MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

15th Annual

AMC 10 A

American Mathematics Contest 10 A

Tuesday, February 4, 2014

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom

1. **Answer (C):** Note that

$$10 \cdot \left(\frac{1}{2} + \frac{1}{5} + \frac{1}{10}\right)^{-1} = 10 \cdot \left(\frac{8}{10}\right)^{-1} = \frac{25}{2}.$$

2. **Answer (C):** Because Roy's cat eats $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ of a can of cat food each day, the cat eats 7 cans of cat food in 12 days. Therefore the cat eats $7 - \frac{7}{12} = 6\frac{5}{12}$ cans in 11 days and $6\frac{5}{12} - \frac{7}{12} = 5\frac{5}{6}$ cans in 10 days. The cat finishes the cat food in the box on the 11th day, which is Thursday.

3. **Answer (E):** In the morning, Bridget sells half of her loaves of bread for $\frac{1}{2} \cdot 48 \cdot \$2.50 = \$60$. In the afternoon, she sells $\frac{2}{3} \cdot 24 = 16$ loaves of bread for $16 \cdot \frac{1}{2} \cdot \$2.50 = \$20$. Finally, she sells the remaining 8 loaves of bread for \$8. Her total cost is $48 \cdot \$0.75 = \$36$. Her profit is $60 + 20 + 8 - 36 = 52$ dollars.

4. **Answer (B):** If Ralph passed the orange house first, then because the blue and yellow houses are not neighbors, the house color ordering must be orange, blue, red, yellow. If Ralph passed the blue house first, then there are 2 possible placements for the yellow house, and each choice determines the placement of the orange and red houses. These 2 house color orderings are blue, orange, yellow, red, and blue, orange, red, yellow. There are 3 possible orderings for the colored houses.

5. **Answer (C):** Because over 50% of the students scored 90 or lower, and over 50% of the students scored 90 or higher, the median score is 90. The mean score is

$$\frac{10}{100} \cdot 70 + \frac{35}{100} \cdot 80 + \frac{30}{100} \cdot 90 + \frac{25}{100} \cdot 100 = 87,$$

for a difference of $90 - 87 = 3$.

6. **Answer (A):** One cow gives $\frac{b}{a}$ gallons in c days, so one cow gives $\frac{b}{ac}$ gallons in 1 day. Thus d cows will give $\frac{bd}{ac}$ gallons in 1 day. In e days d cows will give $\frac{bde}{ac}$ gallons of milk.

7. **Answer (B):** Note that $x + y < a + y < a + b$, so inequality I is true. If $x = -2$, $y = -2$, $a = -1$, and $b = -1$, then none of the other three inequalities is true.

8. **Answer (D):** Note that $\frac{n!(n+1)!}{2} = (n!)^2 \cdot \frac{n+1}{2}$, which is a perfect square if and only if $\frac{n+1}{2}$ is a perfect square. Only choice D satisfies this condition.

9. **Answer (C):** The area of the triangle is $\frac{1}{2} \cdot 2\sqrt{3} \cdot 6 = 6\sqrt{3}$. By the Pythagorean Theorem, the hypotenuse has length $4\sqrt{3}$. The desired altitude has length $\frac{6\sqrt{3}}{\frac{1}{2} \cdot 4\sqrt{3}} = 3$.

10. **Answer (B):** The five consecutive integers starting with a are a, $a+1$, $a+2$, $a+3$, and $a+4$. Their average is $a+2=b$. The average of five consecutive integers starting with b is $b+2=a+4$.

11. **Answer (C):** Let $P > 100$ be the listed price. Then the price reductions in dollars are as follows:

    Coupon 1: $\frac{P}{10}$
    Coupon 2: 20
    Coupon 3: $\frac{18}{100}(P-100)$

    Coupon 1 gives a greater price reduction than coupon 2 when $\frac{P}{10} > 20$, that is, $P > 200$. Coupon 1 gives a greater price reduction than coupon 3 when $\frac{P}{10} > \frac{18}{100}(P-100)$, that is, $P < 225$. The only choice that satisfies these inequalities is \$219.95.

12. **Answer (C):** Each of the 6 sectors has radius 3 and central angle 120°. Their combined area is $6 \cdot \frac{1}{3} \cdot \pi \cdot 3^2 = 18\pi$. The hexagon can be partitioned into 6 equilateral triangles each having side length 6, so the hexagon has area $6 \cdot \frac{\sqrt{3}}{4} \cdot 6^2 = 54\sqrt{3}$. The shaded region has area $54\sqrt{3} - 18\pi$.

13. **Answer (C):** The three squares each have area 1, and $\triangle ABC$ has area $\frac{\sqrt{3}}{4}$. Note that $\angle EAF = 360^\circ - 60^\circ - 2 \cdot 90^\circ = 120^\circ$. Thus the altitude from A in isosceles $\triangle EAF$ partitions the triangle into two $30-60-90^\circ$ right triangles, each with hypotenuse 1. It follows that $\triangle EAF$ has base $EF = \sqrt{3}$ and altitude $\frac{1}{2}$, so its area is $\frac{\sqrt{3}}{4}$. Similarly, triangles GCH and DBI each have area $\frac{\sqrt{3}}{4}$. Therefore the area of hexagon $DEFGHI$ is $3 \cdot \frac{\sqrt{3}}{4} + 3 \cdot 1 + \frac{\sqrt{3}}{4} = 3 + \sqrt{3}$.

14. **Answer (D):** Let the y-intercepts of lines PA and QA be $\pm b$. Then their slopes are $\frac{8 \pm b}{6}$. Setting the product of the slopes to -1 and solving yields $b = \pm 10$. Therefore $\triangle APQ$ has base 20 and altitude 6, for an area of 60.

15. **Answer (C):** Let d be the remaining distance after one hour of driving, and let t be the remaining time until his flight. Then $d = 35(t+1)$, and $d = 50(t-0.5)$. Solving gives $t = 4$ and $d = 175$. The total distance from home to the airport is $175 + 35 = 210$ miles.

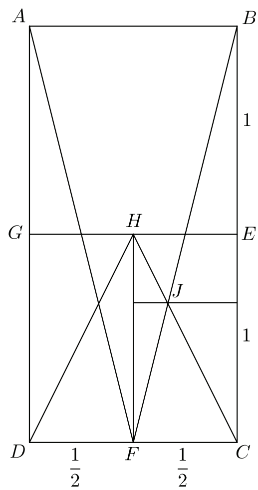
OR

Let d be the distance between David's home and the airport. The time required to drive the entire distance at 35 MPH is $\frac{d}{35}$ hours. The time required to drive at 35 MPH for the first 35 miles and 50 MPH for the remaining $d - 35$ miles is $1 + \frac{d-35}{50}$. The second trip is 1.5 hours quicker than the first, so

$$\frac{d}{35} - \left(1 + \frac{d-35}{50}\right) = 1.5.$$

Solving yields $d = 210$ miles.

16. **Answer (E):** Let J be the intersection point of $\overline{BF}$ and $\overline{HC}$. Then $\triangle JHF$ is similar to $\triangle JCB$ with ratio $1 : 2$. The length of the altitude of $\triangle JHF$ to $\overline{HF}$ plus the length of the altitude of $\triangle JCB$ to $\overline{CB}$ is $FC = \frac{1}{2}$. Thus $\triangle JHF$ has altitude $\frac{1}{6}$ and base 1, and its area is $\frac{1}{12}$. The shaded area is twice the area of $\triangle JHF$, or $\frac{1}{6}$.



OR

Place the figure on the coordinate plane with H at the origin. Then the equation of line DH is $y = 2x$, and the equation of line AF is $y = -4x - 1$. Solving the equations simultaneously shows that the leftmost point of the shaded region has x-coordinate $-\frac{1}{6}$. The kite therefore has diagonals $\frac{1}{3}$ and 1, so its area is $\frac{1}{2} \cdot \frac{1}{3} \cdot 1 = \frac{1}{6}$.

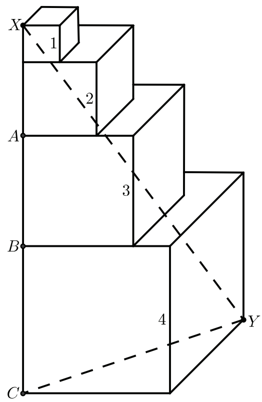
17. **Answer (D):** Each roll of the three dice can be recorded as an ordered triple (a, b, c) of the three values appearing on the dice. There are 6^3 equally likely triples possible. For the sum of two of the values in the triple to equal the third value, the triple must be a permutation of one of the triples $(1, 1, 2)$, $(1, 2, 3)$, $(1, 3, 4)$, $(1, 4, 5)$, $(1, 5, 6)$, $(2, 2, 4)$, $(2, 3, 5)$, $(2, 4, 6)$, or $(3, 3, 6)$. There are $3! = 6$ permutations of the values (a, b, c) when a, b, and c are distinct, and 3 permutations of the values when two of the values are equal. Thus there are $6 \cdot 6 + 3 \cdot 3 = 45$ triples where the sum of two of the values equals the third. The requested probability is $\frac{45}{6^3} = \frac{5}{24}$.

OR

There are 36 outcomes when a pair of dice are rolled, and the probability of rolling a total of $2, 3, 4, 5$, or 6 is $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}$, and $\frac{5}{36}$, respectively. The probability that another die matches this total is $\frac{1}{6}$, and there are 3 ways to choose the die that matches the total of the other two. Thus the requested probability is $3(\frac{1}{36} \cdot \frac{1}{6} + \frac{2}{36} \cdot \frac{1}{6} + \frac{3}{36} \cdot \frac{1}{6} + \frac{4}{36} \cdot \frac{1}{6} + \frac{5}{36} \cdot \frac{1}{6}) = 3 \cdot \frac{15}{36} \cdot \frac{1}{6} = \frac{5}{24}$.

18. **Answer (B):** Let the square have vertices A, B, C, D in counterclockwise order. Without loss of generality assume that $A = (0, 0)$ and $B = (x, 1)$ for some $x > 0$. Because D is the image of B under a $90°$ counterclockwise rotation about A, the coordinates of D are $(-1, x)$, so $x = 4$. Therefore the area of the square is $(AB)^2 = 4^2 + 1^2 = 17$. Note that $C = (3, 5)$, and $ABCD$ is indeed a square.

19. **Answer (A):** Label vertices A, B, and C as shown. Note that $XC = 10$ and $CY = \sqrt{4^2 + 4^2} = 4\sqrt{2}$. Because $\triangle XYC$ is a right triangle, $XY = \sqrt{10^2 + (4\sqrt{2})^2} = 2\sqrt{33}$. The ratio of BX to CX is $\frac{3}{5}$, so in the top face of the bottom cube the distance from B to $\overline{XY}$ is $4\sqrt{2} \cdot \frac{3}{5} = \frac{12\sqrt{2}}{5}$. This distance is less than $3\sqrt{2}$, so $\overline{XY}$ pierces the top and bottom faces of the cube with side length 3. The ratio of AB to XC is $\frac{3}{10}$, so the length of $\overline{XY}$ that is inside the cube with side length 3 is $\frac{3}{10} \cdot 2\sqrt{33} = \frac{3\sqrt{33}}{5}$.



OR

Place the figure in a 3-dimensional coordinate system with the lower left front corner at $(0,0,0)$, $X=(0,0,10)$, and $Y=(4,4,0)$. Then line XY consists of all points of the form $(4t,4t,10-10t)$. This line intersects the bottom face of the cube with side length 3 when $10-10t=4$, or $t=\frac{3}{5}$; this is the point $(\frac{12}{5},\frac{12}{5},4)$, and because $\frac{12}{5}<3$, the point indeed lies on that face. Similarly, line XY intersects the top face of the cube with side length 3 when $10-10t=7$, or $t=\frac{3}{10}$; this is the point $(\frac{6}{5},\frac{6}{5},7)$. Therefore the desired length is

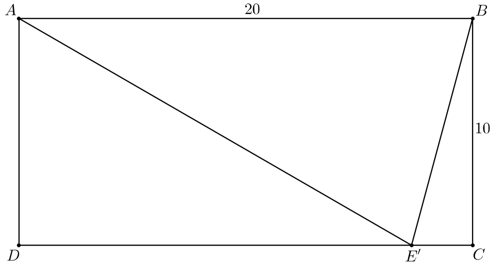
$$\sqrt{\left(\frac{12}{5}-\frac{6}{5}\right)^2+\left(\frac{12}{5}-\frac{6}{5}\right)^2+(4-7)^2}=\frac{3}{5}\sqrt{33}.$$

20. **Answer (D):** By direct multiplication, $8\cdot 888\ldots8=7111\ldots104$, where the product has 2 fewer ones than the number of digits in $888\ldots8$. Because $7+4=11$, the product must have $1000-11=989$ ones, so $k-2=989$ and $k=991$.

21. **Answer (E):** Setting $y=0$ in both equations and solving for x gives $x=-\frac{5}{a}=-\frac{b}{3}$, so $ab=15$. Only four pairs of positive integers (a,b) have product 15, namely $(1,15)$, $(15,1)$, $(3,5)$, and $(5,3)$. Therefore the four possible points on the x-axis have coordinates -5, $-\frac{1}{3}$, $-\frac{5}{3}$, and -1, the sum of which is -8.

22. **Answer (E):** Let E' be the point on $\overline{CD}$ such that $AE'=AB=2AD$. Then $\triangle ADE'$ is a $30-60-90^\circ$ triangle, so $\angle DAE'=60^\circ$. Hence $\angle BAE'=30^\circ$. Also,

$AE' = AB$ implies that $\angle E'BA = \angle BE'A = 75°$, and then $\angle CBE' = 15°$. Thus it follows that E' and E are the same point. Therefore, $AE = AE' = AB = 20$.



23. **Answer (C):** Without loss of generality, assume that the rectangle has dimensions 3 by $\sqrt{3}$. Then the fold has length 2, and the overlapping areas are equilateral triangles each with area $\frac{\sqrt{3}}{4} \cdot 2^2$. The new shape has area $3\sqrt{3} - \frac{\sqrt{3}}{4} \cdot 2^2 = 2\sqrt{3}$, and the desired ratio is $2\sqrt{3} : 3\sqrt{3} = 2 : 3$.

24. **Answer (A):** After the nth iteration there will be $4 + 5 + 6 + \cdots + (n+3) = \frac{(n+3)(n+4)}{2} - 6 = \frac{n(n+7)}{2}$ numbers listed, and $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ numbers skipped. The first number to be listed on the $(n+1)$st iteration will be one more than the sum of these, or $n^2 + 4n + 1$.

    It is necessary to find the greatest integer value of n such that $\frac{n(n+7)}{2} < 500{,}000$. This implies that $n(n+7) < 1{,}000{,}000$. Note that, for $n = 993$, this product becomes $993 \cdot 1000 = 993{,}000$. Next observe that, in general, $(a+k)(b+k) = ab + (a+b)k + k^2$ so $(993+k)(1000+k) = 993{,}000 + 1993k + k^2$. By inspection, the largest integer value of k that will satisfy the above inequality is 3 and the n needed is 996. After the 996th iteration, there will be $\frac{993{,}000+1993\cdot 3+9}{2} = \frac{998{,}988}{2} = 499{,}494$ numbers in the sequence. The 997th iteration will begin with the number $996^2 + 4 \cdot 996 + 1 = 996 \cdot 1000 + 1 = 996{,}001$.

    The 506th number in the 997th iteration will be the 500,000th number in the sequence. This is $996{,}001 + 505 = 996{,}506$.

25. **Answer (B):** Because $2^2 < 5$ and $2^3 > 5$, there are either two or three integer powers of 2 strictly between any two consecutive integer powers of 5. Thus for

each n there is at most one m satisfying the given inequalities, and the question asks for the number of cases in which there are three powers rather than two. Let d (respectively, t) be the number of nonnegative integers n less than 867 such that there are exactly two (respectively, three) powers of 2 strictly between 5^n and 5^{n+1}. Because $2^{2013} < 5^{867} < 2^{2014}$, it follows that $d + t = 867$ and $2d + 3t = 2013$. Solving the system yields $t = 279$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Steve Blasberg, Tom Butts, Steven Davis, Peter Gilchrist, Jerry Grossman, Jon Kane, Joe Kennedy, Gerald Kraus, Roger Waggoner, Kevin Wang, David Wells, LeRoy Wenstrom, and Ronald Yannone.

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