MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

18th Annual

AMC 10A

American Mathematics Competition 10A

Tuesday, February 7, 2017

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

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The problems and solutions for this AMC 10 were prepared by
MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the
co-chairs Jerrold W. Grossman and Carl Yerger.

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1. **Answer (C):**

$$\begin{aligned}(2(2(2(2(2(2+1)+1)+1)+1)+1)+1)\\ &= (2(2(2(2(2(3)+1)+1)+1)+1)+1)\\ &= (2(2(2(2(7)+1)+1)+1)+1)\\ &= (2(2(2(15)+1)+1)+1)\\ &= (2(2(31)+1)+1)\\ &= (2(63)+1)\\ &= 127\end{aligned}$$

   Observe that each intermediate result is 1 less than a power of 2.

2. **Answer (D):** The cheapest popsicles cost $\$3.00 \div 5 = \$0.60$ each. Because $14 \cdot \$0.60 = \$8.40$ and Pablo has just \$8, he could not pay for 14 popsicles even if he were allowed to buy partial boxes. The best he can hope for is 13 popsicles, and he can achieve that by buying two 5-popsicle boxes (for \$6) and one 3-popsicle box (for \$2).

   OR

   If Pablo buys two single popsicles for \$1 each, he could have bought a 3-popsicle box for the same amount of money. Similarly, if Pablo buys three single popsicles or both one 3-popsicle box and one single popsicle, he could have bought a 5-popsicle box for the same amount of money. If Pablo buys two 3-popsicle boxes, he could have bought a 5-popsicle box and a single popsicle for the same amount of money. The previous statements imply that a maximum number of popsicles for a given amount of money can be obtained by buying either at most one single popsicle and the rest 5-popsicle boxes, or a single 3-popsicle box and the rest 5-popsicle boxes. When Pablo has \$8, he can obtain the maximum number of popsicles by buying two 5-popsicle boxes and one 3-popsicle box. This gives a total of $2 \cdot 5 + 1 \cdot 3 = 13$ popsicles.

3. **Answer (B):** The area of the garden is $15 \cdot 10 = 150$ square feet, and the combined area of the six flower beds is $6 \cdot 6 \cdot 2 = 72$ square feet. Therefore the area of the walkways is $150 - 72 = 78$ square feet.

4. **Answer (B):** After exactly half a minute there will be 3 toys in the box and 27 toys outside the box. During the next half-minute, Mia takes 2 toys out and her mom puts 3 toys into the box. This

means that during this half-minute the number of toys in the box was increased by 1. The same argument applies to each of the following half-minutes until all the toys are in the box for the first time. Therefore it takes $1 + 27 \cdot 1 = 28$ half-minutes, which is 14 minutes, to complete the task.

5. **Answer (C):** Let the two numbers be x and y. Then $x + y = 4xy$. Dividing this equation by xy gives $\frac{1}{y} + \frac{1}{x} = 4$. One such pair of numbers is $x = \frac{1}{3}$, $y = 1$.

6. **Answer (B):** The given statement is logically equivalent to its contrapositive: If a student did not receive an A on the exam, then the student did not get all the multiple choice questions right, which means that he got at least one of them wrong. None of the other statements follows logically from the given implication; the teacher made no promises concerning students who did not get all the multiple choice questions right. In particular, a statement does not imply its inverse or its converse; and the negation of the statement that Lewis got all the questions right is not the statement that he got all the questions wrong.

7. **Answer (A):** If the square had side length x, then Jerry's path had length $2x$, and Silvia's path along the diagonal, by the Pythagorean Theorem, had length $\sqrt{2}x$. Therefore Silvia's trip was shorter by $2x - \sqrt{2}x$, and the required percentage is
$$\frac{2x - \sqrt{2}x}{2x} = 1 - \frac{\sqrt{2}}{2} \approx 1 - 0.707 = 0.293 = 29.3\%.$$
The closest of the answer choices is 30%.

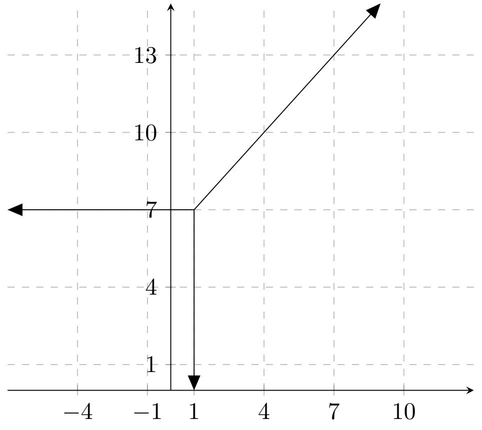
8. **Answer (B):** Each of the 20 people who know each other shakes hands with 10 people. Each of the 10 people who know no one shakes hands with 29 people. Because each handshake involves two people, the number of handshakes is $\frac{1}{2}(20 \cdot 10 + 10 \cdot 29) = 245$.

9. **Answer (C):** Note that Penny is going downhill on the segment on which Minnie is going uphill, and vice versa. Minnie needs $\frac{10}{5}$ hours to go from A to B, $\frac{15}{30}$ hours to go from B to C, and $\frac{20}{20}$ hours to go from C to A, a total of $3\frac{1}{2}$ hours. Penny's time is $\frac{20}{30} + \frac{15}{10} + \frac{10}{40} = 2\frac{5}{12}$ hours. It takes Minnie $3\frac{1}{2} - 2\frac{5}{12} = 1\frac{1}{12}$ hours, which is 65 minutes, longer.

10. **Answer (B):** Four rods can form a quadrilateral with positive area if and only if the length of the longest rod is less than the sum of the lengths of the other three. Therefore if the fourth rod has length n cm, then n must satisfy the inequalities $15 < 3+7+n$ and $n < 3+7+15$, that is, $5 < n < 25$. Because n is an integer, it must be one of the 19 integers from 6 to 24, inclusive. However, the rods of lengths 7 cm and 15 cm have already been chosen, so the number of rods that Joy can choose is $19 - 2 = 17$.

11. **Answer (D):** Let $h = AB$. The region consists of a solid circular cylinder of radius 3 and height h, together with two solid hemispheres of radius 3 centered at A and B. The volume of the cylinder is $\pi \cdot 3^2 \cdot h = 9\pi h$, and the two hemispheres have a combined volume of $\frac{4}{3}\pi \cdot 3^3 = 36\pi$. Therefore $9\pi h + 36\pi = 216\pi$, and $h = 20$.

12. **Answer (E):** Suppose that the two larger quantities are the first and the second. Then $3 = x + 2 \geq y - 4$. This is equivalent to $x = 1$ and $y \leq 7$, and its graph is the downward-pointing ray with endpoint $(1, 7)$. Similarly, if the two larger quantities are the first and third, then $3 = y - 4 \geq x + 2$. This is equivalent to $y = 7$ and $x \leq 1$, and its graph is the leftward-pointing ray with endpoint $(1, 7)$. Finally, if the two larger quantities are the second and third, then $x+2 = y-4 \geq 3$. This is equivalent to $y = x + 6$ and $x \geq 1$, and its graph is the ray with endpoint $(1, 7)$ that points upward and to the right. Thus the graph consists of three rays with common endpoint $(1, 7)$.



Note: This problem is related to a relatively new area of mathematics called tropical geometry.

13. **Answer (D):** The sequence starts $0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, \ldots$. Notice that the pattern repeats and the period is 8. Thus no matter which 8 consecutive numbers are added, the answer will be $0 + 1 + 1 + 2 + 0 + 2 + 2 + 1 = 9$.

14. **Answer (D):** Let M be the cost of Roger's movie ticket, and let S be the cost of Roger's soda. Then $M = 0.20(A - S)$ and $S = 0.05(A - M)$. Thus $5M + S = A$ and $M + 20S = A$. Solving the system for M and S in terms of A gives $M = \frac{19}{99}A$ and $S = \frac{4}{99}A$. The total cost of the movie ticket and soda as a fraction of A is $\frac{23}{99} = 0.2323\ldots \approx 23\%$.

15. **Answer (C):** Half of the time Laurent will pick a number between 2017 and 4034, in which case the probability that his number will be greater than Chloé's number is 1. The other half of the time, he will pick a number between 0 and 2017, and by symmetry his number will be the larger one in half of those cases. Therefore the requested probability is $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$.

OR

The choices of numbers can be represented in the coordinate plane by points in the rectangle with vertices at $(0, 0)$, $(2017, 0)$, $(2017, 4034)$, and $(0, 4034)$. The portion of the rectangle representing the event that Laurent's number is greater than Chloé's number is the portion above the line segment with endpoints $(0, 0)$ and $(2017, 2017)$. This area is $\frac{3}{4}$ of the area of the entire rectangle, so the requested probability is $\frac{3}{4}$.

16. **Answer (B):** Horse k will again be at the starting point after t minutes if and only if k is a divisor of t. Let $I(t)$ be the number of integers k with $1 \le k \le 10$ that divide t. Then $I(1) = 1$, $I(2) = 2$, $I(3) = 2$, $I(4) = 3$, $I(5) = 2$, $I(6) = 4$, $I(7) = 2$, $I(8) = 4$, $I(9) = 3$, $I(10) = 4$, $I(11) = 1$, and $I(12) = 5$. Thus $T = 12$ and the requested sum of digits is $1 + 2 = 3$.

17. **Answer (D):** The ratio $\frac{PQ}{RS}$ has its greatest value when PQ is as large as possible and RS is as small as possible. Points P, Q, R, and S have coordinates among $(\pm 5, 0)$, $(\pm 4, \pm 3)$, $(\pm 4, \mp 3)$, $(\pm 3, \pm 4)$, $(\pm 3, \mp 4)$, and $(0, \pm 5)$. In order for the distance between two of these points to be irrational, the two points must not form a diameter, and they must not have the same x-coordinate or y-coordinate. If

$R = (a, b)$ and $S = (a', b')$, then $|a - a'| \geq 1$ and $|b - b'| \geq 1$. Because $(3, 4)$ and $(4, 3)$ achieve this, they are as close as two points can be, $\sqrt{2}$ units apart. If $P = (a, b)$ and $Q = (a', b')$, then PQ is maximized when the distance from (a', b') to $(-a, -b)$ is minimized. Because $|a + a'| \geq 1$ and $|b + b'| \geq 1$, the points $(3, -4)$ and $(-4, 3)$ are as far apart as possible, $\sqrt{98}$ units. Therefore the greatest possible ratio is $\frac{\sqrt{98}}{\sqrt{2}} = \sqrt{49} = 7$.

18. **Answer (D):** Let x be the probability that Amelia wins. Then $x = \frac{1}{3} + \left(1 - \frac{1}{3}\right)\left(1 - \frac{2}{5}\right)x$, because either Amelia wins on the first toss, or, if she and Blaine both get tails, then the chance of her winning from that point onward is also x. Solving this equation gives $x = \frac{5}{9}$. The requested difference is $9 - 5 = 4$.

OR

The probability that Amelia wins on the first toss is $\frac{1}{3}$, the probability that Amelia wins on the second toss is $\frac{2}{3} \cdot \frac{3}{5} \cdot \frac{1}{3}$, and so on. Therefore the probability that Amelia wins is

$$\begin{aligned}
\frac{1}{3} + \frac{2}{3} \cdot \frac{3}{5} \cdot \frac{1}{3} + \left(\frac{2}{3}\right)^2 \cdot \left(\frac{3}{5}\right)^2 \cdot \frac{1}{3} + \cdots &= \frac{1}{3} \cdot \left(1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \cdots\right) \\
&= \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{5}} \\
&= \frac{5}{9}.
\end{aligned}$$

OR

Let $n \geq 0$ be the greatest integer such that Amelia and Blaine each toss tails n times in a row. Then the game will end in the next round of tosses, either because Amelia tosses a head, which will occur with probability $\frac{1}{3}$, or because Amelia tosses a tail and Blaine tosses a head, which will occur with probability $\frac{2}{3} \cdot \frac{2}{5} = \frac{4}{15}$. The probability that it is Amelia who wins is therefore

$$\frac{\frac{1}{3}}{\frac{1}{3} + \frac{4}{15}} = \frac{5}{9}.$$

19. **Answer (C):** Let X be the set of ways to seat the five people in which Alice sits next to Bob. Let Y be the set of ways to seat the

five people in which Alice sits next to Carla. Let Z be the set of ways to seat the five people in which Derek sits next to Eric. The required answer is $5! - |X \cup Y \cup Z|$. The Inclusion–Exclusion Principle gives

$$|X \cup Y \cup Z| = \big(|X|+|Y|+|Z|\big) - \big(|X \cap Y|+|X \cap Z|+|Y \cap Z|\big) + |X \cap Y \cap Z|.$$

Viewing Alice and Bob as a unit in which either can sit on the other's left side shows that there are $2 \cdot 4! = 48$ elements of X. Similarly there are 48 elements of Y and 48 elements of Z. Viewing Alice, Bob, and Carla as a unit with Alice in the middle shows that $|X \cap Y| = 2 \cdot 3! = 12$. Viewing Alice and Bob as a unit and Derek and Eric as a unit shows that $|X \cap Z| = 2 \cdot 2 \cdot 3! = 24$. Similarly $|Y \cap Z| = 24$. Finally, there are $2 \cdot 2 \cdot 2! = 8$ elements of $X \cap Y \cap Z$. Therefore $|X \cup Y \cup Z| = (48+48+48) - (12+24+24) + 8 = 92$, and the answer is $120 - 92 = 28$.

OR

There are three cases based on where Alice is seated.

• If Alice takes the first or last chair, then Derek or Eric must be seated next to her, Bob or Carla must then take the middle chair, and either of the remaining two individuals can be seated in either of the other two chairs. This gives a total of $2^4 = 16$ arrangements.

• If Alice is seated in the second or fourth chair, then Derek and Eric will take the seats on her two sides, and this can be done in two ways. Bob and Carla can be seated in the two remaining chairs in two ways, which yields a total of $2^3 = 8$ arrangements.

• If Alice sits in the middle chair, then Derek and Eric will be seated on her two sides, with Bob and Carla seated in the first and last chairs. This results in $2^2 = 4$ arrangements.

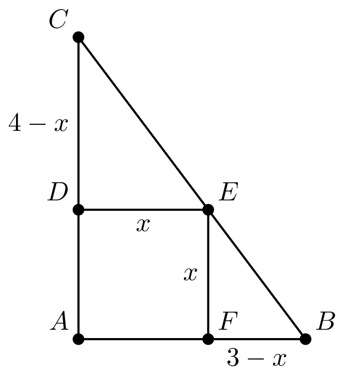
Thus there are $16 + 8 + 4 = 28$ possible arrangements in total.

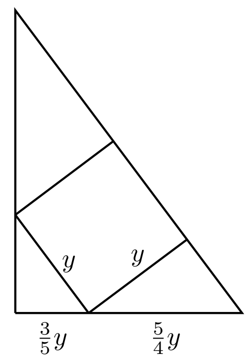
20. **Answer (D):** Note that $S(n+1) = S(n) + 1$ unless the numeral for n ends with a 9. Moreover, if the numeral for n ends with exactly k 9s, then $S(n+1) = S(n) + 1 - 9k$. Thus the possible values of $S(n+1)$ when $S(n) = 1274$ are all of the form $1275 - 9k$, where $k \in \{0, 1, 2, 3, \ldots, 141\}$. Of the choices, only 1239 can be formed in this manner, and $S(n+1)$ will equal 1239 if, for example, n consists of 4 consecutive 9s preceded by 1238 1s.

OR

The value of a positive integer is congruent to the sum of its digits modulo 9. Therefore $n \equiv S(n) = 1274 \equiv 5 \pmod 9$, so $S(n+1) \equiv n+1 \equiv 6 \pmod 9$. Of the given choices, only 1239 meets this requirement.

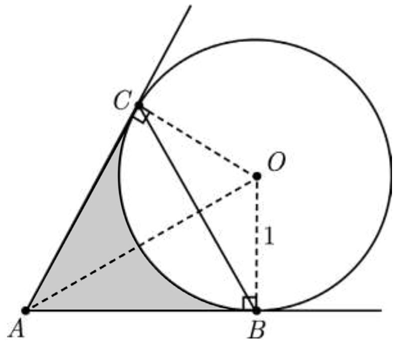
21. **Answer (D):** In the first figure $\triangle FEB \sim \triangle DCE$, so $\frac{x}{3-x} = \frac{4-x}{x}$ and $x = \frac{12}{7}$. In the second figure, the small triangles are similar to the large one, so the lengths of the portions of the side of length 3 are as shown. Solving $\frac{3}{5}y + \frac{5}{4}y = 3$ yields $y = \frac{60}{37}$. Thus $\frac{x}{y} = \frac{12}{7} \cdot \frac{37}{60} = \frac{37}{35}$.





22. **Answer (E):** Let O be the center of the circle, and without loss of generality, assume that radius $OB = 1$. Because $\triangle ABO$ is a $30-60-90^\circ$ right triangle, $AO = 2$ and $AB = BC = \sqrt{3}$. Kite $ABOC$ has diagonals of lengths 2 and $\sqrt{3}$, so its area is $\sqrt{3}$. Because $\angle BOC = 120^\circ$, the area of the sector cut off by $\angle BOC$ is $\frac{1}{3}\pi$. The area of the portion of $\triangle ABC$ lying outside the circle (shaded in the figure) is therefore $\sqrt{3} - \frac{1}{3}\pi$. The area of $\triangle ABC$ is $\frac{1}{4}\sqrt{3}\left(\sqrt{3}\right)^2 = \frac{3}{4}\sqrt{3}$, so the requested fraction is

$$\frac{\sqrt{3} - \frac{1}{3}\pi}{\frac{3}{4}\sqrt{3}} = \frac{4}{3} - \frac{4\sqrt{3}\pi}{27}.$$



23. **Answer (B):** There are $\binom{25}{3} = \frac{25\cdot 24\cdot 23}{6} = 2300$ ways to choose three vertices, but in some cases they will fall on a line. There are $5\cdot\binom{5}{3} = 50$ that fall on a horizontal line, another 50 that fall on a vertical line, $\binom{5}{3}+2\binom{4}{3}+2\binom{3}{3} = 20$ that fall on a line with slope 1, another 20 that fall on a line with slope -1, and 3 each that fall on lines with slopes 2, -2, $\frac{1}{2}$, and $-\frac{1}{2}$. Therefore the answer is $2300-50-50-20-20-12 = 2148$.

24. **Answer (C):** Let q be the additional root of $f(x)$. Then

$$\begin{aligned} f(x) &= (x-q)(x^3+ax^2+x+10) \\ &= x^4+(a-q)x^3+(1-qa)x^2+(10-q)x-10q. \end{aligned}$$

Thus $100 = 10-q$, so $q=-90$ and $c=-10q=900$. Also $1=a-q=a+90$, so $a=-89$. It follows, using the factored form of f shown above, that $f(1) = \big(1-(-90)\big)\cdot(1-89+1+10) = 91\cdot(-77) = -7007$.

25. **Answer (A):** Recall the divisibility test for 11: A three-digit number $\underline{a}\,\underline{b}\,\underline{c}$ is divisible by 11 if and only if $a-b+c$ is divisible by 11. The smallest and largest three-digit multiples of 11 are, respectively, $110 = 10\cdot 11$ and $990 = 90\cdot 11$, so the number of three-digit multiples of 11 is $90-10+1 = 81$. They may be grouped as follows:

- There are 9 multiples of 11 that have the form $\underline{a}\,\underline{a}\,\underline{0}$ for $1 \le a \le 9$. They can each be permuted to form a total of 2 three-digit integers. In each case $\underline{a}\,\underline{a}\,\underline{0}$ is a multiple of 11 and $\underline{a}\,\underline{0}\,\underline{a}$ is not, so these 9 multiples of 11 give 18 integers with the required property.
- There are 8 multiples of 11 that have the form $\underline{a}\,\underline{b}\,\underline{a}$, namely 121, 242, 363, 484, 616, 737, 858, and 979. They can each be permuted to form a total of 3 three-digit integers. In each case $\underline{a}\,\underline{b}\,\underline{a}$ is a multiple

of 11, but neither $\underline{a}\,\underline{a}\,\underline{b}$ nor $\underline{b}\,\underline{a}\,\underline{a}$ is, so these 8 multiples of 11 give 24 integers with the required property.

- If a three-digit multiple of 11 has distinct digits and one digit is 0, it must have the form $\underline{a}\,\underline{0}\,\underline{c}$ with $a+c=11$. There are 8 such integers, namely 209, 308, 407, ..., 902. They can each be permuted to form a total of 4 three-digit integers, but these 8 multiples of 11 give only 4 distinct sets of permutations, leading to $4 \cdot 4 = 16$ integers with the required property.

- The remaining $81-(9+8+8)=56$ three-digit multiples of 11 all have the form $\underline{a}\,\underline{b}\,\underline{c}$, where a, b, and c are distinct nonzero digits. They can each be permuted to form a total of 6 three-digit integers, and in each case both $\underline{a}\,\underline{b}\,\underline{c}$ and $\underline{c}\,\underline{b}\,\underline{a}$—and only these—are multiples of 11. Therefore these 56 multiples of 11 give only 28 distinct sets of permutations, leading to $28 \cdot 6 = 168$ integers with the required property.

The total number of integers with the required property is $18+24+16+168=226$.

Problems and solutions were contributed by Stephen Adams, Paul Baird-Smith, Steven Davis, Marta Eso, Silvia Fernandez, Peter Gilchrist, Jerrold Grossman, Jonathan Kane, Joe Kennedy, Michael Khoury, Steven Miller, Hugh Montgomery, Yasick Nemenov, Mark Saul, Walter Stromquist, Roger Waggoner, Dave Wells, and Carl Yerger.

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