2019

AMC 10B

DO NOT OPEN UNTIL WEDNESDAY, February 13, 2019

****Administration On An Earlier Date Will Disqualify Your School's Results****

1. All the information needed to administer this exam is contained in the AMC 10/12 Teacher's Manual. PLEASE READ THE MANUAL BEFORE FEBRUARY 13, 2019.
2. Your PRINCIPAL or VICE PRINCIPAL must verify on the AMC 10/12 COMPETITION CERTIFICATION FORM (found on **maa.org/amc** under 'AMC 10/12') that you followed all rules associated with the administration of the exam.
3. Answer sheets must be returned to the MAA AMC office the day after the competition. Ship with appropriate postage using a trackable method. FedEx or UPS is strongly recommended.
4. The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

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MAA AMC
American Mathematics Competitions

American Mathematics Competitions

20th Annual

AMC 10B

American Mathematics Competition 10B
Wednesday, February 13, 2019

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR COMPETITION MANAGER TELLS YOU.
2. This is a 25-question multiple-choice exam. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 10 Answer Sheet with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer sheet will be graded. **You must use and submit the original answer sheets provided by the MAA AMC. Photocopies will not be scored.**
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on the exam will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the exam, your competition manager will ask you to record certain information on the answer form.
8. When your competition manager gives the signal, begin working on the problems. You will have **75 minutes** to complete the exam.
9. When you finish the exam, *sign your name* in the space provided at the top of the Answer Sheet.

The Committee on the American Mathematics Competitions reserves the right to disqualify scores from a school if it determines that the required security procedures were not followed.

Students who score well on this AMC 10 will be invited to take the 37th annual American Invitational Mathematics Examination (AIME) on Wednesday, March 13, 2019, or Thursday, March 21, 2019. More details about the AIME are on the back page of this exam booklet.

1. Alicia had two containers. The first was $\frac{5}{6}$ full of water and the second was empty. She poured all the water from the first container into the second container, at which point the second container was $\frac{3}{4}$ full of water. What is the ratio of the volume of the smaller container to the volume of the larger container?

(A) $\frac{5}{8}$ **(B)** $\frac{4}{5}$ **(C)** $\frac{7}{8}$ **(D)** $\frac{9}{10}$ **(E)** $\frac{11}{12}$

2. Consider the statement, "If n is not prime, then $n-2$ is prime." Which of the following values of n is a counterexample to this statement?

(A) 11 **(B)** 15 **(C)** 19 **(D)** 21 **(E)** 27

3. In a high school with 500 students, 40% of the seniors play a musical instrument, while 30% of the non-seniors do not play a musical instrument. In all, 46.8% of the students do not play a musical instrument. How many non-seniors play a musical instrument?

(A) 66 **(B)** 154 **(C)** 186 **(D)** 220 **(E)** 266

4. All lines with equation $ax+by=c$ such that a, b, c form an arithmetic progression pass through a common point. What are the coordinates of that point?

(A) $(-1,2)$ **(B)** $(0,1)$ **(C)** $(1,-2)$ **(D)** $(1,0)$
(E) $(1,2)$

5. Triangle ABC lies in the first quadrant. Points A, B, and C are reflected across the line $y=x$ to points A', B', and C', respectively. Assume that none of the vertices of the triangle lie on the line $y=x$. Which of the following statements is <u>not</u> always true?

(A) Triangle $A'B'C'$ lies in the first quadrant.

(B) Triangles ABC and $A'B'C'$ have the same area.

(C) The slope of line AA' is -1.

(D) The slopes of lines AA' and CC' are the same.

(E) Lines AB and $A'B'$ are perpendicular to each other.

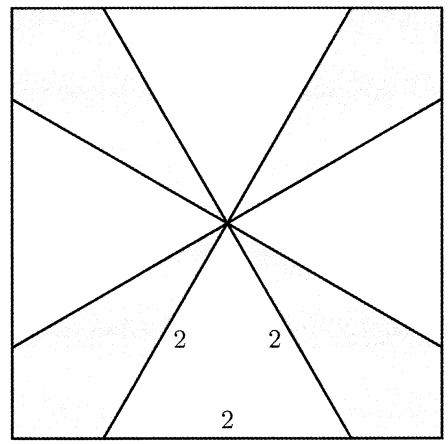
6. A positive integer n satisfies the equation $(n+1)!+(n+2)! = 440 \cdot n!$. What is the sum of the digits of n?

(A) 2 **(B)** 5 **(C)** 10 **(D)** 12 **(E)** 15

7. Each piece of candy in a shop costs a whole number of cents. Casper has exactly enough money to buy either 12 pieces of red candy, 14 pieces of green candy, 15 pieces of blue candy, or n pieces of purple candy. A piece of purple candy costs 20 cents. What is the least possible value of n?

(A) 18 **(B)** 21 **(C)** 24 **(D)** 25 **(E)** 28

8. The figure below shows a square and four equilateral triangles, with each triangle having a side lying on a side of the square, such that each triangle has side length 2 and the third vertices of the triangles meet at the center of the square. The region inside the square but outside the triangles is shaded. What is the area of the shaded region?



(A) 4 **(B)** $12-4\sqrt{3}$ **(C)** $3\sqrt{3}$ **(D)** $4\sqrt{3}$ **(E)** $16-4\sqrt{3}$

9. The function f is defined by

$$f(x) = \lfloor |x| \rfloor - |\lfloor x \rfloor|$$

for all real numbers x, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to the real number r. What is the range of f?

(A) $\{-1,0\}$ **(B)** the set of nonpositive integers **(C)** $\{-1,0,1\}$
(D) $\{0\}$ **(E)** the set of nonnegative integers

10. In a given plane, points A and B are 10 units apart. How many points C are there in the plane such that the perimeter of $\triangle ABC$ is 50 units and the area of $\triangle ABC$ is 100 square units?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** infinitely many

11. Two jars each contain the same number of marbles, and every marble is either blue or green. In Jar 1 the ratio of blue to green marbles is $9:1$, and the ratio of blue to green marbles in Jar 2 is $8:1$. There are 95 green marbles in all. How many more blue marbles are in Jar 1 than in Jar 2?

(A) 5 **(B)** 10 **(C)** 25 **(D)** 45 **(E)** 50

12. What is the greatest possible sum of the digits in the base-seven representation of a positive integer less than 2019 ?

(A) 11 **(B)** 14 **(C)** 22 **(D)** 23 **(E)** 27

13. What is the sum of all real numbers x for which the median of the numbers 4, 6, 8, 17, and x is equal to the mean of those five numbers?

(A) -5 **(B)** 0 **(C)** 5 **(D)** $\frac{15}{4}$ **(E)** $\frac{35}{4}$

14. The base-ten representation for 19! is $121{,}6T5{,}100{,}40M{,}832{,}H00$, where T, M, and H denote digits that are not given. What is $T+M+H$?

(A) 3 **(B)** 8 **(C)** 12 **(D)** 14 **(E)** 17

15. Right triangles T_1 and T_2 have areas 1 and 2, respectively. A side of T_1 is congruent to a side of T_2, and a different side of T_1 is congruent to a different side of T_2. What is the square of the product of the lengths of the other (third) sides of T_1 and T_2 ?

(A) $\frac{28}{3}$ **(B)** 10 **(C)** $\frac{21}{2}$ **(D)** $\frac{32}{3}$ **(E)** 12

16. In $\triangle ABC$ with a right angle at C, point D lies in the interior of $\overline{AB}$ and point E lies in the interior of $\overline{BC}$ so that $AC=CD$, $DE=EB$, and the ratio $AC:DE=4:3$. What is the ratio $AD:DB$?

(A) $2:3$ **(B)** $2:\sqrt{5}$ **(C)** $1:1$ **(D)** $3:\sqrt{5}$ **(E)** $3:2$

17. A red ball and a green ball are randomly and independently tossed into bins numbered with the positive integers so that for each ball, the probability that it is tossed into bin k is 2^{-k} for $k=1,2,3,\ldots$. What is the probability that the red ball is tossed into a higher-numbered bin than the green ball?

(A) $\frac{1}{4}$ **(B)** $\frac{2}{7}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{3}{7}$

18. Henry decides one morning to do a workout, and he walks $\frac{3}{4}$ of the way from his home to his gym. The gym is 2 kilometers away from Henry's home. At that point, he changes his mind and walks $\frac{3}{4}$ of the way from where he is back toward home. When he reaches that point, he changes his mind again and walks $\frac{3}{4}$ of the distance from there back toward the gym. If Henry keeps changing his mind when he has walked $\frac{3}{4}$ of the distance toward either the gym or home from the point where he last changed his mind, he will get very close to walking back and forth between a point A kilometers from home and a point B kilometers from home. What is $|A-B|$?

(A) $\frac{2}{3}$ **(B)** 1 **(C)** $1\frac{1}{5}$ **(D)** $1\frac{1}{4}$ **(E)** $1\frac{1}{2}$

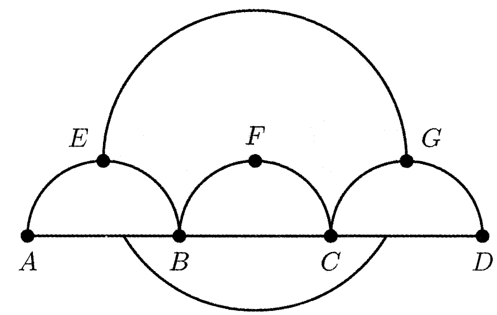
19. Let S be the set of all positive integer divisors of 100,000. How many numbers are the product of two distinct elements of S ?

(A) 98 **(B)** 100 **(C)** 117 **(D)** 119 **(E)** 121

20. As shown in the figure, line segment $\overline{AD}$ is trisected by points B and C so that $AB=BC=CD=2$. Three semicircles of radius 1, $\overarc{AEB}$, $\overarc{BFC}$, and $\overarc{CGD}$, have their diameters on $\overline{AD}$, lie in the same halfplane determined by line AD, and are tangent to line EG at E, F, and G, respectively. A circle of radius 2 has its center at F. The area of the region inside the circle but outside the three semicircles, shaded in the figure, can be expressed in the form

$$\frac{a}{b}\cdot\pi-\sqrt{c}+d,$$

where a, b, c, and d are positive integers and a and b are relatively prime. What is $a+b+c+d$?



(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

21. Debra flips a fair coin repeatedly, keeping track of how many heads and how many tails she has seen in total, until she gets either two heads in a row or two tails in a row, at which point she stops flipping. What is the probability that she gets two heads in a row but she sees a second tail before she sees a second head?

(A) $\frac{1}{36}$ **(B)** $\frac{1}{24}$ **(C)** $\frac{1}{18}$ **(D)** $\frac{1}{12}$ **(E)** $\frac{1}{6}$

22. Raashan, Sylvia, and Ted play the following game. Each starts with \$1. A bell rings every 15 seconds, at which time each of the players who currently has money simultaneously chooses one of the other two players independently and at random and gives \$1 to that player. What is the probability that after the bell has rung 2019 times, each player will have \$1? (For example, Raashan and Ted may each decide to give \$1 to Sylvia, and Sylvia may decide to give her dollar to Ted, at which point Raashan will have \$0, Sylvia will have \$2, and Ted will have \$1, and that is the end of the first round of play. In the second round Raashan has no money to give, but Sylvia and Ted might choose each other to give their \$1 to, and the holdings will be the same at the end of the second round.)

(A) $\frac{1}{7}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

23. Points $A(6,13)$ and $B(12,11)$ lie on a circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω?

(A) $\frac{83\pi}{8}$ **(B)** $\frac{21\pi}{2}$ **(C)** $\frac{85\pi}{8}$ **(D)** $\frac{43\pi}{4}$ **(E)** $\frac{87\pi}{8}$

24. Define a sequence recursively by $x_0 = 5$ and

$$x_{n+1} = \frac{x_n^2 + 5x_n + 4}{x_n + 6}$$

for all nonnegative integers n. Let m be the least positive integer such that

$$x_m \le 4 + \frac{1}{2^{20}}.$$

In which of the following intervals does m lie?

(A) $[9,26]$ **(B)** $[27,80]$ **(C)** $[81,242]$ **(D)** $[243,728]$

(E) $[729,\infty)$

25. How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

(A) 55 **(B)** 60 **(C)** 65 **(D)** 70 **(E)** 75

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Questions and comments about problems
and solutions for this exam should be sent to:

amchq@maa.org

Send questions and comments about administrative arrangements to:

amcinfo@maa.org

or

MAA American Mathematics Competitions
P.O. Box 471
Annapolis Junction, MD 20701

The problems and solutions for this AMC 10 were prepared by MAA's Subcommittee on the AMC 10/AMC 12 Exams, under the direction of the co-chairs Jerrold W. Grossman and Carl Yerger.

2019 AIME

The 37th annual AIME will be held on Wednesday, March 13, 2019, with the alternate on Thursday, March 21, 2019. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate if you achieve a high score on this competition. Top-scoring students on the AMC 10/12/AIME will be selected to take the 48th Annual USA Mathematical Olympiad (USAMO) on April 17–18, 2019.