MAA AMC

American Mathematics Competitions

Official Solutions

MAA American Mathematics Competitions

39th Annual

AIME I

American Invitational Mathematics Examination I

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This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

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Problem 1:

Zou and Chou are practicing their 100-meter sprints by running 6 races against each other. Zou wins the first race, and after that, the probability that one of them wins a race is $\frac{2}{3}$ if they won the previous race but only $\frac{1}{3}$ if they lost the previous race. The probability that Zou will win exactly 5 of the 6 races is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Answer (097):

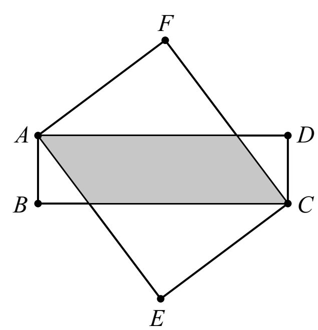
Zou will win exactly 5 out of 6 races if her record over the last 5 races is WWWWL, WWWLW, WWLWW, WLWWW, or LWWWW, where W represents a race won and L represents a race lost. In the first case, the sequence of winners changes once, and in the other four cases, the sequence of winners changes twice. Thus the probability that one of these sequences occurs is

$$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4 + 4\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = \frac{16+4\cdot 8}{3^5} = \frac{16}{81}.$$

The requested sum is $16+81=97$.

Problem 2:

In the diagram below, $ABCD$ is a rectangle with side lengths $AB=3$ and $BC=11$, and $AECF$ is a rectangle with side lengths $AF=7$ and $FC=9$. The area of the shaded region common to the interiors of both rectangles is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Solution:

Answer (109):

Let G be the intersection of $\overline{AD}$ and $\overline{CF}$. Then $\triangle AGF \sim \triangle CGD$, so

$$\frac{AG}{CG} = \frac{FG}{DG} = \frac{AF}{CD} = \frac{7}{3}.$$

It follows that there are constants x and y such that $AG=7x$, $CG=3x$, $FG=7y$, and $DG=3y$. Thus

$$7x+3y=11 \text{ and}$$
$$7y+3x=9.$$

Adding the two equations and dividing by 10 gives $x+y=2$. Subtracting the second equation from the first and dividing by 4 gives $x-y=\frac{1}{2}$. Hence $x=\frac{5}{4}$ and $y=\frac{3}{4}$. Because $\overline{AD} \parallel \overline{BC}$ and $\overline{AE} \parallel \overline{CF}$, the region interior to the two rectangles is a parallelogram, and thus the required area is $AG\cdot AB = 7x\cdot 3 = 7\cdot\frac{5}{4}\cdot 3 = \frac{105}{4}$. The requested sum is $105+4=109$.

OR

Defining G as above, let $t = DG$ so that $AG = 11 - t$ and, by the Pythagorean Theorem, $CG = \sqrt{3^2 + t^2}$. Because $\triangle AGF \sim \triangle CGD$, it follows that

$$\frac{AG}{AF} = \frac{CG}{CD} \text{ and } \frac{11-t}{7} = \frac{\sqrt{9+t^2}}{3}.$$

Solving for t gives $t = \frac{9}{4}$, from which the required area is $3 \cdot \left(11 - \frac{9}{4}\right) = \frac{105}{4}$, as above.

Problem 3:

Find the number of positive integers less than 1000 that can be expressed as the difference of two integral powers of 2.

Solution:

Answer (050):

Assume that positive integer N can be represented as $N = 2^x - 2^y$ for integers x and y. Because $N < 2^x$, x must be a positive integer which implies that y is nonnegative.

If $2^r - 2^s = 2^u - 2^v$ for nonnegative integers r, s, u, and v with $r > s$ and $u > v$, then the greatest power of 2 that divides the left side is 2^s, while the greatest power of 2 that divides the right side is 2^v. Hence $s = v$ and $r = u$. Therefore no positive integer can be represented as the difference of two integral powers of 2 in two distinct ways.

If $N = 2^x - 2^y$, then $N \geq 2^{x-1}$, so $x \leq 10$.

- If $1 \leq x \leq 9$, there are x choices for y, namely $y = 0, 1, \ldots, x-1$.
- If $x = 10$, there are 5 choices for y, namely $y = 5, 6, 7, 8, 9$.

Therefore there are $1 + 2 + \cdots + 9 + 5 = 50$ positive integers N that can be expressed as a difference of two integral powers of 2.

Problem 4:

Find the number of ways 66 identical coins can be separated into three nonempty piles so that there are fewer coins in the first pile than in the second pile and fewer coins in the second pile than in the third pile.

Solution:

Answer (331):

Assume that there are three distinct piles with x coins in the first, y coins in the second, and z coins in the third. The answer is the number of solutions to $x + y + z = 66$, where x, y, and z are integers satisfying $0 < x < y < z$.

Without the restriction $x < y < z$, the number of solutions in positive integers is the same as the number of solutions to $x + y + z = 63$ in nonnegative integers, which by the sticks-and-stones technique with 63 stones and 2 sticks is $\binom{63+2}{2} = 32 \cdot 65$.

The number of solutions where $x = y$ is the number of solutions in nonnegative integers of $2x + z = 63$. This equation has one solution for each odd number z from 0 to 63, which gives 32 solutions. Similarly, there are 32 solutions where $x = z$ and 32 solutions where $y = z$. In addition, there is 1 solution where $x = y = z$. Altogether there are $32 + 32 + 32 - 2 = 3 \cdot 32 - 2$ solutions where at least two of the variables are equal.

Therefore there are $32 \cdot 65 - 32 \cdot 3 + 2 = 32 \cdot 62 + 2$ solutions where all three variables assume distinct values. The number of unordered solutions is then

$$\frac{32 \cdot 62 + 2}{3!} = \frac{32 \cdot 60 + 66}{6} = 32 \cdot 10 + 11 = 331.$$

OR

Note that there are $\left\lfloor \frac{n-1}{2} \right\rfloor$ ways to place n coins into two piles with a different number of coins in each pile. Consider the size of the smallest pile among the three piles. If it has 1 coin in it, then removing a coin from each pile reduces the problem to the case of two piles with $n - 3$ coins.

If all three piles have at least two coins, then removing a coin from each of them reduces the problem to the case with $n-3$ coins and three piles. Thus if a_n is the number of ways to build three piles using n coins, then for $n \geq 4$,

$$a_n = a_{n-3} + \left\lfloor \frac{n-4}{2} \right\rfloor.$$

Applying the recursion twice gives for $n \geq 7$:

$$a_n = a_{n-6} + \left\lfloor \frac{n-7}{2} \right\rfloor + \left\lfloor \frac{n-4}{2} \right\rfloor.$$

This form has the advantage that the sum of the last two terms simplifies to $n-6$. Thus

$$a_n = a_{n-6} + (n-6).$$

Repeated application of this last recursion yields

$$\begin{aligned} a_{66} &= a_{60} + 60 = a_{54} + 54 + 60 = \cdots = a_6 + 6 + 12 + \cdots + 60 \\ &= 1 + 6(1 + 2 + \cdots + 10) = 1 + 3 \cdot 11 \cdot 10 = 331. \end{aligned}$$

OR

The result is the number of positive integer solutions to $x + y + z = 66$ with $0 < x < y < z$. By letting $a = x - 1$, $b = y - x - 1$, and $c = z - y - 1$, this is equivalent to counting solutions to $3a + 2b + c = 60$ with $a, b, c \geq 0$. Because $c = 60 - 3a - 2b$, the count equals the number of lattice points (a, b) in the triangle defined by $3a + 2b \leq 60$, $a \geq 0$, and $b \geq 0$.

This triangle is formed by taking the rectangle given by $0 \leq a \leq 20$ and $0 \leq b \leq 30$ and cutting it in half along the diagonal. The total number of lattice points in the rectangle is $31 \cdot 21$, while the number of points along the diagonal is $\gcd(20, 30) + 1 = 11$. Hence the total number of lattice points in this triangle is

$$\frac{31 \cdot 21 + 11}{2} = 331.$$

Problem 5:

Call a three-term strictly increasing arithmetic sequence of integers *special* if the sum of the squares of the three terms equals the product of the middle term and the square of the common difference. Find the sum of the third terms of all special sequences.

Solution:

Answer (031):

Let $a - d$, a, and $a + d$ be the three terms, with $d > 0$. The given condition is

$$(a-d)^2 + a^2 + (a+d)^2 = d^2 a, \quad \text{so} \quad 3a^2 - d^2 a + 2d^2 = 0.$$

Consider the equation as a quadratic in a and apply the quadratic formula to get

$$a = \frac{d^2 \pm d \cdot \sqrt{d^2 - 24}}{6}.$$

Because a is an integer, the discriminant must be a perfect square. Hence

$$d^2 - 24 = x^2, \quad \text{so} \quad 24 = (d - x)(d + x)$$

for some nonnegative integer x. Because $d - x$ and $d + x$ have the same parity, they are both even. There are two ways to factor 24 into two positive even integers. Thus either $d - x = 4$ and $d + x = 6$ or $d - x = 2$ and $d + x = 12$, implying that $d = 5$ or $d = 7$.

- For $d = 5$, the sequence is $0, 5, 10$.
- For $d = 7$, the sequence is $7, 14, 21$.

The requested sum is $10 + 21 = 31$.

OR

As in the previous solution, $3a^2 - d^2a + 2d^2 = 0$, from which $a - 2 = \frac{3a^2}{d^2}$.

Because $a - 2$ is an integer, $a = 3k^2 + 2$ for some positive integer k. Hence $d = \frac{3k^2+2}{k} = 3k + \frac{2}{k}$, which implies that $k = 1$ or $k = 2$. Checking these two cases yields the same two possible values for d as above.

Problem 6:

Segments $\overline{AB}$, $\overline{AC}$, and $\overline{AD}$ are edges of a cube and segment $\overline{AG}$ is a diagonal through the center of the cube. Point P satisfies $BP = 60\sqrt{10}$, $CP = 60\sqrt{5}$, $DP = 120\sqrt{2}$, and $GP = 36\sqrt{7}$. Find AP.

Solution:

Answer (192):

Let the cube have side length s and place the cube in Cartesian 3-space with vertices $A(0, 0, 0)$, $B(s, 0, 0)$, $C(0, s, 0)$, $D(0, 0, s)$, and $G(s, s, s)$. Let P have coordinates (x, y, z). Then

$$\begin{array}{ccccccccc} BP^2 & = & (x-s)^2 & + & y^2 & + & z^2 & = & \left(60\sqrt{10}\right)^2 \\ CP^2 & = & x^2 & + & (y-s)^2 & + & z^2 & = & \left(60\sqrt{5}\right)^2 \\ DP^2 & = & x^2 & + & y^2 & + & (z-s)^2 & = & \left(120\sqrt{2}\right)^2 \\ GP^2 & = & (x-s)^2 & + & (y-s)^2 & + & (z-s)^2 & = & \left(36\sqrt{7}\right)^2. \end{array}$$

Adding the first three equations and subtracting the fourth equation yields

$$\begin{aligned} 2\left(x^2 + y^2 + z^2\right) &= 12^2 \cdot \left[\left(5\sqrt{10}\right)^2 + \left(5\sqrt{5}\right)^2 + \left(10\sqrt{2}\right)^2 - \left(3\sqrt{7}\right)^2\right] \\ &= 12^2 \cdot (250 + 125 + 200 - 63) = 2 \cdot 12^2 \cdot 16^2 = 2 \cdot 192^2. \end{aligned}$$

Therefore $AP = \sqrt{x^2 + y^2 + z^2} = 192$.

Problem 7:

Find the number of pairs (m, n) of positive integers with $1 \le m < n \le 30$ such that there exists a real number x satisfying

$$\sin(mx) + \sin(nx) = 2.$$

Solution:

Answer (063):

Note that the maximum of $\sin(mx)$ is 1 and is achieved when $x = \frac{360^\circ k + 90^\circ}{m}$ for any integer k. If $\sin(mx) + \sin(nx) = 2$, then there exists a real number x such that $\sin(mx) = \sin(nx) = 1$. Thus $x = \frac{1}{m}(360^\circ k + 90^\circ) = \frac{1}{n}(360^\circ \ell + 90^\circ)$ for some integers k and ℓ. Hence

$$\frac{4k+1}{m} = \frac{4\ell+1}{n},$$

which is equivalent to $(4k+1) \cdot n = (4\ell+1) \cdot m$. Because $4k+1$ and $4\ell+1$ are odd, the greatest power of 2 dividing m must be equal to the greatest power of 2 dividing n. Let $m = 2^t \cdot m'$ and $n = 2^t \cdot n'$, where m' and n' are both odd. Then $(4k+1) \cdot n' = (4\ell+1) \cdot m'$ and $4 \mid (m' - n')$.

Conversely, if m' and n' are odd positive integers satisfying $m' \equiv n' \pmod 4$, then there exist positive integers k and ℓ such that the above equation holds:

- If m' and n' are congruent to 1 modulo 4, then setting $4k + 1 = m'$ and $4\ell + 1 = n'$ leads to integer values for k and ℓ.
- If m' and n' are congruent to 3 modulo 4, then setting $4k + 1 = 3m'$ and $4\ell + 1 = 3n'$ leads to integer values for k and ℓ.

Therefore the required integers k and ℓ exist if and only if m' and n' are both odd and $4 \mid (m' - n')$, which means that either $m', n' \in \{1, 5, 9, \ldots, 29\}$ or $m', n' \in \{3, 7, 11, \ldots, 27\}$. In the first case, m and n are distinct integers from $\{1, 5, 9, \ldots, 29\}$, from $\{2, 10, 18, 26\}$, or from $\{4, 20\}$. In the second case, m and n are distinct integers from $\{3, 7, 11, \ldots, 27\}$, from $\{6, 14, 22, 30\}$, or from $\{12, 28\}$. Hence there are

$$\binom{8}{2} + \binom{4}{2} + \binom{2}{2} + \binom{7}{2} + \binom{4}{2} + \binom{2}{2} = 63$$

ordered pairs with the required properties.

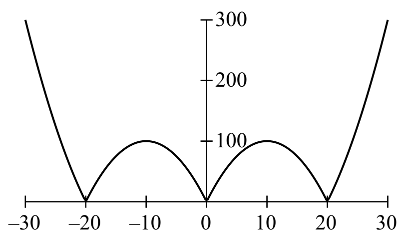
Problem 8:
Find the number of integers c such that the equation

$$\Big|\big|20|x| - x^2\big| - c\Big| = 21$$

has 12 distinct real solutions.

Solution:

Answer (057):
The equation $y = \big|20|x| - x^2\big|$ has the following graph.

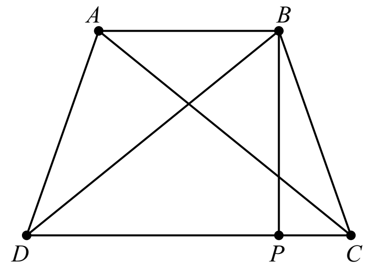


On the interval $[-20, 20]$, the graph reaches a maximum of 100 at $x = \pm 10$. The solutions to the equation $\Big|\big|20|x| - x^2\big| - c\Big| = 21$ are the x-coordinates of the points of intersection of this graph with the lines $y = c + 21$ and $y = c - 21$. Each of these lines intersects the graph exactly 6 times when y is in the range $0 < y < 100$, and no two of these intersections have the same x-coordinate. Thus in order for the given equation to have 12 real solutions, c must satisfy $21 < c < 79$. There are $78 - 21 = 57$ integers in this range.

Problem 9:
Let $ABCD$ be an isosceles trapezoid with $AD = BC$ and $AB < CD$. Suppose that the distances from A to the lines BC, CD, and BD are 15, 18, and 10, respectively. Let K be the area of $ABCD$. Find $\sqrt{2} \cdot K$.

Solution:

Answer (567):



By symmetry, $\triangle ABC$ is congruent to $\triangle BAD$. The areas of these two triangles can be calculated in three ways as

$$\frac{15 \cdot BC}{2} = \frac{18 \cdot AB}{2} = \frac{10 \cdot BD}{2}.$$

Thus there is a constant k such that $AB = 5k$, $AD = BC = 6k$, and $AC = BD = 9k$. Because $ABCD$ is an isosceles trapezoid, it is cyclic, so by Ptolemy's Theorem, $AC \cdot BD = AB \cdot CD + AD \cdot BC$. Thus $CD = 9k$. Let P be the foot of the perpendicular from B to $\overline{CD}$. Then $CP = \frac{CD-AB}{2} = 2k$. By the Pythagorean Theorem, $BC^2 = BP^2 + CP^2$, which implies that $(6k)^2 = 18^2 + (2k)^2$ and $k = \frac{9\sqrt{2}}{4}$. Therefore the area of $ABCD$ is

$$\frac{AB + CD}{2} \cdot BP = \frac{5k + 9k}{2} \cdot 18 = 7 \cdot 18k = 7 \cdot 18 \cdot \frac{9\sqrt{2}}{4} = \frac{567\sqrt{2}}{2}.$$

The requested product is $\sqrt{2} \cdot \frac{567\sqrt{2}}{2} = 567$.

Problem 10:

Consider the sequence $(a_k)_{k\geq 1}$ of positive rational numbers defined by $a_1 = \frac{2020}{2021}$ and for $k \geq 1$, if $a_k = \frac{m}{n}$ for relatively prime positive integers m and n, then

$$a_{k+1} = \frac{m + 18}{n + 19}.$$

Determine the sum of all positive integers j such that the rational number a_j can be written in the form $\frac{t}{t+1}$ for some positive integer t.

Solution:

Answer (059):

Note that all the terms in the sequence $(a_k)_{k\geq 1}$ are strictly between $\frac{18}{19}$ and 1. Call an integer j *simple* if the rational number a_j can be written in the form $\frac{t}{t+1}$ for some integer $t > 18$. Suppose j is a simple positive integer and term j of the sequence is $a_j = \frac{t}{t+1}$. Let $t = p_1 p_2 \cdots p_\ell + 18$ with $p_1 \leq p_2 \leq \cdots \leq p_\ell$ being the primes in the prime factorization of $t - 18$. Note that for any positive integer k, the greatest common divisor of $t + 18k$ and $t + 1 + 19k$ is

$$\gcd(t + 18k, t + 1 + 19k) = \gcd(t + 18k, k + 1) = \gcd(t - 18, k + 1).$$

Thus this greatest common divisor is first greater than 1 when $k = p_1 - 1$, in which case the greatest common divisor is equal to p_1. At that point,

$$a_{j+p_1-1} = \frac{t + 18(p_1 - 1)}{t + 1 + 19(p_1 - 1)} = \frac{p_1 p_2 \cdots p_\ell + 18p_1}{p_1 p_2 \cdots p_\ell + 19p_1} = \frac{p_2 p_3 \cdots p_\ell + 18}{p_2 p_3 \cdots p_\ell + 19},$$

so $j + (p_1 - 1)$ is the next integer greater than j that is simple. By the same reasoning, the numbers

$$j + (p_1 - 1) + (p_2 - 1), \ldots, j + (p_1 - 1) + (p_2 - 1) + (p_3 - 1) + \cdots + (p_\ell - 1)$$

are all the simple numbers exceeding $j + (p_1 - 1)$.

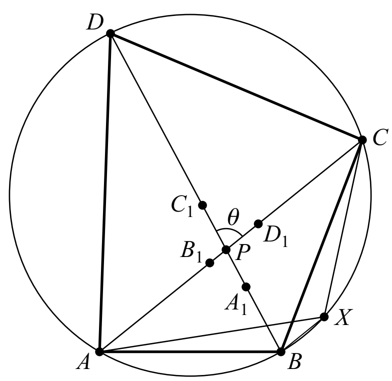
The first simple number is $j = 1$ for which $t = 2020 = 2002 + 18 = 2 \cdot 7 \cdot 11 \cdot 13 + 18$. Therefore the sequence of simple numbers is $1, 2, 8, 18$, and 30. The requested sum is

$$1 + 2 + 8 + 18 + 30 = 59.$$

Problem 11:
Let $ABCD$ be a cyclic quadrilateral with $AB = 4$, $BC = 5$, $CD = 6$, and $DA = 7$. Let A_1 and C_1 be the feet of the perpendiculars from A and C, respectively, to line BD, and let B_1 and D_1 be the feet of the perpendiculars from B and D, respectively, to line AC. The perimeter of $A_1B_1C_1D_1$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Answer (301):
Let P denote the intersection point of diagonals $\overline{AC}$ and $\overline{BD}$, and let θ be the acute angle formed by $\overline{AC}$ and $\overline{BD}$. Because $\angle DC_1C = \angle DD_1C = 90°$, it follows that CDC_1D_1 is cyclic, implying that $\angle PD_1C_1 = \angle PDC$ and $\angle PC_1D_1 = \angle PCD$.



It follows that $\triangle PD_1C_1 \sim \triangle PDC$, and so

$$\frac{C_1D_1}{CD} = \frac{PC_1}{PC} = \cos\theta.$$

Similarly,

$$\frac{A_1B_1}{AB} = \frac{B_1C_1}{BC} = \frac{D_1A_1}{DA} = \cos\theta.$$

Therefore

$$C_1D_1 + B_1C_1 + A_1B_1 + A_1D_1 = (CD + BC + AB + AD)\cos\theta = 22\cos\theta.$$

Let X be the reflection of B across the perpendicular bisector of diagonal $\overline{AC}$. Then $ABXC$ is an isosceles trapezoid, so A, B, X, C, and D lie on a circle. Because $\widehat{AB} = \widehat{XC}$,

$$\angle XAD = \frac{\widehat{XC} + \widehat{CD}}{2} = \frac{\widehat{AB} + \widehat{CD}}{2} = \angle APB = \theta.$$

Similarly, $\angle XCD = \angle APD = 180° - \theta$. Applying the Law of Cosines to $\triangle XCD$ and $\triangle XAD$ gives

$$XD^2 = 4^2 + 6^2 + 2 \cdot 4 \cdot 6 \cos\theta = 5^2 + 7^2 - 2 \cdot 5 \cdot 7 \cos\theta,$$

so $\cos\theta = \frac{11}{59}$. Therefore the perimeter of $A_1B_1C_1D_1$ is $22 \cdot \frac{11}{59} = \frac{242}{59}$. The requested sum is $242 + 59 = 301$.

OR

As in the first solution, the perimeter of $A_1B_1C_1D_1$ equals $22\cos\theta$.

Note that the area of $ABCD$ equals $\frac{AC\cdot BD}{2}\cdot\sin\theta$. On the other hand, by Brahmagupta's formula, area of cyclic quadrilateral $ABCD$ equals

$$\sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where a, b, c, d are side lengths and s is the semiperimeter. In this case,

$$\frac{AC\cdot BD}{2}\cdot\sin\theta=\sqrt{4\cdot 5\cdot 6\cdot 7}.$$

By Ptolemy's Theorem, $AC\cdot BD=4\cdot 6+5\cdot 7=59$. Hence

$$\sin\theta=\frac{2\cdot\sqrt{840}}{59},$$

from which $\cos\theta=\frac{11}{59}$ and the solution finishes as above.

Problem 12:

Let $A_1A_2A_3\ldots A_{12}$ be a dodecagon (12-gon). Three frogs initially sit at A_4, A_8, and A_{12}. At the end of each minute, simultaneously each of the three frogs jumps to one of the two vertices adjacent to its current position, chosen randomly and independently with both choices being equally likely. All three frogs stop jumping as soon as two frogs arrive at the same vertex at the same time. The expected number of minutes until the frogs stop jumping is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Answer (019):

Define the distance between two frogs as the number of sides between them that do not contain the third frog. At any moment before the frogs stop jumping, the three distances, in nondecreasing order, can only be the following triples: $(4,4,4)$, $(2,4,6)$, and $(2,2,8)$. Let A, B, and C be the expected number of minutes from each status to the stopping time, respectively. Then A, B, and C satisfy the following system of equations.

$$\begin{aligned}
A&=1+\frac{1}{4}A+\frac{3}{4}B\\
B&=1+\frac{1}{4}\cdot 0+\frac{1}{8}A+\frac{1}{2}B+\frac{1}{8}C\\
C&=1+\frac{1}{2}\cdot 0+\frac{1}{4}B+\frac{1}{4}C
\end{aligned}$$

Solving the system yields $A=\frac{16}{3}$, $B=4$, and $C=\frac{8}{3}$. Because the frogs start in configuration A, the requested sum is $16+3=19$.

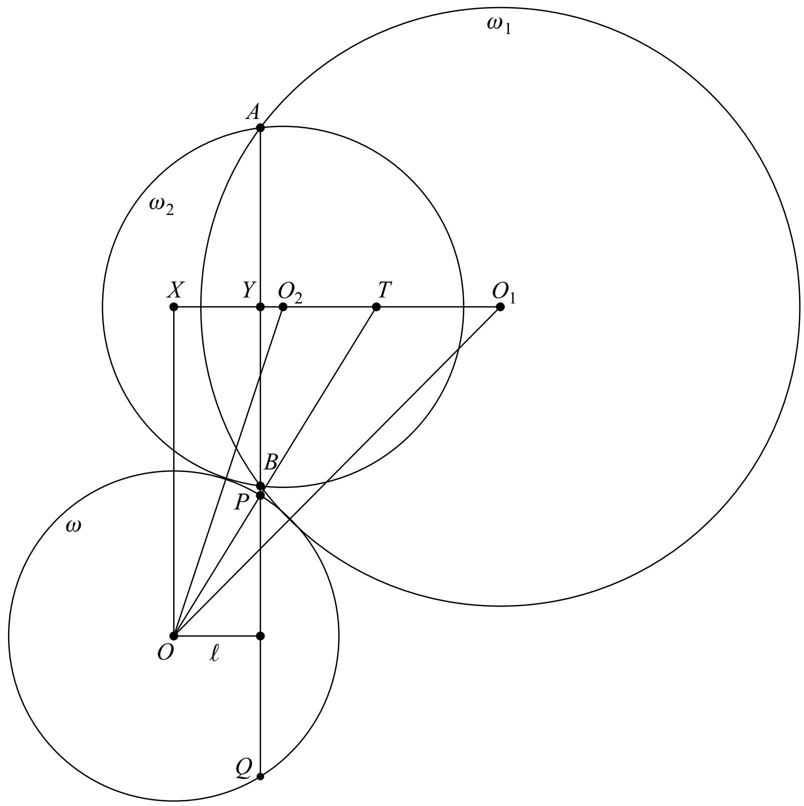
Problem 13:

Circles ω_1 and ω_2 with radii 961 and 625, respectively, intersect at distinct points A and B. A third circle ω is externally tangent to both ω_1 and ω_2. Suppose line AB intersects ω at two points P and Q such that the measure of minor arc $\overparen{PQ}$ is $120°$. Find the distance between the centers of ω_1 and ω_2.

Solution:

Answer (672):

Let $R_1=961$ and $R_2=625$ be the radii of ω_1 and ω_2, respectively, r be the radius of ω, and ℓ be the distance from the center O of ω to the line AB. Let O_1 and O_2 be the centers of ω_1 and ω_2, respectively. Let X be the projection of O onto line O_1O_2, and let Y be the intersection of $\overline{AB}$ with line O_1O_2.



Let the distance between O_1 and O_2 be d. Then $d = O_1Y - O_2Y$. Because $\overline{AB}$ is a chord in both ω_1 and ω_2, the power of point Y is the same with respect to both circles. Thus

$$R_1^2 - R_2^2 = O_1Y^2 - O_2Y^2 = d(O_1Y + O_2Y).$$

Furthermore, note that

$$\begin{aligned} d(O_1X + O_2X) &= O_1X^2 - O_2X^2 = O_1O^2 - O_2O^2 \\ &= (R_1 + r)^2 - (R_2 + r)^2 \\ &= \left(R_1^2 - R_2^2\right) + 2r(R_1 - R_2). \end{aligned}$$

Substituting the first equality into the second one and subtracting yields

$$2r(R_1 - R_2) = d(O_1X + O_2X) - d(O_1Y + O_2Y) = 2dXY = 2d\ell,$$

which shows that

$$\frac{R_1 - R_2}{d} = \frac{\ell}{r} = \cos(60^\circ) = \frac{1}{2}.$$

Therefore $d = 2(R_1 - R_2) = 2(961 - 625) = 672$.

Note: In the figure shown, it is assumed that the points X, Y, O_2, and O_1 occur in that order along the line containing the centers of ω_1 and ω_2. If the order were different, the same argument with appropriate sign changes would yield the same answer.

OR

Let O, O_1, O_2, R_1, R_2, and r be as in the first solution. Let line OP intersect line O_1O_2 at T, and let $u = TO_2$, $v = TO_1$ and $x = PT$. Because lines PQ and O_1O_2 are perpendicular, lines OT and O_1O_2 meet at a 60° angle.

Applying the Law of Cosines four times gives

$$\begin{aligned}
\triangle O_2PT : O_2P^2 &= u^2 + x^2 - ux \\
\triangle O_1PT : O_1P^2 &= v^2 + x^2 + vx \\
\triangle O_2OT : (r + R_2)^2 &= u^2 + (r + x)^2 - u(r + x) \\
\triangle O_1OT : (r + R_1)^2 &= v^2 + (r + x)^2 + v(r + x).
\end{aligned}$$

Adding the first and fourth equations, and subtracting the second and third equations gives

$$\left(O_2P^2 - O_1P^2\right) + \left(R_1^2 - R_2^2\right) + 2r(R_1 - R_2) = r(u + v).$$

Because point P is on the radical axis of ω_1 and ω_2, the power of point P with respect to either circle is

$$O_2P^2 - R_2^2 = O_1P^2 - R_1^2.$$

Hence $2r(R_1 - R_2) = r(u + v)$ which simplifies to

$$u + v = 2(R_1 - R_2).$$

The requested distance

$$O_1O_2 = O_1T + O_2T = u + v$$

is therefore equal to $2 \cdot (961 - 625) = 672$.

Problem 14:

For any positive integer a, $\sigma(a)$ denotes the sum of the positive integer divisors of a. Let n be the least positive integer such that $\sigma(a^n) - 1$ is divisible by 2021 for all positive integers a. Find the sum of the prime factors in the prime factorization of n.

Solution:

Answer (125):

If a has prime factorization $p_1^{\alpha_1} p_2^{\alpha_2} \cdots$, then $\sigma(a) = \sigma(p_1^{\alpha_1})\sigma(p_2^{\alpha_2}) \cdots$ and hence $\sigma(a^n) = \sigma(p_1^{n\alpha_1})\sigma(p_2^{n\alpha_2}) \cdots$. Therefore it suffices to find the least positive integer n such that $\sigma(p^{n\alpha}) \equiv 1 \pmod{2021}$ for all prime powers p^α. Because $2021 = 43 \cdot 47$, by the Chinese Remainder Theorem, it is sufficient that $\sigma(p^{n\alpha}) \equiv 1 \pmod{43}$ and $\sigma(p^{n\alpha}) \equiv 1 \pmod{47}$ for all prime powers p^α.

Assume that n satisfies the required condition. In particular, for all p and α, n must satisfy

$$\sigma(p^{n\alpha}) = 1 + p + p^2 + \cdots + p^{n\alpha} \equiv 1 \pmod{q},$$

where $q = 43$ or $q = 47$.

- If $p = q$, this sum will always be congruent to $1 \pmod{q}$.
- If $p \equiv 1 \pmod{q}$, then each term in the sum is $1 \pmod{q}$, so

  $$\sigma(p^{n\alpha}) \equiv n\alpha + 1 \equiv 1 \pmod{q}.$$

  Thus the required n must satisfy $q \mid n\alpha$ for all α, so $q \mid n$.

  Note that $43 \cdot 4 + 1 = 173$ and $47 \cdot 6 + 1 = 283$ are both prime numbers, so such p exist for both $q = 43$ and $q = 47$.
- If p is a prime such that $p \neq q$ and $\gcd(p - 1, q) = 1$, then

  $$1 + p + p^2 + \cdots + p^{n\alpha} = \frac{p^{n\alpha+1} - 1}{p - 1} \equiv 1 \pmod{q},$$

which, after clearing the denominators and canceling a factor of p, reduces to

$$p^{n\alpha} \equiv 1 \pmod{q}.$$

By Fermat's Little Theorem, it is sufficient to have $q - 1 \mid n$. However, for both $q = 43$ and $q = 47$ there exists a prime p such that p is a primitive root modulo q. For example, $p = 5$ is a primitive root modulo both 43 and 47. Therefore the condition that $q - 1 \mid n$ is also necessary.

It follows that n must be divisible by 42, 43, 46, and 47. The requested sum is $2 + 3 + 7 + 23 + 43 + 47 = 125$.

Problem 15:
Let S be the set of positive integers k such that the two parabolas

$$y = x^2 - k \qquad \text{and} \qquad x = 2(y - 20)^2 - k$$

intersect in four distinct points, and these four points lie on a circle with radius at most 21. Find the sum of the least element of S and the greatest element of S.

Solution:

Answer (285):
Note that $y = x^2 - k$ has its vertex at $(0, -k)$, which is below the line $y = 20$, and opens upwards. Parabola $x = 2(y - 20)^2 - k$ has its vertex at $(-k, 20)$ and opens to the right.

- If $0 < k \le 4$, then for $-k \le x \le 0$, the maximum value of y on the first parabola is $k^2 - k \le 12$. However, for $x \le 0$, the minimum value of y for the second parabola is $20 - \sqrt{\frac{k}{2}} > 18$. Thus if $0 < k \le 4$, the second parabola does not intersect the left half of the first parabola.
- If $k > 5$, then at $x = -k$ the first parabola has y value $(-k)^2 - k > 20$, and hence the vertex of the second parabola lies to the left of the first parabola. The lower half of the second parabola intersects the y-axis at $20 - \sqrt{\frac{k}{2}}$, which is above the vertex of the first parabola. Hence if $k > 5$, the two parabolas intersect at four points.
- If $k = 5$, then the first parabola passes through the vertex $(-5, 20)$ of the second parabola. If $x = -5 + \epsilon$, then the y-coordinate of the first parabola is $20 - 10\epsilon + \epsilon^2$, while the y-coordinate of the lower half of the second parabola is $20 - \sqrt{\frac{\epsilon}{2}}$. Because $\sqrt{\frac{\epsilon}{2}} > 10\epsilon - \epsilon^2$ for small positive values of ϵ, the lower half of the second parabola lies below and to the left of the left half of the first parabola. Similarly to the previous case, the lower half of the second parabola intersects the y-axis above the vertex of the first parabola. Thus for $k = 5$, the two parabolas intersect at four points.

Adding the first equation given in the problem to half of the second equation yields

$$y + \frac{x}{2} = (y - 20)^2 + x^2 - \frac{3}{2}k,$$

which, upon completing the square, gives

$$\left(y - \frac{41}{2}\right)^2 + \left(x - \frac{1}{4}\right)^2 = \frac{325}{16} + \frac{3}{2}k.$$

All four intersection points satisfy this equation, which is an equation of a circle. Hence as long as the two parabolas intersect in four distinct points, these four points are concyclic. Moreover, the square of the radius of this circle is $\frac{325}{16} + \frac{3}{2}k$.
Thus the desired condition is that

$$\frac{325}{16} + \frac{3}{2}k \le 441,$$

which holds when $k \le 280$. The requested sum is $5 + 280 = 285$.

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