MAA AMC
American Mathematics Competitions

MAA American Mathematics Competitions

38th Annual

AIME I

American Invitational Mathematics Examination I
Wednesday, March 11, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR COMPETITION MANAGER TELLS YOU TO BEGIN.
2. This is a 15-question competition. All answers are integers ranging from 000 to 999, inclusive.
3. Mark your answer to each problem on the answer sheet with a #2 pencil. Check blackened answers for accuracy and erase errors completely. Only answers that are properly marked on the answer sheet will be scored.
4. SCORING: You will receive 1 point for each correct answer, 0 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only blank scratch paper, blank graph paper, rulers, compasses, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on the competition will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the competition, your competition manager will ask you to record your name on the answer sheet.
8. You will have 3 hours to complete the competition once your competition manager tells you to begin.
9. When you finish the competition, sign your name in the space provided on the answer sheet.

The MAA AMC Office reserves the right to disqualify scores from a school if it determines that the rules or the required security procedures were not followed.

The publication, reproduction, or communication of the problems or solutions of this competition during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

A combination of your AIME score and your AMC 10/12 score is used to determine eligibility for participation in the USA (Junior) Mathematical Olympiad. The USA(J)MO will be given on Tuesday and Wednesday, April 14 and 15, 2020.

1. In $\triangle ABC$ with $AB = AC$, point D lies strictly between A and C on side $\overline{AC}$, and point E lies strictly between A and B on side $\overline{AB}$ such that $AE = ED = DB = BC$. The degree measure of $\angle ABC$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

2. There is a unique positive real number x such that the three numbers $\log_8(2x)$, $\log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

3. A positive integer N has base-eleven representation $\underline{a}\,\underline{b}\,\underline{c}$ and base-eight representation $\underline{1}\,\underline{b}\,\underline{c}\,\underline{a}$, where a, b, and c represent (not necessarily distinct) digits. Find the least such N expressed in base ten.

4. Let S be the set of positive integers N with the property that the last four digits of N are 2020, and when the last four digits are removed, the result is a divisor of N. For example, 42,020 is in S because 4 is a divisor of 42,020. Find the sum of all the digits of all the numbers in S. For example, the number 42,020 contributes $4 + 2 + 0 + 2 + 0 = 8$ to this total.

5. Six cards numbered 1 through 6 are to be lined up in a row. Find the number of arrangements of these six cards where one of the cards can be removed leaving the remaining five cards in either ascending or descending order.

6. A flat board has a circular hole with radius 1 and a circular hole with radius 2 such that the distance between the centers of the two holes is 7. Two spheres with equal radii sit in the two holes such that the spheres are tangent to each other. The square of the radius of the spheres is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

7. A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committee could have as few as 1 member or as many as 23 members. Let N be the number of such committees that can be formed. Find the sum of the prime numbers that divide N.

8. A bug walks all day and sleeps all night. On the first day, it starts at point O, faces east, and walks a distance of 5 units due east. Each night the bug rotates 60° counterclockwise. Each day it walks in this new direction half as far as it walked the previous day. The bug gets arbitrarily close to point P. Then $OP^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

9. Let S be the set of positive integer divisors of 20^9. Three numbers are chosen independently and at random with replacement from the set S and labeled a_1, a_2, and a_3 in the order they are chosen. The probability that both a_1 divides a_2 and a_2 divides a_3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

10. Let m and n be positive integers satisfying the conditions

    - $\gcd(m+n, 210) = 1$,
    - m^m is a multiple of n^n, and
    - m is not a multiple of n.

    Find the least possible value of $m + n$.

11. For integers a, b, c, and d, let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$. Find the number of ordered triples (a, b, c) of integers with absolute values not exceeding 10 for which there is an integer d such that $g(f(2)) = g(f(4)) = 0$.

12. Let n be the least positive integer for which $149^n - 2^n$ is divisible by $3^3 \cdot 5^5 \cdot 7^7$. Find the number of positive integer divisors of n.

13. Point D lies on side $\overline{BC}$ of $\triangle ABC$ so that $\overline{AD}$ bisects $\angle BAC$. The perpendicular bisector of $\overline{AD}$ intersects the bisectors of $\angle ABC$ and $\angle ACB$ in points E and F, respectively. Given that $AB = 4$, $BC = 5$, and $CA = 6$, the area of $\triangle AEF$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m + n + p$.

14. Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P\big(P(x)\big) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a + b)^2$.

15. Let $\triangle ABC$ be an acute triangle with circumcircle ω, and let H be the intersection of the altitudes of $\triangle ABC$. Suppose the tangent to the circumcircle of $\triangle HBC$ at H intersects ω at points X and Y with $HA = 3$, $HX = 2$, and $HY = 6$. The area of $\triangle ABC$ can be written as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

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Scores and official competition solutions will be sent to your competition manager who can share that information with you.

For more information about the MAA American Mathematics Competitions program, please visit maa.org/amc.

Questions and comments about this competition should be sent to:

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The problems and solutions for this AIME were prepared by the MAA AIME Editorial Board under the direction of:

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