MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

18th Annual

AMC 10B

American Mathematics Competition 10B

Wednesday, February 15, 2017

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (B):** Working backwards, switching the digits of the numbers 71, 72, 73, 74, and 75 and subtracting 11 gives, respectively, 6, 16, 26, 36, and 46. Only 6 and 36 are divisible by 3, and only $36 \div 3 = 12$ is a two-digit number.

2. **Answer (C):** Each lap took Sofia $\frac{100 \text{ m}}{4 \text{ m/s}} + \frac{300 \text{ m}}{5 \text{ m/s}} = 85$ seconds, so 5 laps took her $5 \cdot 85 = 425$ seconds, which is 7 minutes and 5 seconds.

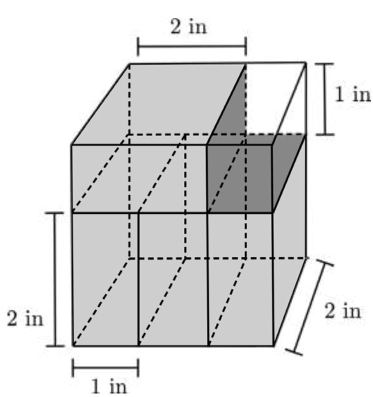
3. **Answer (E):** Adding the inequalities $y > -1$ and $z > 1$ yields $y + z > 0$. The other four choices give negative values if, for example, $x = \frac{1}{8}$, $y = -\frac{1}{4}$, and $z = \frac{3}{2}$.

4. **Answer (D):** The given equation implies that $3x + y = -2(x - 3y)$, which is equivalent to $x = y$. Therefore

$$\frac{x+3y}{3x-y} = \frac{4y}{2y} = 2.$$

5. **Answer (D):** Suppose Camilla originally had b blueberry jelly beans and c cherry jelly beans. After eating 10 pieces of each kind, she now has $b - 10$ blueberry jelly beans and $c - 10$ cherry jelly beans. The conditions of the problem are equivalent to the equations $b = 2c$ and $b - 10 = 3(c - 10)$. Then $2c - 10 = 3c - 30$, which means that $c = 20$ and $b = 2 \cdot 20 = 40$.

6. **Answer (B):** A possible arrangement of 4 blocks is shown by the figure.



Four blocks do not completely fill the box because the combined volume of the blocks is only $4(2 \cdot 2 \cdot 1) = 16$ cubic inches, whereas the volume of the box is $3 \cdot 2 \cdot 3 = 18$ cubic inches. Because the unused space, $18 - 16 = 2$ cubic inches, is less than the volume of a block, 4 cubic inches, no more than 4 blocks can fit in the box.

7. **Answer (C):** Let $2d$ be the distance in kilometers to the friend's house. Then Samia bicycled distance d at rate 17 and walked distance d at rate 5, for a total time of

$$\frac{d}{17} + \frac{d}{5} = \frac{44}{60}$$

hours. Solving this equation yields $d = \frac{17}{6} = 2.833\ldots$. Therefore Samia walked about 2.8 kilometers.

8. **Answer (C):** The altitude $\overline{AD}$ lies on a line of symmetry for the isosceles triangle. Under reflection about this line, B will be sent to C. Because B is obtained from D by adding 3 to the x-coordinate and subtracting 6 from the y-coordinate, C is obtained from D by subtracting 3 from the x-coordinate and adding 6 to the y-coordinate. Thus the third vertex C has coordinates $(-1 - 3, 3 + 6) = (-4, 9)$.

OR

To find the coordinates of $C(x, y)$, note that D is the midpoint of $\overline{BC}$. Therefore

$$\frac{x+2}{2} = -1 \qquad \text{and} \qquad \frac{y-3}{2} = 3.$$

Solving these equations gives $x = -4$ and $y = 9$, so $C = (-4, 9)$.

9. **Answer (D):** The probability of getting all 3 questions right is $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$. Because there are 3 ways to get 2 of the questions right and 1 wrong, the probability of getting exactly 2 right is $3\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = \frac{6}{27}$. Therefore the probability of winning is $\frac{1}{27} + \frac{6}{27} = \frac{7}{27}$.

10. **Answer (E):** Because the lines are perpendicular, their slopes, $\frac{a}{2}$ and $-\frac{2}{b}$, are negative reciprocals, so $a = b$. Substituting b for a and using the point $(1, -5)$ yields the equations $b + 10 = c$ and $2 - 5b = -c$. Adding the two equations yields $12 - 4b = 0$, so $b = 3$. Thus $c = 3 + 10 = 13$.

11. **Answer (D):** The students who like dancing but say they dislike it constitute $60\% \cdot (100\% - 80\%) = 12\%$ of the students. Similarly, the students who dislike dancing and say they dislike it constitute $(100\% - 60\%) \cdot 90\% = 36\%$ of the students. Therefore the requested fraction is $\frac{12}{12+36} = \frac{1}{4} = 25\%$.

12. **Answer (A):** For Elmer's old car, let M be the fuel efficiency in kilometers per liter, and let C be the cost of fuel in dollars per liter. Then for his new car, the fuel efficiency is $1.5M$, and the cost of fuel is $1.2C$. The cost in dollars per kilometer for the old car is $\frac{C}{M}$, and for the new car it is $\frac{1.2C}{1.5M} = 0.8\frac{C}{M}$. Therefore, fuel for the long trip will cost 20% less in Elmer's new car.

13. **Answer (C):** Let x, y, and z be the number of people taking exactly one, two, and three classes, respectively. The condition that each student in the program takes at least one class is equivalent to the equation $x + y + z = 20$. The condition that there are 9 students taking at least two classes is equivalent to the equation $y + z = 9$. The sum $10 + 13 + 9 = 32$ counts once the students taking one class, twice the students taking two classes, and three times the students taking three classes. Then $x + 2y + 3z = 32$, which is equivalent to $z = 32 - (x + y + z) - (y + z) = 32 - 20 - 9 = 3$.

OR

Let Y, B, and P be the sets of students taking yoga, bridge, and painting, respectively. By the Inclusion–Exclusion Principle,

$$|Y \cup B \cup P| = |Y| + |B| + |P| - (|Y \cap B| + |Y \cap P| + |B \cap P|) + |Y \cap B \cap P|.$$

Furthermore, $|Y \cap B| + |Y \cap P| + |B \cap P| = 9 + 2|Y \cap B \cap P|$, because in tabulating the students taking at least two classes by considering the pairs of classes one by one, the students taking all three classes are counted three times rather than just once. Thus

$$20 = 10 + 13 + 9 - (9 + 2|Y \cap B \cap P|) + |Y \cap B \cap P| = 23 - |Y \cap B \cap P|,$$

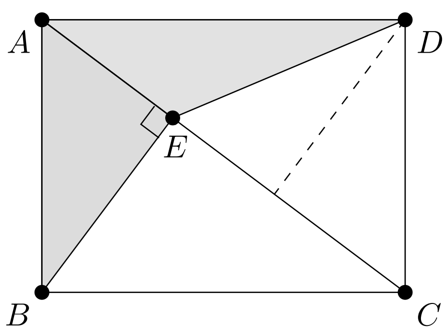
so the number of students taking all three classes is $|Y \cap B \cap P| = 3$.

14. **Answer (D):** An integer will have a remainder of 1 when divided by 5 if and only if the units digit is either 1 or 6. The randomly selected positive integer will itself have a units digit of each of the numbers

from 0 through 9 with equal probability. This digit of N alone will determine the units digit of N^{16}. Computing the 16th power of each of these 10 digits by squaring the units digit four times yields one 0, one 5, four 1s, and four 6s. The probability is therefore $\frac{8}{10} = \frac{4}{5}$.

Note: This result also follows from Fermat's Little Theorem.

15. **Answer (E):** Triangles ADE and ABE have the same area because they share the base $\overline{AE}$ and, by symmetry, they have the same height. By the Pythagorean Theorem, $AC = 5$. Because $\triangle ABE \sim \triangle ACB$, the ratio of their areas is the square of the ratio of their corresponding sides. Their hypotenuses have lengths 3 and 5, respectively, so their areas are in the ratio 9 to 25. The area of $\triangle ACB$ is half that of the rectangle, so the area of $\triangle ABE$ is $\frac{9}{25} \cdot 6 = \frac{54}{25}$. Thus the area of $\triangle ADE$ is also $\frac{54}{25}$.

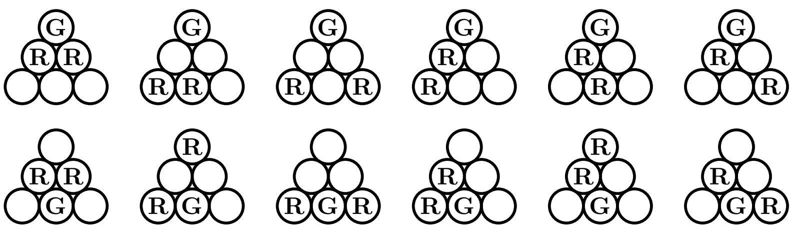


16. **Answer (A):** It will be easier to count the complementary set. There are 9 one-digit numerals that do not contain the digit 0, $9 \cdot 9 = 81$ two-digit numerals that do not contain the digit 0, $9 \cdot 9 \cdot 9 = 729$ three-digit numerals that do not contain the digit 0, and $1 \cdot 9 \cdot 9 \cdot 9 = 729$ four-digit numerals starting with 1 that do not contain the digit 0, a total of 1548. All four-digit numerals between 2000 and 2017, inclusive, contain the digit 0. Therefore $2017 - 1548 = 469$ numerals in the required range do contain the digit 0.

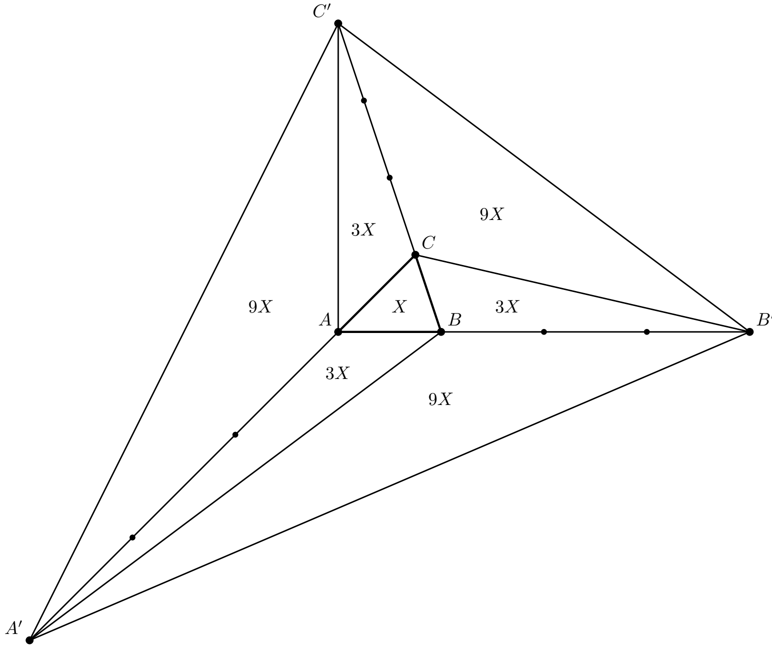
17. **Answer (B):** The monotonous positive integers with one digit or increasing digits can be put into a one-to-one correspondence with the nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The number of such subsets is $2^9 - 1 = 511$. The monotonous positive integers with one digit or decreasing digits can be put into a one-to-one correspondence

with the subsets of $\{0,1,2,3,4,5,6,7,8,9\}$ other than $\emptyset$ and $\{0\}$. The number of these is $2^{10}-2=1022$. The single-digit numbers are included in both sets, so there are $511+1022-9=1524$ monotonous positive integers.

18. **Answer (D):** By symmetry, there are just two cases for the position of the green disk: corner or non-corner. If a corner disk is painted green, then there is 1 case in which both red disks are adjacent to the green disk, there are 2 cases in which neither red disk is adjacent to the green disk, and there are 3 cases in which exactly one of the red disks is adjacent to the green disk. Similarly, if a non-corner disk is painted green, then there is 1 case in which neither red disk is in a corner, there are 2 cases in which both red disks are in a corner, and there are 3 cases in which exactly one of the red disks is in a corner. The total number of paintings is $1+2+3+1+2+3=12$.

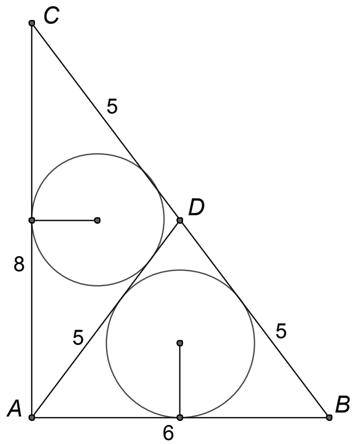


19. **Answer (E):** Draw segments $\overline{CB'}$, $\overline{AC'}$, and $\overline{BA'}$. Let X be the area of $\triangle ABC$. Because $\triangle BB'C$ has a base 3 times as long and the same altitude, its area is $3X$. Similarly, the areas of $\triangle AA'B$ and $\triangle CC'A$ are also $3X$. Furthermore, $\triangle AA'C'$ has 3 times the base and the same height as $\triangle ACC'$, so its area is $9X$. The areas of $\triangle CC'B'$ and $\triangle BB'A'$ are also $9X$ by the same reasoning. Therefore the area of $\triangle A'B'C'$ is $X+3(3X)+3(9X)=37X$, and the requested ratio is $37:1$. Note that nothing in this argument requires $\triangle ABC$ to be equilateral.



20. **Answer (B):** There are $\left\lfloor \frac{21}{2} \right\rfloor + \left\lfloor \frac{21}{4} \right\rfloor + \left\lfloor \frac{21}{8} \right\rfloor + \left\lfloor \frac{21}{16} \right\rfloor = 10+5+2+1 = 18$ powers of 2 in the prime factorization of 21!. Thus $21! = 2^{18}k$, where k is odd. A divisor of 21! must be of the form $2^i b$ where $0 \le i \le 18$ and b is a divisor of k. For each choice of b, there is one odd divisor of 21! and 18 even divisors. Therefore the probability that a randomly chosen divisor is odd is $\frac{1}{19}$. In fact, $21! = 2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, so it has $19 \cdot 10 \cdot 5 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 60{,}800$ positive integer divisors, of which $10 \cdot 5 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 3{,}200$ are odd.

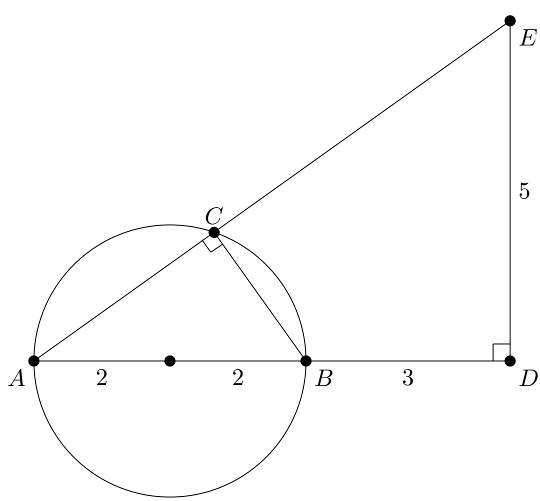
21. **Answer (D):** By the converse of the Pythagorean Theorem, $\angle BAC$ is a right angle, so $BD = CD = AD = 5$, and the area of each of the small triangles is 12 (half the area of $\triangle ABC$). The area of $\triangle ABD$ is equal to its semiperimeter, $\frac{1}{2} \cdot (5 + 5 + 6) = 8$, multiplied by the radius of the inscribed circle, so the radius is $\frac{12}{8} = \frac{3}{2}$. Similarly, the radius of the inscribed circle of $\triangle ACD$ is $\frac{4}{3}$. The requested sum is $\frac{3}{2} + \frac{4}{3} = \frac{17}{6}$.



22. **Answer (D):** Because $\angle ACB$ is inscribed in a semicircle, it is a right angle. Therefore $\triangle ABC$ is similar to $\triangle AED$, so their areas are related as AB^2 is to AE^2. Because $AB^2 = 4^2 = 16$ and, by the Pythagorean Theorem,

$$AE^2 = (4+3)^2 + 5^2 = 74,$$

this ratio is $\frac{16}{74} = \frac{8}{37}$. The area of $\triangle AED$ is $\frac{35}{2}$, so the area of $\triangle ABC$ is $\frac{35}{2} \cdot \frac{8}{37} = \frac{140}{37}$.



23. **Answer (C):** The remainder when N is divided by 5 is clearly 4. A positive integer is divisible by 9 if and only if the sum of its digits is divisible by 9. The sum of the digits of N is $4(0+1+2+\cdots+9)+10\cdot1+10\cdot2+10\cdot3+(4+0)+(4+1)+(4+2)+(4+3)+(4+4)=270$, so N must be a multiple of 9. Then $N-9$ must also be a multiple of 9, and the last digit of $N-9$ is 5, so it is also a multiple of 5. Thus $N-9$ is a multiple of 45, and N leaves a remainder of 9 when divided by 45.

24. **Answer (C):** Assume without loss of generality that two of the vertices of the triangle are on the branch of the hyperbola in the first quadrant. This forces the centroid of the triangle to be the vertex $(1,1)$ of the hyperbola. Because the vertices of the triangle are equidistant from the centroid, the first-quadrant vertices must be $\left(a,\frac{1}{a}\right)$ and $\left(\frac{1}{a},a\right)$ for some positive number a. By symmetry, the third vertex must be $(-1,-1)$. The distance between the vertex $(-1,-1)$ and the centroid $(1,1)$ is $2\sqrt{2}$, so the altitude of the triangle must be $\frac{3}{2}\cdot2\sqrt{2}=3\sqrt{2}$, which makes the side length of the triangle $s=\frac{2}{\sqrt{3}}\cdot3\sqrt{2}=2\sqrt{6}$. The required area is $\frac{\sqrt{3}}{4}s^2=6\sqrt{3}$. The requested value is $(6\sqrt{3})^2=108$. In fact, the vertices of the equilateral triangle are $(-1,-1)$, $(2+\sqrt{3},2-\sqrt{3})$, and $(2-\sqrt{3},2+\sqrt{3})$.

25. **Answer (E):** Let S be the sum of Isabella's 7 scores. Then S is a multiple of 7, and

$$658=91+92+93+\cdots+97\le S\le 94+95+96+\cdots+100=679,$$

so S is one of 658, 665, 672, or 679. Because $S-95$ is a multiple of 6, it follows that $S=665$. Thus the sum of Isabella's first 6 scores was $665-95=570$, which is a multiple of 5, and the sum of her first 5 scores was also a multiple of 5. Therefore her sixth score must have been a multiple of 5. Because her seventh score was 95 and her scores were all different, her sixth score was 100. One possible sequence of scores is $91, 93, 92, 96, 98, 100, 95$.

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