MAA

MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

13th Annual

AMC 10 A

American Mathematics Contest 10 A

Tuesday, February 7, 2012

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom

1. **Answer (D):** Because 20 seconds is $\frac{1}{3}$ of a minute, Cagney can frost $5 \div \frac{1}{3} = 15$ cupcakes in five minutes. Because 30 seconds is $\frac{1}{2}$ of a minute, Lacey can frost $5 \div \frac{1}{2} = 10$ cupcakes in five minutes. Altogether they can frost $15 + 10 = 25$ cupcakes in five minutes.

2. **Answer (E):** The length of each rectangle is equal to the side length of the square. The width of each rectangle is half the side length of the square, so the rectangle's dimensions are 4 by 8.

3. **Answer (E):** The distance from -2 to -6 is $|(-6) - (-2)| = 4$ units. The distance from -6 to 5 is $|5 - (-6)| = 11$ units. Altogether the bug crawls $4 + 11 = 15$ units.

4. **Answer (C):** Ray AB is common to both angles, so the degree measure of $\angle CBD$ is either $24 + 20 = 44$ or $24 - 20 = 4$. The smallest possible degree measure is 4.

5. **Answer (B):** The number of female adult cats was 50, and 25 of those were accompanied by an average of 4 kittens each. Thus the total number of kittens was $25 \cdot 4 = 100$, and the total number of cats and kittens was $100 + 100 = 200$.

6. **Answer (D):** Let $x > 0$ be the first number, and let $y > 0$ be the second number. The first statement implies $xy = 9$. The second statement implies $\frac{1}{x} = \frac{4}{y}$, so $y = 4x$. Substitution yields $x \cdot (4x) = 9$, so $x = \sqrt{\frac{9}{4}} = \frac{3}{2}$. Therefore $x + y = \frac{3}{2} + 4 \cdot \frac{3}{2} = \frac{15}{2}$.

7. **Answer (C):** The ratio of blue marbles to red marbles is $3 : 2$. If the number of red marbles is doubled, the ratio will be $3 : 4$, and the fraction of marbles that are red will be $\frac{4}{3+4} = \frac{4}{7}$.

8. **Answer (D):** Let the three whole numbers be $a < b < c$. The set of sums of pairs of these numbers is $(a + b, a + c, b + c) = (12, 17, 19)$. Thus $2(a + b + c) = (a + b) + (a + c) + (b + c) = 12 + 17 + 19 = 48$, and $a + b + c = 24$. If follows that $(a, b, c) = (24 - 19, 24 - 17, 24 - 12) = (5, 7, 12)$. Therefore the middle number is 7.

9. **Answer (D):** The sum could be 7 only if the even die showed 2 and the odd showed 5, the even showed 4 and the odd showed 3, or the even showed 6 and the odd showed 1. Each of these events can occur in $2 \cdot 2 = 4$ ways. Hence there are 12 ways for a 7 to occur. There are $6 \cdot 6 = 36$ possible outcomes, so the probability that a 7 occurs is $\frac{12}{36} = \frac{1}{3}$.

10. **Answer (C):** Let a be the initial term and d the common difference for the arithmetic sequence. Then the sum of the degree measures of the central angles is

$$a + (a + d) + \cdots + (a + 11d) = 12a + 66d = 360,$$

so $2a + 11d = 60$. Letting $d = 4$ yields the smallest possible positive integer value for a, namely $a = 8$.

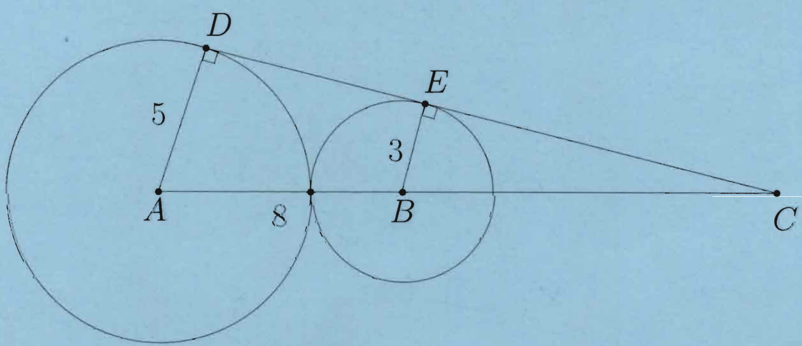
11. **Answer (D):** Let D and E be the points of tangency to circles A and B, respectively, of the common tangent line that intersects ray AB at point C. Then $AD = 5$, $BE = 3$, and $AB = 5 + 3 = 8$. Because right triangles ADC and BEC are similar, it follows that

$$\frac{BC}{AC} = \frac{BE}{AD},$$

so

$$\frac{BC}{BC + 8} = \frac{3}{5}.$$

Solving gives $BC = 12$.



12. **Answer (A):** There were $200 \cdot 365 = 73000$ non-leap days in the 200-year time period from February 7, 1812 to February 7, 2012. One fourth of those years contained a leap day, except for 1900, so there were $\frac{1}{4} \cdot 200 - 1 = 49$ leap days during that time. Therefore Dickens was born 73049 days before a Tuesday. Because the same day of the week occurs every 7 days and $73049 = 7 \cdot 10435 + 4$, the day of Dickens' birth (February 7, 1812) was 4 days before a Tuesday, which was a Friday.

13. **Answer (C):** If the numbers are arranged in the order a, b, c, d, e, then the iterative average is

$$\frac{\frac{\frac{\frac{a+b}{2}+c}{2}+d}{2}+e}{2} = \frac{a+b+2c+4d+8e}{16}.$$

The largest value is obtained by letting $(a,b,c,d,e) = (1,2,3,4,5)$ or $(2,1,3,4,5)$, and the smallest value is obtained by letting $(a,b,c,d,e) = (5,4,3,2,1)$ or $(4,5,3,2,1)$. In the former case the iterative average is $65/16$, and in the latter case the iterative average is $31/16$, so the desired difference is

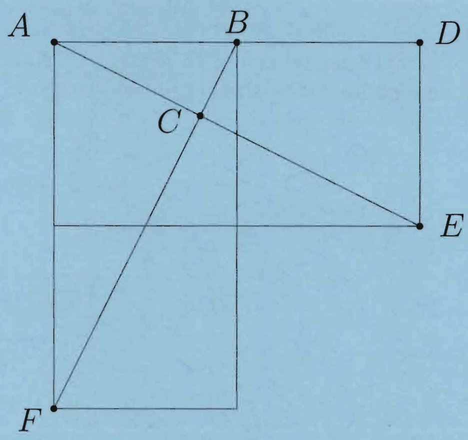
$$\frac{65}{16} - \frac{31}{16} = \frac{34}{16} = \frac{17}{8}.$$

14. **Answer (B):** Separate the modified checkerboard into two parts: the first 30 columns and the last column. The larger section consists of rows, each containing 15 black squares. The last column contains 16 black squares. Thus the total number of black squares is $31 \cdot 15 + 16 = 481$.

OR

There are 16 rows that have 16 black squares and 15 rows that have 15 black squares, so the total number of black squares is $16^2 + 15^2 = 481$.

15. **Answer (B):** Place the figure on the coordinate plane with A at the origin, B on the positive x-axis, and label the other points as shown. Then the equation of line AE is $y = -\frac{1}{2}x$, and the equation of line BF is $y = 2x - 2$. Solving the simultaneous equations shows that $C = (\frac{4}{5}, -\frac{2}{5})$. Therefore $\triangle ABC$ has base $AB = 1$ and altitude $\frac{2}{5}$, so its area is $\frac{1}{5}$.



OR

Congruent right triangles AED and FBA have the property that their corresponding legs are perpendicular; hence their hypotenuses are perpendicular. Therefore $\angle ACB$ is a right angle and $\triangle ACB$ is similar to $\triangle FAB$. Because $AB = 1$ and $BF = \sqrt{5}$, the ratio of the area of $\triangle ACB$ to that of $\triangle FAB$ is 1 to 5. The area of $\triangle FAB$ is 1, so the area of $\triangle ACB$ is $\frac{1}{5}$.

16. **Answer (C):** Label the runners A, B, and C in increasing order of speed. After the start, runner B and runner C will be together again once runner C has run an extra 500 meters. Hence it takes $\frac{500}{5.0-4.8} = 2500$ seconds for runners B and C to be together again. Similarly, it takes $\frac{500}{4.8-4.4} = 1250$ seconds for runner A and runner B to be together again. Runners A and B will also be together at $2 \cdot 1250 = 2500$ seconds, at which time all three runners will be together.

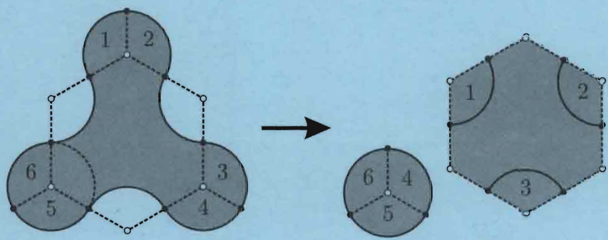
17. **Answer (C):** Note that

$$\frac{a^3 - b^3}{(a-b)^3} = \frac{a^2 + ab + b^2}{a^2 - 2ab + b^2}.$$

Hence the given equation may be written as $3a^2+3ab+3b^2 = 73a^2-146ab+73b^2$. Combining like terms and factoring gives $(10a-7b)(7a-10b) = 0$. Because $a > b$, and a and b are relatively prime, $a = 10$ and $b = 7$. Thus $a - b = 3$.

18. **Answer (E):** The labeled circular sectors in the figure each have the same area because they are all $\frac{2\pi}{3}$-sectors of a circle of radius 1. Therefore the area enclosed by the curve is equal to the area of a circle of radius 1 plus the area of a regular hexagon of side 2. Because the regular hexagon can be partitioned into 6 congruent equilateral triangles of side 2, it follows that the required area is

$$\pi + 6\left(\frac{\sqrt{3}}{4}\cdot 2^2\right) = \pi + 6\sqrt{3}\,.$$



19. **Answer (D):** Let the length of the lunch break be m minutes. Then the three painters each worked $480 - m$ minutes on Monday, the two helpers worked $372 - m$ minutes on Tuesday, and Paula worked $672 - m$ minutes on Wednesday. If Paula paints $p\%$ of the house per minute and her helpers paint a total of $h\%$ of the house per minute, then

$$\begin{aligned}(p+h)(480-m) &= 50,\\ h(372-m) &= 24, \text{ and}\\ p(672-m) &= 26.\end{aligned}$$

Adding the last two equations gives $672p + 372h - mp - mh = 50$, and subtracting this equation from the first one gives $108h - 192p = 0$, so $h = \frac{16p}{9}$. Substitution into the first equation then leads to the system

$$\begin{aligned}\frac{25p}{9}(480-m) &= 50,\\ p(672-m) &= 26.\end{aligned}$$

The solution of this system is $p = \frac{1}{24}$ and $m = 48$. Note that $h = \frac{2}{27}$.

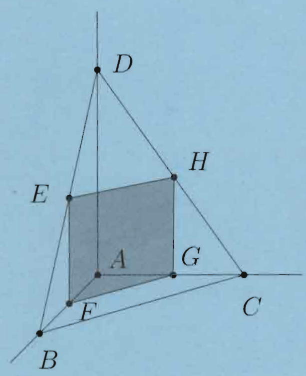
20. **Answer (A):** There are $2^4 = 16$ possible initial colorings for the four corner squares. If their initial coloring is $BBBB$, one of the four cyclic permutations of $BBBW$, or one of the two cyclic permutations of $BWBW$, then all four corner squares are black at the end. If the initial coloring is $WWWW$, one of the four cyclic permutations of $BWWW$, or one of the four cyclic permutations of

$BBWW$, then at least one corner square is white at the end. Hence all four corner squares are black at the end with probability $\frac{7}{16}$. Similarly, all four edge squares are black at the end with probability $\frac{7}{16}$. The center square is black at the end if and only if it was initially black, so it is black at the end with probability $\frac{1}{2}$. The probability that all nine squares are black at the end is $\frac{1}{2} \cdot \left(\frac{7}{16}\right)^2 = \frac{49}{512}$.

21. **Answer (C):** The midpoint formula gives $E = (\frac{1}{2}, 0, \frac{3}{2})$, $F = (\frac{1}{2}, 0, 0)$, $G = (0, 1, 0)$, and $H = (0, 1, \frac{3}{2})$. Note that $EF = GH = \frac{3}{2}$, $\overline{EF} \perp \overline{EH}$, $\overline{GF} \perp \overline{GH}$, and

$$EH = FG = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{5}}{2}.$$

Therefore $EFGH$ is a rectangle with area $\frac{3}{2} \cdot \frac{\sqrt{5}}{2} = \frac{3\sqrt{5}}{4}$.



22. **Answer (A):** The sum of the first k positive integers is $\frac{k(k+1)}{2}$. Therefore the sum of the first k even integers is

$$2 + 4 + 6 + \cdots + 2k = 2(1 + 2 + 3 + \cdots + k) = 2 \cdot \frac{k(k+1)}{2} = k(k+1).$$

The sum of the first k odd integers is

$$(1 + 2 + 3 + \cdots + 2k) - (2 + 4 + 6 + \cdots + 2k) = \frac{2k(2k+1)}{2} - k(k+1) = k^2.$$

The given conditions imply that $m^2 - 212 = n(n+1)$, which may be rewritten as $n^2 + n + (212 - m^2) = 0$. The discriminant for n in this quadratic equation is $1 - 4(212 - m^2) = 4m^2 - 847$, and this must be the square of an odd integer. Let $p^2 = 4m^2 - 847$, and rearrange this equation so that $(2m + p)(2m - p) = 847$.

The only factor pairs for 847 are $847 \cdot 1$, $121 \cdot 7$, and $77 \cdot 11$. Equating these pairs to $2m + p$ and $2m - p$ yields $(m, p) = (212, 423)$, $(32, 57)$, and $(22, 33)$. Note that the corresponding values of n are found using $n = \frac{-1+p}{2}$, which yields 211, 28, and 16, respectively. The sum of the possible values of n is 255.

23. **Answer (B):** This situation can be modeled with a graph having these six people as vertices, in which two vertices are joined by an edge if and only if the corresponding people are internet friends. Let n be the number of friends each person has; then $1 \le n \le 4$. If $n = 1$, then the graph consists of three edges sharing no endpoints. There are 5 choices for Adam's friend and then 3 ways to partition the remaining 4 people into 2 pairs of friends, for a total of $5 \cdot 3 = 15$ possibilities. The case $n = 4$ is complementary, with non-friendship playing the role of friendship, so there are 15 possibilities in that case as well.

    For $n = 2$, the graph must consist of cycles, and the only two choices are two triangles (3-cycles) and a hexagon (6-cycle). In the former case, there are $\binom{5}{2} = 10$ ways to choose two friends for Adam and that choice uniquely determines the triangles. In the latter case, every permutation of the six vertices determines a hexagon, but each hexagon is counted $6 \cdot 2 = 12$ times, because the hexagon can start at any vertex and be traversed in either direction. This gives $\frac{6!}{12} = 60$ hexagons, for a total of $10 + 60 = 70$ possibilities. The complementary case $n = 3$ provides 70 more. The total is therefore $15 + 15 + 70 + 70 = 170$.

24. **Answer (E):** Adding the two equations gives

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac = 14,$$

so

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 14.$$

Note that there is a unique way to express 14 as the sum of perfect squares (up to permutations), namely, $14 = 3^2 + 2^2 + 1^2$. Because $a - b$, $b - c$, and $c - a$ are integers with their sum equal to 0 and $a \ge b \ge c$, it follows that $a - c = 3$ and either $a - b = 2$ and $b - c = 1$, or $a - b = 1$ and $b - c = 2$. Therefore either $(a, b, c) = (c + 3, c + 1, c)$ or $(a, b, c) = (c + 3, c + 2, c)$. Substituting the relations in the first case into the first given equation yields $2011 = a^2 - c^2 + ab - b^2 = (a - c)(a + c) + (a - b)b = 3(2c + 3) + 2(c + 1)$. Solving gives $(a, b, c) = (253, 251, 250)$. The second case does not yield an integer solution. Therefore $a = 253$.

25. **Answer (D):** It may be assumed that $x \le y \le z$. Because there are six possible ways of permuting the triple (x, y, z), it follows that the set of all triples (x, y, z) with $0 \le x \le y \le z \le n$ is a region whose volume is $\frac{1}{6}$ of the volume of the cube $[0, n]^3$, that is $\frac{1}{6}n^3$. Let S be the set of triples meeting the required condition. For every $(x, y, z) \in S$ consider the translation $(x, y, z) \mapsto (x', y', z') = (x, y-1, z-2)$. Note that $y' = y - 1 > x = x'$ and $z' = z - 2 > y - 1 = y'$. Thus the image of S under this translation is equal to $\{(x', y', z') : 0 \le x' < y' < z' \le n - 2\}$. Again by symmetry of the possible permutations of the triples (x', y', z'), the volume of this set is $\frac{1}{6}(n-2)^3$. Because $\frac{7^3}{9^3} = \frac{343}{729} < \frac{1}{2}$ and $\frac{8^3}{10^3} = \frac{512}{1000} > \frac{1}{2}$, the smallest possible value of n is 10.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steve Dunbar, Michelle Ghrist, Peter Gilchrist, Jerrold Grossman, Dan Kennedy, Joe Kennedy, David Torney, David Wells, LeRoy Wenstrom, and Ron Yannone.

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