdot 4^k}. \end{aligned}$$

As k approaches infinity, these values rapidly converge to $\frac{2}{5}$ and $\frac{8}{5}$, respectively, so Henry is essentially walking back and forth between two points that are $\frac{8}{5} - \frac{2}{5} = 1\frac{1}{5}$ kilometers apart.

19. **Answer (C):** Note that 100,000 $= 2^5 \cdot 5^5$. This implies that for a number to be a product of two elements in S it must be of the form $2^a \cdot 5^b$ with $0 \le a \le 10$ and $0 \le b \le 10$. The corresponding product for the remainder of this solution will be denoted (a, b). Note that the pairs $(0,0)$, $(0,10)$, $(10,0)$, and $(10,10)$ cannot be obtained as the product of two distinct elements of S; these products can be obtained only as $1 \cdot 1 = 1$, $5^5 \cdot 5^5 = 5^{10}$, $2^5 \cdot 2^5 = 2^{10}$, and $10^5 \cdot 10^5 = 10^{10}$, respectively. This gives at most $11 \cdot 11 - 4 = 117$ possible products. To see that all these pairs can be achieved, consider four cases:

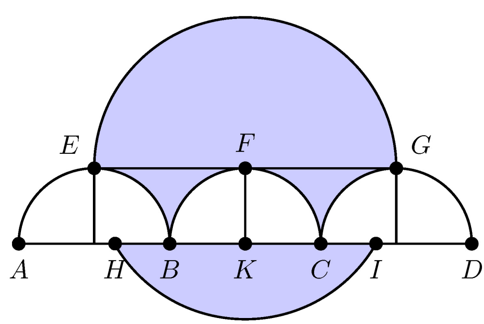
    If $0 \le a \le 5$ and $0 \le b \le 5$, other than $(0,0)$, then (a,b) can be achieved with the divisors 1 and $2^a \cdot 5^b$.

    If $6 \le a \le 10$ and $0 \le b \le 5$, other than $(10,0)$, then (a,b) can be achieved with the divisors 2^5 and $2^{a-5} \cdot 5^b$.

    If $0 \le a \le 5$ and $6 \le b \le 10$, other than $(0,10)$, then (a,b) can be achieved with the divisors 5^5 and $2^a \cdot 5^{b-5}$.

    Finally, if $6 \le a \le 10$ and $6 \le b \le 10$, other than $(10,10)$, then (a,b) can be achieved with the divisors $2^5 \cdot 5^5$ and $2^{a-5} \cdot 5^{b-5}$.

20. **Answer (E):** Let H and I be the intersections of $\overline{AD}$ with the circle centered at F, where H lies between A and B, and I lies between C and D; and let K be the foot of the perpendicular line segment from F to $\overline{AD}$. The specified region consists of three subregions: a semicircle of radius 2, a 4×1 rectangle with 4 quarter circles of radius 1 removed, and the segment of the circle cut off by chord $\overline{HI}$, as shown in the figure below.



    The semicircle of radius 2 has area 2π. The rectangle minus the 4 quarter circles has area $4 - \pi$. Because $FK = 1$ and $FI = 2$, it follows that $\angle KFI$ has measure $60°$, and therefore the segment of the circle is a third of the circle with $\triangle HFI$ removed. The area of the segment is

$$\frac{4}{3}\pi - \frac{1}{2}\cdot 2\sqrt{3}\cdot 1 = \frac{4}{3}\pi - \sqrt{3}.$$

Adding the areas of the three subregions gives $\frac{7}{3}\pi - \sqrt{3} + 4$, and the requested sum is $7 + 3 + 3 + 4 = 17$.

21. **Answer (B):** With probability 1, either HH or TT will occur after a finite number of flips. The desired event will occur if and only if the sequence of flips is THTHH or THTHTHH or THTHTHTHH or .... The probabilities of these outcomes are $\left(\frac{1}{2}\right)^5$, $\left(\frac{1}{2}\right)^7$, $\left(\frac{1}{2}\right)^9$, ..., a geometric sequence with common ratio $\frac{1}{4}$. The requested probability is the sum of these probabilities,

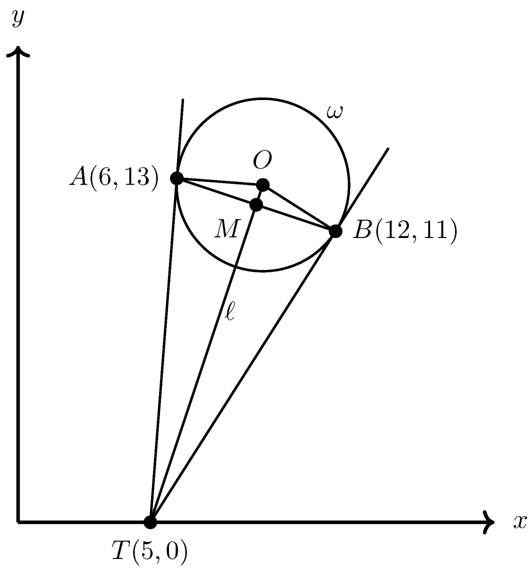
$$\frac{\left(\frac{1}{2}\right)^5}{1-\frac{1}{4}} = \frac{1}{32}\cdot\frac{4}{3} = \frac{1}{24}.$$

22. **Answer (B):** No player can ever end up with \$3 at the end of a round, because that player had to give away one of the dollars in play. Therefore the only two possible distributions of the money are 1-1-1 and 2-1-0. Suppose that a round of the game starts at 1-1-1. Without loss of generality, assume that Raashan gives his dollar to Sylvia. Then the only way for the round to end at 1-1-1 is for Ted to give his dollar to Raashan (otherwise Sylvia would end up with \$2) and for Sylvia to give her dollar to Ted; the probability of this is $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$. Next suppose that a round starts at 2-1-0; without loss of generality, assume that Raashan has \$2 and Sylvia has \$1. Then the only way for the round to end at 1-1-1 is for Sylvia to give her dollar to Ted (otherwise Raashan would end up with \$2) and for Raashan to give his dollar to Sylvia; the probability of this is $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$. Thus no matter how the round starts, the probability that the round will end at 1-1-1 is $\frac{1}{4}$. In particular, the probability is $\frac{1}{4}$ that at the end of the 2019th round each player will have \$1.

Note: It may seem counterintuitive that an uneven distribution is more likely than an even distribution. But in a situation with larger initial bankrolls evenly distributed to a larger number of players, inequality reigns after many rounds. See this website:

`http://www.decisionsciencenews.com/?s=happens+next`

23. **Answer (C):** Let T be the point where the tangents at A and B intersect. By symmetry T lies on the perpendicular bisector ℓ of $\overline{AB}$, so in fact T is the (unique) intersection point of line ℓ with the x-axis. Computing the midpoint M of $\overline{AB}$ gives $(9,12)$, and computing the slope of $\overline{AB}$ gives $\frac{13-11}{6-12} = -\frac{1}{3}$. This means that the slope of ℓ is 3, so the equation of ℓ is given by $y-12 = 3(x-9)$. Setting $y=0$ yields that $T=(5,0)$.



Now let O be the center of circle ω. Note that $\overline{OA} \perp \overline{AT}$ and $\overline{OB} \perp \overline{BT}$, so in fact M is the foot of the altitude from A to the hypotenuse of $\triangle OAT$. By the distance formula, $TM = 4\sqrt{10}$ and $MA = \sqrt{10}$. Then by the Altitude on Hypotenuse Theorem, $MO = \frac{1}{4}\sqrt{10}$, so by the Pythagorean Theorem radius AO of circle ω is $\frac{1}{4}\sqrt{170}$. As a result, the area of the circle is

$$\frac{1}{16} \cdot 170 \cdot \pi = \frac{85\pi}{8}.$$

OR

As above, the line $y-12 = 3(x-9)$ passes through $T(5,0)$ and the center of the circle O. The slope of line AT is 13, so the slope of AO is $-\frac{1}{13}$. The equation of line OA is $13(y-13) = 6-x$. Thus the intersection of line AO and line OT is $O\left(\frac{37}{4}, \frac{51}{4}\right)$. Then the radius of the circle is

$$OA = \sqrt{\left(\frac{37}{4} - 6\right)^2 + \left(\frac{51}{4} - 13\right)^2} = \frac{1}{4}\sqrt{170}$$

as in the solution above.

24. **Answer (C):** First note that it suffices to study $y_n = x_n - 4$ and find the least positive integer m such that $y_m \le \frac{1}{2^{20}}$. Now $y_0 = 1$ and

$$y_{n+1} = \frac{y_n(y_n + 9)}{y_n + 10}.$$

Observe that (y_n) is a strictly decreasing sequence of positive numbers. Because

$$\frac{y_{n+1}}{y_n} = 1 - \frac{1}{y_n + 10},$$

it follows that

$$\frac{9}{10} \le \frac{y_{n+1}}{y_n} \le \frac{10}{11},$$

and because $y_0 = 1$,

$$\left(\frac{9}{10}\right)^k \le y_k \le \left(\frac{10}{11}\right)^k$$

for all integers $k \ge 2$.

Now note that

$$\left(\frac{1}{2}\right)^{\frac{1}{4}} < \frac{9}{10}$$

because this is equivalent to $0.5 < (0.9)^4 = (0.81)^2$. Therefore

$$\left(\frac{1}{2}\right)^{\frac{m}{4}} < y_m \le \frac{1}{2^{20}},$$

so $m > 80$. Now note that

$$\left(\frac{11}{10}\right)^{10} = \left(1 + \frac{1}{10}\right)^{10} > 1 + 10 \cdot \frac{1}{10} = 2,$$

so

$$\frac{10}{11} < \left(\frac{1}{2}\right)^{\frac{1}{10}}.$$

Thus
$$\frac{1}{2^{20}} < y_{m-1} < \left(\frac{10}{11}\right)^{m-1} < \left(\frac{1}{2}\right)^{\frac{m-1}{10}},$$
so $m < 201$. Thus m lies in the range **(C)**. (Numerical calculations will show that $m = 133$.)

25. **Answer (C):** For $n \geq 2$, let a_n be the number of sequences of length n that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s. In order for the sequence to end with a 0 and satisfy the conditions, it must end either 010 or 0110. Thus $a_n = a_{n-2} + a_{n-3}$. The initial conditions for this recurrence relation are $a_2 = 0$, $a_3 = 1$ (the sequence 010), and $a_4 = 1$ (the sequence 0110). Then $a_5 = a_3 + a_2 = 1 + 0 = 1$, $a_6 = a_4 + a_3 = 1 + 1 = 2$, $a_7 = a_5 + a_4 = 1 + 1 = 2$, and so on. A bit of calculation produces the display below; $a_{19} = 65$.

n	2	3	4	5	6	7	8	9	10	11
a_n	0	1	1	1	2	2	3	4	5	7

n	12	13	14	15	16	17	18	19
a_n	9	12	16	21	28	37	49	65

OR

There are four cases, depending on the number of 0s in the string. If there are 10 0s, then there are 9 1s, and because no two 0s can be consecutive, the string must be 0101010101010101010. If there are 9 0s and 10 1s, then there are 8 gaps between the 0s into which at least one but no more than two 1s must be placed, and there are $\binom{8}{2}$ ways to choose the gaps into which to place two 1s. Similarly, if there are 8 0s and 11 1s, then there are $\binom{7}{4}$ ways to choose the gaps into which to place two 1s; and if there are 7 0s and 12 1s, then there are $\binom{6}{6}$ ways to choose the gaps into which to place two 1s. There cannot be more than 10 nor fewer than 7 0s. The number of possible strings is
$$1 + \binom{8}{2} + \binom{7}{4} + \binom{6}{6} = 1 + 28 + 35 + 1 = 65.$$

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