MAA AMC

American Mathematics Competitions

Official Solutions

MAA American Mathematics Competitions

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AOIME

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This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

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1. Find the number of ordered pairs of positive integers (m, n) such that $m^2n = 20^{20}$.

Answer (231):

Because $20^{20} = 2^{40}5^{20}$, if $m^2n = 20^{20}$, there must be nonnegative integers a, b, c, and d such that $m = 2^a5^b$ and $n = 2^c5^d$. Then

$$2a + c = 40 \quad \text{and}$$
$$2b + d = 20.$$

The first equation has 21 solutions corresponding to $a = 0, 1, 2, \ldots, 20$, and the second equation has 11 solutions corresponding to $b = 0, 1, 2, \ldots, 10$. Therefore there are a total of $21 \cdot 11 = 231$ ordered pairs (m, n) such that $m^2n = 20^{20}$.

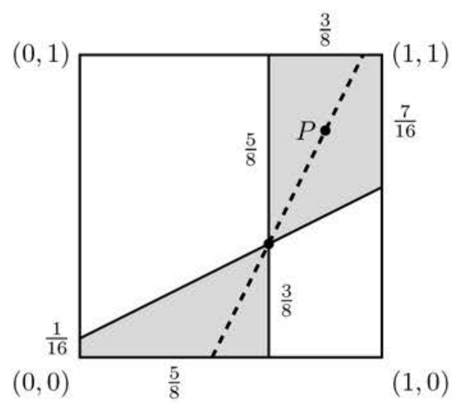
2. Let P be a point chosen uniformly at random in the interior of the unit square with vertices at $(0,0), (1,0), (1,1)$, and $(0,1)$. The probability that the slope of the line determined by P and the point $\left(\frac{5}{8}, \frac{3}{8}\right)$ is greater than $\frac{1}{2}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer (171):

The line through the fixed point $\left(\frac{5}{8}, \frac{3}{8}\right)$ with slope $\frac{1}{2}$ has equation $y = \frac{1}{2}x + \frac{1}{16}$. The slope between P and the fixed point exceeds $\frac{1}{2}$ if P falls within the shaded region in the diagram below consisting of two trapezoids with area

$$\frac{\frac{1}{16} + \frac{3}{8}}{2} \cdot \frac{5}{8} + \frac{\frac{5}{8} + \frac{7}{16}}{2} \cdot \frac{3}{8} = \frac{43}{128}.$$

Because the entire square has area 1, the required probability is $\frac{43}{128}$. The requested sum is $43 + 128 = 171$.



3. The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer (103):

Using the Change of Base Formula to convert the logarithms in the given equation to base 2 yields

$$\frac{\log_2 3^{20}}{\log_2 2^x} = \frac{\log_2 3^{2020}}{\log_2 2^{x+3}}, \quad \text{and then} \quad \frac{20 \log_2 3}{x \cdot \log_2 2} = \frac{2020 \log_2 3}{(x+3)\log_2 2}.$$

Canceling the logarithm factors then yields

$$\frac{20}{x} = \frac{2020}{x+3},$$

which has solution $x = \frac{3}{100}$. The requested sum is $3 + 100 = 103$.

4. Triangles $\triangle ABC$ and $\triangle A'B'C'$ lie in the coordinate plane with vertices $A(0,0)$, $B(0,12)$, $C(16,0)$, $A'(24,18)$, $B'(36,18)$, and $C'(24,2)$. A rotation of m degrees clockwise around the point (x,y), where $0 < m < 180$, will transform $\triangle ABC$ to $\triangle A'B'C'$. Find $m+x+y$.

Answer (108):

Because the rotation sends the vertical segment $\overline{AB}$ to the horizontal segment $\overline{A'B'}$, the angle of rotation is 90° clockwise. For any point (x,y) not at the origin, the line segments from $(0,0)$ to (x,y) and from (x,y) to $(x-y, y+x)$ are perpendicular and are the same length. Thus a 90° clockwise rotation around the point (x,y) sends the point $A(0,0)$ to the point $(x-y, y+x) = A'(24,18)$. This has the solution $(x,y) = (21,-3)$. The requested sum is $90+21-3 = 108$.

5. For each positive integer n, let $f(n)$ be the sum of the digits in the base-four representation of n, and let $g(n)$ be the sum of the digits in the base-eight representation of $f(n)$. For example, $f(2020) = f(133210_{\text{four}}) = 10 = 12_{\text{eight}}$, and $g(2020) =$ the digit sum of $12_{\text{eight}} = 3$. Let N be the least value of n such that the base-sixteen representation of $g(n)$ cannot be expressed using only the digits 0 through 9. Find the remainder when N is divided by 1000.

Answer (151):

First note that if $h_b(s)$ is the least positive integer whose digit sum, in some fixed base b, is s, then h_b is a strictly increasing function. This together with the fact that $g(N) \geq 10$ shows that $f(N)$ is the least positive integer whose base-eight digit sum is 10. Thus $f(N) = 37_{\text{eight}} = 31$, and N is the least positive integer whose base-four digit sum is 31. Therefore

$$\begin{aligned} N &= 13333333333_{\text{four}} = 2 \cdot 4^{10} - 1 = 2 \cdot 1024^2 - 1 \\ &\equiv 2 \cdot 24^2 - 1 \equiv 151 \pmod{1000}. \end{aligned}$$

6. Define a sequence recursively by $t_1 = 20$, $t_2 = 21$, and

$$t_n = \frac{5t_{n-1}+1}{25t_{n-2}}$$

for all $n \geq 3$. Then t_{2020} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Answer (626):

More generally, let the first two terms be a and b and replace 5 and 25 in the recursive formula by k and k^2, respectively. Then some algebraic calculation shows that

$$t_3 = \frac{b\,k+1}{a\,k^2}, \quad t_4 = \frac{a\,k+b\,k+1}{a\,b\,k^3}, \quad t_5 = \frac{a\,k+1}{b\,k^2}, \quad t_6 = a, \quad \text{and} \quad t_7 = b,$$

so the sequence is periodic with period 5. Therefore

$$t_{2020} = t_5 = \frac{20 \cdot 5 + 1}{21 \cdot 25} = \frac{101}{525}.$$

The requested sum is $101 + 525 = 626$.

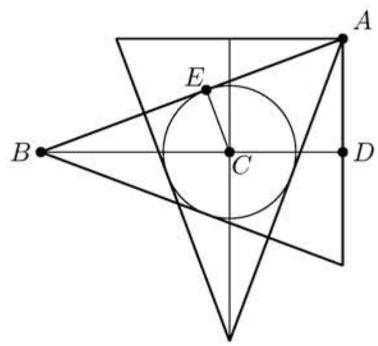
7. Two congruent right circular cones each with base radius 3 and height 8 have axes of symmetry that intersect at right angles at a point in the interior of the cones a distance 3 from the base of each cone. A sphere with radius r lies inside both cones. The maximum possible value for r^2 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer (298):

Consider the cross section of the cones and sphere by a plane that contains the two axes of symmetry of the cones as shown below. The sphere with maximum radius will be tangent to the sides of each of the cones. The center of that sphere must be on the axis of symmetry of each of the cones and thus must be at the intersection of their axes of symmetry. Let A be the point in the cross section where the bases of the cones meet, and let C be the center of the sphere. Let the axis of symmetry of one of the cones extend from its vertex, B, to the center of its base, D. Let the sphere be tangent to $\overline{AB}$ at E. The right triangles $\triangle ABD$ and $\triangle CBE$ are similar, implying that the radius of the sphere is

$$CE = AD \cdot \frac{BC}{AB} = AD \cdot \frac{BD - CD}{AB} = 3 \cdot \frac{5}{\sqrt{8^2+3^2}} = \frac{15}{\sqrt{73}} = \sqrt{\frac{225}{73}}.$$

The requested sum is $225 + 73 = 298$.



8. Define a sequence of functions recursively by $f_1(x) = |x-1|$ and $f_n(x) = f_{n-1}(|x-n|)$ for integers $n > 1$. Find the least value of n such that the sum of the zeros of f_n exceeds 500,000.

Answer (101):

First it will be shown by induction that the zeros of f_n are the integers $a, a+2, a+4, \ldots, a+n(n-1)$, where $a = n - \frac{n(n-1)}{2}$.

This is certainly true for $n = 1$. Suppose that it is true for $n = m-1 \geq 1$, and note that the zeros of f_m are the solutions of $|x-m| = k$, where k is a nonnegative zero of f_{m-1}. Because the zeros of f_{m-1} form an arithmetic sequence with common difference 2, so do the zeros of f_m. The greatest zero of f_{m-1} is

$$m - 1 + \frac{(m-1)(m-2)}{2} = \frac{m(m-1)}{2},$$

so the greatest zero of f_m is $m + \frac{m(m-1)}{2}$ and the least is $m - \frac{m(m-1)}{2}$.

It follows that the number of zeros of f_n is $\frac{n(n-1)}{2} + 1 = \frac{n^2-n+2}{2}$, and their average value is n. The sum of the zeros of f_n is

$$\frac{n^3 - n^2 + 2n}{2}.$$

Let $S(n) = n^3 - n^2 + 2n = n(n-2)(n+1)$, so the sum of the zeros exceeds 500,000 if and only if $S(n) > 1{,}000{,}000 = 100^3$. Because $S(n)$ is increasing for $n > 2$, the values $S(100) = 1{,}000{,}000 - 10{,}000 + 200 = 990{,}200$ and $S(101) = 1{,}030{,}301 - 10{,}201 + 202 = 1{,}020{,}302$ show that the requested value of n is 101.

9. While watching a show, Ayako, Billy, Carlos, Dahlia, Ehuang, and Frank sat in that order in a row of six chairs. During the break, they went to the kitchen for a snack. When they came back, they sat on those six chairs in such a way that if two of them sat next to each other before the break, then they did not sit next to each other after the break. Find the number of possible seating orders they could have chosen after the break.

Answer (090):

Ayako (A), Billy (B), Carlos (C), Dahlia (D), Ehuang (E), and Frank (F) originally sat in the order $ABCDEF$. Let $T(XY)$ denote the set of seatings where X and Y sit next to each other after the break. Then the required number of seating orders is given by the Inclusion-Exclusion Principle as

$$6! - \big(|T(AB)| + |T(BC)| + |T(CD)| + |T(DE)| + |T(EF)|\big) +$$
$$\big(|T(AB) \cap T(BC)| + |T(AB) \cap T(CD)| + \cdots\big) - \cdots .$$

Each term can be calculated separately.

(a) $|T(AB)| = |T(BC)| = |T(CD)| = |T(DE)| = |T(EF)| = 2 \cdot 5! = 240$. Because there are 5 terms, the sum is $5 \cdot 240 = 1200$.

(b) For $|T(XY) \cap T(ZW)|$, if $Y = Z$, then XYW must sit consecutively, so $|T(XY) \cap T(ZW)| = 2 \cdot 4! = 48$. There are 4 terms that satisfy $Y = Z$, so the sum is $4 \cdot 48 = 192$. If XY and ZW are pairwise disjoint, then $|T(XY) \cap T(ZW)| = 2^2 \cdot 4! = 96$. There are 6 terms, so the sum is $6 \cdot 96 = 576$.

(c) If there are at least three pairs that sit next to each other, consider these three subcases:

   i. If the three pairs are consecutive, the sum is $3 \cdot 2 \cdot 3! = 36$.

   ii. If exactly two of the pairs are consecutive, the sum is $6 \cdot 2^2 \cdot 3! = 144$.

   iii. If none of the three pairs is consecutive, the sum is $1 \cdot 2^3 \cdot 3! = 48$.

(d) If there are at least four pairs that sit next to each other, then if the pairs are consecutive, the sum is $2 \cdot 2 \cdot 2! = 8$. If the pairs are not consecutive, then the sum is $3 \cdot 2^2 \cdot 2! = 24$.

(e) If all five pairs sit next to each other, the number is $1 \cdot 2 \cdot 1! = 2$.

Therefore the required number of seating orders is

$$6! - 1200 + (192 + 576) - (36 + 144 + 48) + (8 + 24) - 2 = 90.$$

Note: See A002464 of the On-Line Encyclopedia of Integer Sequences for equivalent formulations.

10. Find the sum of all positive integers n such that when $1^3 + 2^3 + 3^3 + \cdots + n^3$ is divided by $n + 5$, the remainder is 17.

Answer (239):

The sum of the cubes from 1 to n is

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

For this to be equal to $(n + 5)q + 17$ for some integer q, it must be that

$$n^2(n+1)^2 = 4(n+5)q + 4 \cdot 17,$$

so

$$n^2(n+1)^2 \equiv 4 \cdot 17 = 68 \pmod{n+5}.$$

But $n^2(n+1)^2 \equiv (-5)^2(-4)^2 = 400 \pmod{n+5}$. Thus $n^2(n+1)^2$ is congruent to both 68 and 400, which implies that $n+5$ divides $400-68 = 332 = 2^2 \cdot 83$. Because $n+5 > 17$, the only choices for $n+5$ are 83, 166, and 332. Checking all three cases verifies that $n = 78$ and $n = 161$ work, but $n = 327$ does not. The requested sum is $78 + 161 = 239$.

OR

The sum of the cubes of the integers from 1 through n is

$$\frac{n^2(n+1)^2}{4},$$

which, when divided by $n+5$, has quotient

$$Q = \frac{1}{4}n^3 - \frac{3}{4}n^2 + 4n - 20 = \frac{n^2(n-3)}{4} + 4n - 20$$

with remainder 100. If n is not congruent to 1 (mod 4), then Q is an integer, and

$$\frac{n^2(n+1)^2}{4} = (n+5)Q + 100 \equiv 17 \pmod{n+5},$$

so $n+5$ divides $100 - 17 = 83$, and $n = 78$. If $n \equiv 1 \pmod 4$, then Q is half of an integer, and letting $n = 4k+1$ for some integer k gives

$$\frac{n^2(n+1)^2}{4} = 2(2k+3)Q + 100 \equiv 17 \pmod{n+5}.$$

Thus $2k+3$ divides $100 - 17 = 83$. It follows that $k = 40$, and $n = 161$. The requested sum is $161 + 78 = 239$.

11. Let $P(x) = x^2 - 3x - 7$, and let $Q(x)$ and $R(x)$ be two quadratic polynomials also with the coefficient of x^2 equal to 1. David computes each of the three sums $P+Q$, $P+R$, and $Q+R$ and is surprised to find that each pair of these sums has a common root, and these three common roots are distinct. If $Q(0) = 2$, then $R(0) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer (071):

Let the common root of $P+Q$ and $P+R$ be p, the common root of $P+Q$ and $Q+R$ be q, and the common root of $Q+R$ and $P+R$ be r. Because p and q are both roots of $P+Q$ and $P+Q$ has leading coefficient 2, it follows that $P(x)+Q(x) = 2(x-p)(x-q)$. Similarly, $P(x)+R(x) = 2(x-p)(x-r)$ and $Q(x)+R(x) = 2(x-q)(x-r)$. Adding these three equations together and dividing by 2 yields

$$P(x)+Q(x)+R(x) = (x-p)(x-q) + (x-p)(x-r) + (x-q)(x-r),$$

so

$$\begin{aligned} P(x) &= (P(x)+Q(x)+R(x)) - (Q(x)+R(x)) \\ &= (x-p)(x-q) + (x-p)(x-r) - (x-q)(x-r) \\ &= x^2 - 2px + (pq + pr - qr). \end{aligned}$$

Similarly,

$$\begin{aligned} Q(x) &= x^2 - 2qx + (pq + qr - pr) \ \text{ and} \\ R(x) &= x^2 - 2rx + (pr + qr - pq). \end{aligned}$$

Comparing the x coefficients yields $p = \frac{3}{2}$, and comparing the constant coefficients yields $-7 = pq + pr - qr = \frac{3}{2}(q+r) - qr$. The fact that $Q(0) = 2$ implies that $\frac{3}{2}(q-r) + qr = 2$. Adding these two equations yields $q = -\frac{5}{3}$, and so substituting back in to solve for r gives $r = -\frac{27}{19}$. Finally,

$$R(0) = pr + qr - pq = \left(-\frac{27}{19}\right)\left(\frac{3}{2} - \frac{5}{3}\right) + \frac{5}{2} = \frac{9}{38} + \frac{5}{2} = \frac{52}{19}.$$

The requested sum is $52 + 19 = 71$. Note that $Q(x) = x^2 + \frac{10}{3}x + 2$ and $R(x) = x^2 + \frac{54}{19}x + \frac{52}{19}$.

12. Let m and n be odd integers greater than 1. An $m \times n$ rectangle is made up of unit squares where the squares in the top row are numbered left to right with the integers 1 through n, those in the second row are numbered left to right with the integers $n+1$ through $2n$, and so on. Square 200 is in the top row, and square 2000 is in the bottom row. Find the number of ordered pairs (m, n) of odd integers greater than 1 with the property that, in the $m \times n$ rectangle, the line through the centers of squares 200 and 2000 intersects the interior of square 1099.

Answer (248):

Because square 2000 is in the bottom row, it follows that $\frac{2000}{m} \le n < \frac{2000}{m-1}$. Moreover, because square 200 is in the top row, and square 2000 is not in the top row, $1 < m \le 10$. In particular, because the number of rows in the rectangle must be odd, m must be one of 3, 5, 7, or 9.

For each possible choice of m and n, let $\ell_{m,n}$ denote the line through the centers of squares 200 and 2000. Note that for odd values of m, the line $\ell_{m,n}$ passes through the center of square 1100. Thus $\ell_{m,n}$ intersects the interior of cell 1099 exactly when its slope is strictly between -1 and 1. The line $\ell_{m,n}$ is vertical whenever square 2000 is the 200th square in the bottom row of the rectangle. This would happen for $m = 3, 5, 7, 9$ when $n = 900, 450, 300, 225$, respectively. When n is 1 greater than or 1 less than these numbers, the slope of $\ell_{m,n}$ is 1 or -1, respectively. In all other cases the slope is strictly between -1 and 1. The admissible values for n for each possible value of m are given in the following table.

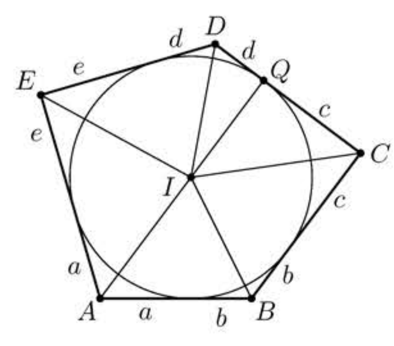
m	minimum n	maximum n	avoided n	number of odd n
3	667	999	$899, 900, 901$	165
5	400	499	$449, 450, 451$	48
7	286	333	$299, 300, 301$	22
9	223	249	$224, 225, 226$	13

This accounts for $165 + 48 + 22 + 13 = 248$ rectangles.

13. Convex pentagon $ABCDE$ has side lengths $AB = 5$, $BC = CD = DE = 6$, and $EA = 7$. Moreover, the pentagon has an inscribed circle (a circle tangent to each side of the pentagon). Find the area of $ABCDE$.

Answer (060):

Let ω be the inscribed circle, I be its center, and r be its radius. The area of $ABCDE$ is equal to its semiperimeter, 15, times r, so the problem is reduced to finding r. Let a be the length of the tangent segment from A to ω, and analogously define b, c, d, and e. Then $a + b = 5$, $b + c = c + d = d + e = 6$, and $e + a = 7$, with a total of $a + b + c + d + e = 15$. Hence $a = 3$, $b = d = 2$, and $c = e = 4$. It follows that $\angle B = \angle D$ and $\angle C = \angle E$. Let Q be the point where ω is tangent to $\overline{CD}$. Then $\angle IAE = \angle IAB = \frac{1}{2}\angle A$. The sum of the internal angles in polygons $ABCQI$ and $AIQDE$ are equal, so $\angle IAE + \angle AIQ + \angle IQD + \angle D + \angle E = \angle IAB + \angle B + \angle C + \angle CQI + \angle QIA$, which implies that $\angle AIQ$ must be $180°$. Therefore points A, I, and Q are collinear.



Because $\overline{AQ} \perp \overline{CD}$, it follows that

$$AC^2 - AD^2 = CQ^2 - DQ^2 = c^2 - d^2 = 12.$$

Another expression for $AC^2 - AD^2$ can be found as follows. Note that $\tan\left(\frac{\angle B}{2}\right) = \frac{r}{2}$ and $\tan\left(\frac{\angle E}{2}\right) = \frac{r}{4}$, so

$$\cos(\angle B) = \frac{1 - \tan^2\left(\frac{\angle B}{2}\right)}{1 + \tan^2\left(\frac{\angle B}{2}\right)} = \frac{4 - r^2}{4 + r^2}$$

and

$$\cos(\angle E) = \frac{1 - \tan^2\left(\frac{\angle E}{2}\right)}{1 + \tan^2\left(\frac{\angle E}{2}\right)} = \frac{16 - r^2}{16 + r^2}.$$

Applying the Law of Cosines to $\triangle ABC$ and $\triangle AED$ gives

$$AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos(\angle B) = 5^2 + 6^2 - 2 \cdot 5 \cdot 6 \cdot \frac{4 - r^2}{4 + r^2}$$

and

$$AD^2 = AE^2 + DE^2 - 2 \cdot AE \cdot DE \cdot \cos(\angle E) = 7^2 + 6^2 - 2 \cdot 7 \cdot 6 \cdot \frac{16 - r^2}{16 + r^2}.$$

Hence

$$12 = AC^2 - AD^2 = 5^2 - 2 \cdot 5 \cdot 6 \cdot \frac{4 - r^2}{4 + r^2} - 7^2 + 2 \cdot 7 \cdot 6 \cdot \frac{16 - r^2}{16 + r^2},$$

yielding

$$2 \cdot 7 \cdot 6 \cdot \frac{16 - r^2}{16 + r^2} - 2 \cdot 5 \cdot 6 \cdot \frac{4 - r^2}{4 + r^2} = 36;$$

equivalently

$$7(16 - r^2)(4 + r^2) - 5(4 - r^2)(16 + r^2) = 3(16 + r^2)(4 + r^2).$$

Substituting $x = r^2$ gives the quadratic equation $5x^2 - 84x + 64 = 0$, with solutions $\frac{42-38}{5} = \frac{4}{5}$ and $\frac{42+38}{5} = 16$. The solution $r^2 = \frac{4}{5}$ corresponds to a five-pointed star, which is not convex. Indeed, if $r < 3$, then $\tan\left(\frac{\angle A}{2}\right)$, $\tan\left(\frac{\angle C}{2}\right)$, and $\tan\left(\frac{\angle E}{2}\right)$ are less than 1, implying that $\angle A$, $\angle C$, and $\angle E$ are acute, which cannot happen in a convex pentagon. Thus $r^2 = 16$ and $r = 4$. The requested area is $15 \cdot 4 = 60$.

OR

Define a, b, c, d, e, and r as in the first solution. Then, as above, $a = 3$, $b = d = 2$, $c = e = 4$, $\angle B = \angle D$, and $\angle C = \angle E$. Let $\alpha = \frac{\angle A}{2}$, $\beta = \frac{\angle B}{2}$, and $\gamma = \frac{\angle C}{2}$. It follows that $540^\circ = 2\alpha + 4\beta + 4\gamma$, so $270^\circ = \alpha + 2\beta + 2\gamma$. Thus

$$\tan(2\beta + 2\gamma) = \frac{1}{\tan\alpha},$$

$\tan(\beta) = \frac{r}{2}$, $\tan(\gamma) = \frac{r}{4}$, and $\tan(\alpha) = \frac{r}{3}$. By the Tangent Addition Formula,

$$\tan(\beta + \gamma) = \frac{6r}{8 - r^2}$$

and

$$\tan(2\beta + 2\gamma) = \frac{\frac{12r}{8-r^2}}{1 - \frac{36r^2}{(8-r^2)^2}} = \frac{12r(8 - r^2)}{(8 - r^2)^2 - 36r^2}.$$

Therefore

$$\frac{12r(8 - r^2)}{(8 - r^2)^2 - 36r^2} = \frac{3}{r},$$

which simplifies to $5r^4 - 84r^2 + 64 = 0$. Then the solution proceeds as in the first solution.

OR

Define a, b, c, d, e, and r as in the first solution. Note that

$$\arctan\left(\frac{a}{r}\right) + \arctan\left(\frac{b}{r}\right) + \arctan\left(\frac{c}{r}\right) + \arctan\left(\frac{d}{r}\right) + \arctan\left(\frac{e}{r}\right) = 180^\circ.$$

Hence

$$\operatorname{Arg}(r+3i) + 2\cdot\operatorname{Arg}(r+2i) + 2\cdot\operatorname{Arg}(r+4i) = 180^\circ.$$

Therefore

$$\operatorname{Im}\left((r+3i)(r+2i)^2(r+4i)^2\right) = 0.$$

Simplifying this equation gives the same quadratic equation in r^2 as above.

14. For real number x let $\lfloor x\rfloor$ be the greatest integer less than or equal to x, and define $\{x\} = x - \lfloor x\rfloor$ to be the fractional part of x. For example, $\{3\} = 0$ and $\{4.56\} = 0.56$. Define $f(x) = x\{x\}$, and let N be the number of real-valued solutions to the equation $f(f(f(x))) = 17$ for $0 \le x \le 2020$. Find the remainder when N is divided by 1000.

Answer (010):

For any nonnegative integer n, the function f increases on the interval $[n, n+1)$, with $f(n) = 0$ and $f(x) < n+1$ for every x in this interval. On this interval $f(x) = x(x-n)$, which takes on every real value in the interval $[0, n+1)$ exactly once. Thus for each nonnegative real number y, the equation $f(x) = y$ has exactly one solution $x \in [n, n+1)$ for every $n \ge \lfloor y\rfloor$.

For each integer $a \ge 17$ there is exactly one x with $\lfloor x\rfloor = a$ such that $f(x) = 17$; likewise for each integer $b \ge a \ge 17$ there is exactly one x with $\lfloor f(x)\rfloor = a$ and $\lfloor x\rfloor = b$ such that $f(f(x)) = 17$. Finally, for each integer $c \ge b \ge a \ge 17$ there is exactly one x with $\lfloor f(f(x))\rfloor = a$, $\lfloor f(x)\rfloor = b$, and $\lfloor x\rfloor = c$ such that $f(f(f(x))) = 17$. Thus $f(f(f(x))) = 17$ has exactly one solution x with $0 \le x \le 2020$ for each triple of integers (a, b, c) with $17 \le a \le b \le c < 2020$, noting that $x = 2020$ is not a solution. This nondecreasing ordered triple can be identified with a multiset of three elements of the set of 2003 integers $\{17, 18, 19, \ldots, 2019\}$, which can be selected in $\binom{2005}{3}$ ways. Thus

$$N = \frac{2005\cdot 2004\cdot 2003}{6} \equiv 10 \pmod{1000}.$$

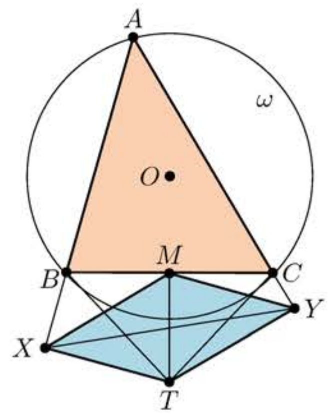
15. Let $\triangle ABC$ be an acute scalene triangle with circumcircle ω. The tangents to ω at B and C intersect at T. Let X and Y be the projections of T onto lines AB and AC, respectively. Suppose $BT = CT = 16$, $BC = 22$, and $TX^2 + TY^2 + XY^2 = 1143$. Find XY^2.

Answer (717):

Let M denote the midpoint of $\overline{BC}$. The critical claim is that M is the orthocenter of $\triangle AXY$, which has the circle with diameter $\overline{AT}$ as its circumcircle. To see this, note that because $\angle BXT = \angle BMT = 90^\circ$, the quadrilateral $MBXT$ is cyclic. Thus

$$\angle MXA = \angle MXB = \angle MTB = 90^\circ - \angle TBM = 90^\circ - \angle A,$$

implying that $\overline{MX} \perp \overline{AC}$. Similarly, $\overline{MY} \perp \overline{AB}$. In particular, $MXTY$ is a parallelogram.



Hence, by the Parallelogram Law,

$$TM^2 + XY^2 = 2(TX^2 + TY^2) = 2(1143 - XY^2).$$

But $TM^2 = TB^2 - BM^2 = 16^2 - 11^2 = 135$. Therefore

$$XY^2 = \frac{1}{3}(2 \cdot 1143 - 135) = 717.$$

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