MAA AMC

American Mathematics Competitions

Solutions Pamphlet

MAA American Mathematics Competitions

37th Annual

AIME I

American Invitational Mathematics Examination I

Wednesday, March 13, 2019

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Correspondence about the problems/solutions for this AIME and orders for any publications should be addressed to:

MAA American Mathematics Competitions
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The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

Jonathan M. Kane
AIME Chair

1. **Answer (342):**

   Write

$$\begin{aligned} N &= (10-1)+(10^2-1)+\cdots+(10^{321}-1) \\ &= 10+10^2+10^3+10^4+10^5+10^6+\cdots+10^{321}-321 \\ &= 1110-321+10^4+10^5+10^6+\cdots+10^{321} \\ &= 789+10^4+10^5+10^6+\cdots+10^{321}. \end{aligned}$$

   The sum of the digits of N is therefore equal to $7+8+9+(321-3)=342$.

2. **Answer (029):**

   There are $\binom{20}{2}=190$ equally likely pairs $\{J,B\}$. In 19 of those pairs ($\{1,2\}$, $\{2,3\}$, $\{3,4\}\ldots$, $\{19,20\}$), the numbers differ by less than 2, so the probability that the numbers differ by at least 2 is $1-\frac{19}{190}=\frac{9}{10}$. Then $B-J\geq 2$ holds in exactly half of these cases, so it has probability $\frac{1}{2}\cdot\frac{9}{10}=\frac{9}{20}$. The requested sum is $9+20=29$.

3. **Answer (120):**

   Triangle PQR is a right triangle with area $\frac{1}{2}\cdot 15\cdot 20=150$. Each of $\triangle PAF$, $\triangle QCB$, and $\triangle RED$ shares an angle with $\triangle PQR$. Because the area of a triangle with sides a, b, and included angle γ is $\frac{1}{2}a\cdot b\cdot\sin\gamma$, it follows that the areas of $\triangle PAF$, $\triangle QCB$, and $\triangle RED$ are each $\frac{1}{2}\cdot 5\cdot 5\cdot\frac{150}{ab}$, where a and b are the lengths of the sides of $\triangle PQR$ adjacent to the shared angle. Thus the sum of the areas of $\triangle PAF$, $\triangle QCB$, and $\triangle RED$ is

$$5\cdot 5\cdot\frac{150}{15\cdot 25}+5\cdot 5\cdot\frac{150}{25\cdot 20}+5\cdot 5\cdot\frac{150}{20\cdot 15}=25\left(\frac{2}{5}+\frac{3}{10}+\frac{1}{2}\right)=30.$$

   Therefore $ABCDEF$ has area $150-30=120$.

4. **Answer (122):**

   There is 1 way of making no substitutions to the starting lineup. If the coach makes exactly 1 substitution, this can be done in 11^2 ways. Two substitutions can happen in $11^2\cdot 11\cdot 10$ ways. Similarly, three substitutions can happen in $11^2\cdot 11\cdot 10\cdot 11\cdot 9$ ways. The total number of possibilities is $1+11^2+11^2\cdot 11\cdot 10+11^2\cdot 11\cdot 10\cdot 11\cdot 9=122+11^3(10+990)\equiv 122 \pmod{1000}$.

5. **Answer (252):**

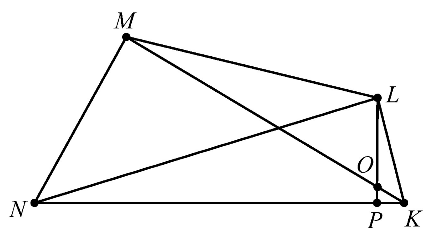
   All paths that first hit the axes at the origin must pass through the point $(1,1)$. There are 63 paths from the point $(4,4)$ to the point $(1,1)$: $\binom{6}{3}=20$ that take 3 steps left and 3 steps down, $\binom{5}{2\ 2\ 1} = 30$ that take 2 steps left, 2 steps down, and 1 diagonal step, $\binom{4}{1\ 1\ 2} = 12$ that take 1 step left, 1 step down, and 2 diagonal steps, and 1 that takes 3 diagonal steps. The total probability of moving from $(4,4)$ to $(1,1)$ is therefore

$$\frac{1}{3^6} \cdot 20 + \frac{1}{3^5} \cdot 30 + \frac{1}{3^4} \cdot 12 + \frac{1}{3^3} \cdot 1 = \frac{245}{3^6}.$$

Multiplying by $\frac{1}{3}$ gives $\frac{245}{3^7}$, the probability that the path first reaches the axes at the origin. The requested sum is $245 + 7 = 252$.

6. **Answer (090):**

Let P be the intersection of line LO with line KN. Then $\triangle KPL \sim \triangle KLN$, so $\frac{28}{KP} = \frac{KN}{28}$ and $KP \cdot KN = 28^2$. Also $\triangle KPO \sim \triangle KMN$, so $\frac{KP}{8} = \frac{8+MO}{KN}$ and $KP \cdot KN = 8(8 + MO)$. Thus $28^2 = 8(8 + MO)$, from which $MO = 90$.



Note that the value of MN is irrelevant.

7. **Answer (880):**

The two equations are equivalent to $x(\gcd(x,y))^2 = 10^{60}$ and $y(\operatorname{lcm}(x,y))^2 = 10^{570}$, respectively. Multiplying corresponding sides of the equations leads to $xy\,(\gcd(x,y)\operatorname{lcm}(x,y))^2 = (xy)^3 = 10^{630}$, so $xy = 10^{210}$. It follows that there are nonnegative integers a, b, c, and d such that $(x,y) = (2^a 5^b, 2^c 5^d)$ with $a + c = b + d = 210$. Furthermore,

$$\frac{(\operatorname{lcm}(x,y))^2}{x} = \frac{y\,(\operatorname{lcm}(x,y))^2}{xy} = \frac{10^{570}}{10^{210}} = 10^{360}.$$

Thus $\max(2a, 2c) - a = \max(2b, 2d) - b = 360$. Because neither $2a - a$ nor $2b - b$ can equal 360 when $a + c = b + d = 210$, it follows that $2c - a = 2d - b = 360$. Hence $(a,b,c,d) = (20, 20, 190, 190)$, so the prime factorization of x has $20 + 20 = 40$ prime factors, and the prime factorization of y has $190 + 190 = 380$ prime factors. The requested sum is $3 \cdot 40 + 2 \cdot 380 = 880$.

8. **Answer (067):**

Let $c = \sin^2 x \cdot \cos^2 x$, and let $S(n) = \sin^{2n} x + \cos^{2n} x$. Then for $n \geq 1$

$$\begin{aligned} S(n) &= (\sin^{2n} x + \cos^{2n} x) \cdot (\sin^2 x + \cos^2 x) \\ &= \sin^{2n+2} x + \cos^{2n+2} x + \sin^2 x \cdot \cos^2 x (\sin^{2n-2} x + \cos^{2n-2} x) \\ &= S(n+1) + cS(n-1). \end{aligned}$$

Because $S(0) = 2$ and $S(1) = 1$, it follows that $S(2) = 1-2c$, $S(3) = 1-3c$, $S(4) = 2c^2 - 4c + 1$, and $\frac{11}{36} = S(5) = 5c^2 - 5c + 1$. Hence $c = \frac{1}{6}$ or $\frac{5}{6}$, and because $4c = \sin^2 2x$, the only possible value of c is $\frac{1}{6}$. Therefore

$$S(6) = S(5) - cS(4) = \frac{11}{36} - \frac{1}{6}\left(2\left(\frac{1}{6}\right)^2 - 4\left(\frac{1}{6}\right) + 1\right) = \frac{13}{54}.$$

The requested sum is $13 + 54 = 67$.

9. **Answer (540):**

Let p, q, and r represent primes. Because $\tau(n) = 1$ only for $n = 1$, there is no n for which $\{\tau(n), \tau(n+1)\} = \{1, 6\}$. If $\{\tau(n), \tau(n+1)\} = \{2, 5\}$, then $\{n, n+1\} = \{p, q^4\}$, so $|p - q^4| = 1$. Checking $q = 2$ and $p = 17$ yields the solution $n = 16$. If $q > 2$, then q is odd, and $p = q^4 \pm 1$ is even, so p cannot be prime.

If $\{\tau(n), \tau(n+1)\} = \{3, 4\}$, then $\{n, n+1\} = \{p^2, q^3\}$ or $\{p^2, qr\}$. Consider $|p^2 - q^3| = 1$. If $p^2 - 1 = (p-1)(p+1) = q^3$, then $q = 2$. This yields the solution $p = 3$ and $q = 2$, so $n = 8$. If $q^3 - 1 = (q-1)(q^2 + q + 1) = p^2$, then $q - 1 = 1$, which does not give a solution. Consider $|p^2 - qr| = 1$. If $p^2 - 1 = (p-1)(p+1) = qr$, then if $p > 2$, the left side is divisible by 8, so there are no solutions. Finding the smallest four primes such that $p^2 + 1 = qr$ gives $3^2 + 1 = 10$, $5^2 + 1 = 26$, $11^2 + 1 = 122$, and $19^2 + 1 = 362$. The six least values of n are 8, 9, 16, 25, 121, and 361, whose sum is 540.

10. **Answer (352):**

Because each root of the polynomial appears with multiplicity 3, Viète's Formulas show that

$$z_1 + z_2 + \cdots + z_{673} = -\frac{20}{3}$$

and

$$z_1^2 + z_2^2 + \cdots + z_{673}^2 = \frac{1}{3}\left((-20)^2 - 2 \cdot 19\right) = \frac{362}{3}.$$

Then the identity

$$\left(\sum_{i=1}^{673} z_i\right)^2 = \sum_{i=1}^{673} z_i^2 + 2\left(\sum_{1 \leq j < k \leq 673} z_j z_k\right)$$

shows that

$$\sum_{1\le j<k\le 673} z_j z_k = \frac{\left(-\frac{20}{3}\right)^2 - \frac{362}{3}}{2} = -\frac{343}{9}.$$

The requested sum is $343 + 9 = 352$.

Note that such a polynomial does exist. For example, let $z_{673} = -\frac{20}{3}$, and for $i = 1, 2, 3, \ldots, 336$, let

$$z_i = \sqrt{\frac{343i}{9\sum\limits_{j=1}^{336} j}} \quad \text{and} \quad z_{i+336} = -z_i.$$

Then

$$\sum_{i=1}^{673} z_i = -\frac{20}{3} \quad \text{and} \quad \sum_{i=1}^{673} z_i^2 = 2\sum_{i=1}^{336} \frac{343i}{9\sum\limits_{j=1}^{336} j} + \left(\frac{20}{3}\right)^2 = \frac{362}{3},$$

as required.

OR

There are constants a and b such that

$$(x - z_1)(x - z_2)(x - z_3)\cdots(x - z_{673}) = x^{673} + ax^{672} + bx^{671} + \cdots.$$

Then

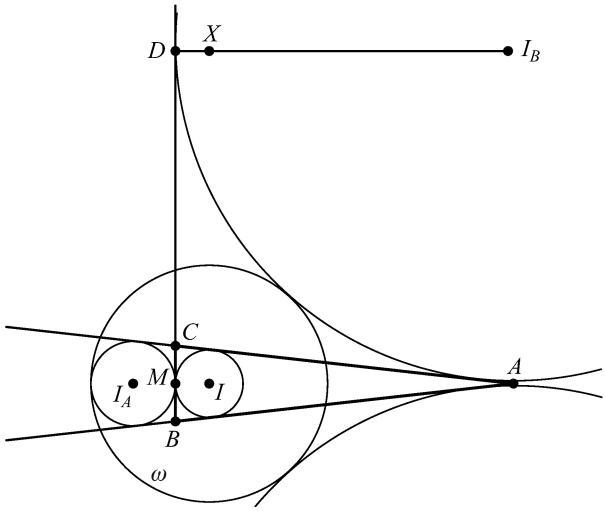
$$(x^{673} + ax^{672} + bx^{671} + \cdots)^3 = x^{2019} + 20x^{2018} + 19x^{2017} + \cdots.$$

Comparing the x^{2018} and x^{2017} coefficients shows that $3a = 20$ and $3a^2 + 3b = 19$. Solving this system yields $a = \frac{20}{3}$ and $b = -\frac{343}{9}$. Viète's Formulas then give $\left|\sum_{1\le j<k\le 673} z_j z_k\right| = |b| = \frac{343}{9}$, as above.

11. **Answer (020):**

Rescale the triangle so that $BC = 1$ and $AB = AC = x$. Then $[ABC] = \frac{1}{2}\sqrt{x^2 - \frac{1}{4}}$. Let the incircle of $\triangle ABC$ have center I and radius r. Let the excircles opposite A and B have centers I_A and I_B and radii r_A and r_B, respectively. For any triangle ABC with $a = BC$, $b = AC$, $c = AB$, inradius r, and the radius of the excircle opposite A, r_A, the area of $\triangle ABC$ is given by $r \cdot \frac{a+b+c}{2}$ and by $r_A \cdot \frac{b+c-a}{2}$. It follows that

$$r = \frac{1}{2}\sqrt{\frac{x - \frac{1}{2}}{x + \frac{1}{2}}}, \quad r_A = \frac{1}{2}\sqrt{\frac{x + \frac{1}{2}}{x - \frac{1}{2}}}, \quad \text{and} \quad r_B = \sqrt{x^2 - \frac{1}{4}}.$$



Let M be the midpoint of $\overline{BC}$, and let D be the point of tangency of the excircle opposite B with line BC. Let point X lie on line I_BD so that $\overline{IX} \perp \overline{I_BD}$. Note that the radius of ω is equal to $r + 2r_A$. Note also that BD is the semiperimeter of $\triangle ABC$; that is, $BD = x + \frac{1}{2}$, and so $IX = MD = BD - BM = x$. The Pythagorean Theorem applied to $\triangle IXI_B$ yields

$$x^2 + (r_B - r)^2 = (r + 2r_A + r_B)^2.$$

Expressing each term in the above equation in terms in x gives

$$x^2 = 4rr_A + 4rr_B + 4r_Ar_B + 4r_A^2 = 1 + (2x - 1) + (2x + 1) + \frac{x + \frac{1}{2}}{x - \frac{1}{2}}$$

$$= 4x + 1 + \frac{2x + 1}{2x - 1} = \frac{8x^2}{2x - 1},$$

implying that $x = \frac{9}{2}$. Thus the minimum possible perimeter with integer side lengths occurs when $BC = 2$ and $AB = AC = 9$, giving a perimeter of 20.

OR

Set T_A and T_B to be the tangency points of ω with the excircle opposite A, ω_A, and excircle opposite B, ω_B, respectively. Note that the homothety $\mathcal{H}_A$ sending ω_A to ω has positive scale factor and is centered at T_A, while the homothety $\mathcal{H}_B$ sending ω to ω_B has negative scale factor and is centered at T_B. Thus the composition $\mathcal{H}_B \circ \mathcal{H}_A$ is another homothety with negative scale factor and sends ω_A to ω_B. Because ω_A and ω_B have common tangent lines

AC and BC, the center of this homothety is C and therefore T_A, T_B, and C are collinear. In turn,

$$\angle BT_AI = 90^\circ - \frac{\angle T_BIT_A}{2} = 90^\circ - \frac{\angle AIC}{2} = 45^\circ - \frac{\angle B}{4}.$$

Now note that $\overline{BI_A}$ is a median of $\triangle BT_AM$, where M is the midpoint of $\overline{BC}$; combining this with $\angle I_ABM = 90^\circ - \frac{\angle B}{2}$ yields the trigonometric equation

$$\cot\left(45^\circ - \frac{\angle B}{4}\right) = 2\tan\left(90^\circ - \frac{\angle B}{2}\right) = \frac{2}{\tan\frac{\angle B}{2}}.$$

Let $\beta = \frac{\angle B}{4}$. Then

$$\cot(45^\circ - \beta) = \tan(45^\circ + \beta) = \frac{1+\tan\beta}{1-\tan\beta}$$

and

$$\frac{2}{\tan(2\beta)} = \frac{1-\tan^2\beta}{\tan\beta}.$$

This shows that $\tan\beta = \frac{3-\sqrt{5}}{2}$ and $\tan(4\beta) = \tan(\angle B) = 4\sqrt{5}$. It follows that $AB = \frac{9}{2}BC$ as above.

12. **Answer (230):**

The arguments of two complex numbers differ by 90° if the ratio of the numbers is a pure imaginary number. Thus three distinct complex numbers A, B, and C form a right triangle in the complex plane with right angle at B if and only if $\frac{C-B}{B-A}$ has real part equal to 0. Hence

$$\begin{aligned}\frac{f(f(z)) - f(z)}{f(z) - z} &= \frac{(z^2-19z)^2 - 19(z^2-19z) - (z^2-19z)}{(z^2-19z) - z} \\ &= \frac{(z^2-19z)(z^2-19z-19-1)}{z^2-20z} \\ &= \frac{z(z-19)(z+1)(z-20)}{z(z-20)} \\ &= z^2 - 18z - 19\end{aligned}$$

must have real part equal to 0. If $z = x + 11i$, the real part of $z^2 - 18z - 19$ is $x^2 - 11^2 - 18x - 19$, which is 0 when $x = 9 \pm \sqrt{221}$. The requested sum is $9 + 221 = 230$.

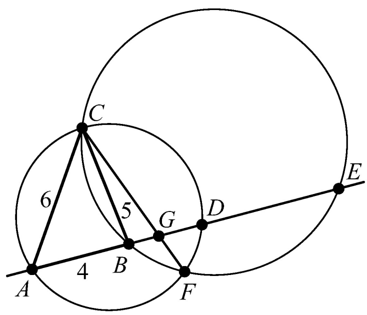
13. **Answer (032):**

Because quadrilateral $DFAC$ is cyclic, $\angle DFC = \angle DAC = \angle BAC$. Because quadrilateral $EFBC$ is cyclic, $\angle EFC = \angle EBC = 180^\circ - \angle ABC$. Hence $\angle EFD = \angle EFC - \angle DFC = 180^\circ - \angle ABC - \angle BAC = \angle ACB$. Applying the Law of Cosines to $\triangle DEF$ and $\triangle ABC$ gives

$$\cos\angle EFD = \cos\angle ACB = \frac{6^2 + 5^2 - 4^2}{2\cdot 6\cdot 5} = \frac{3}{4}$$

and

$$DE = \sqrt{7^2 + 2^2 - 2\cdot 2\cdot 7\cdot \frac{3}{4}} = 4\sqrt{2}.$$



Let $\overline{CF}$ intersect line AB at G. Because $\triangle ACG \sim \triangle FDG$ and $\triangle BCG \sim \triangle FEG$,

$$3 = \frac{AC}{FD} = \frac{CG}{DG} = \frac{GA}{GF} \quad \text{and} \quad \frac{5}{7} = \frac{BC}{FE} = \frac{CG}{EG} = \frac{GB}{GF}.$$

Therefore

$$\frac{21}{5} = \frac{EG}{DG} = \frac{ED + DG}{DG} \quad \text{and} \quad \frac{21}{5} = \frac{GA}{GB} = \frac{GB + BA}{GB}.$$

Solving for DG and GB yields $GD = \frac{5\sqrt{2}}{4}$ and $BG = \frac{5}{4}$, so

$$BE = BG + GD + DE = \frac{5}{4} + \frac{5\sqrt{2}}{4} + 4\sqrt{2} = \frac{5 + 21\sqrt{2}}{4}.$$

The requested sum is $5 + 21 + 2 + 4 = 32$.

14. **Answer (097):**

Suppose prime $p > 2$ divides $2019^8 + 1$. Then $2019^8 \equiv -1 \pmod{p}$. Squaring gives $2019^{16} \equiv 1 \pmod{p}$. If $2019^m \equiv 1 \pmod{p}$ for some $0 < m < 16$, it follows that

$$2019^{\gcd(m,16)} \equiv 1 \pmod{p}.$$

But $2019^8 \equiv -1 \pmod{p}$, so $\gcd(m, 16)$ cannot divide 8, which is a contradiction. Thus 2019^{16} is the least positive power of 2019 congruent to 1 $\pmod{p}$. By Fermat's Little Theorem, $2019^{p-1} \equiv 1 \pmod{p}$. It follows that $p = 16k + 1$ for some positive integer k. The least two primes of this form are 17 and 97. The least odd prime factor of $2019^8 + 1$ is not 17 because

$$2019 \equiv 13 \pmod{17} \quad \text{and} \quad 13^2 \equiv 169 \equiv -1 \pmod{17},$$

which implies $2019^8 \equiv 1 \not\equiv -1 \pmod{17}$. But $2019 \equiv -18 \pmod{97}$, so

$$\begin{aligned}(-18)^2 &= 324 \equiv 33 \pmod{97},\\ 33^2 &= 1089 \equiv 22 \pmod{97}, \text{ and}\\ 22^2 &= 484 \equiv -1 \pmod{97}.\end{aligned}$$

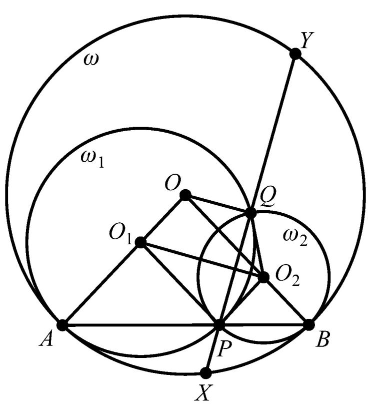
Thus the least odd prime factor is 97.

In fact, $2019^8 + 1 = 2 \cdot 97 \cdot p$, where p is the 25-digit prime

$$1423275002072658812388593.$$

15. **Answer (065):**

Let O, O_1, and O_2 denote the centers of ω, ω_1, and ω_2, respectively. Points O_1 and O_2 lie on $\overline{AO}$ and $\overline{BO}$, respectively, as shown in the figure below. It is clear that $\triangle AOB$, $\triangle AO_1P$, and $\triangle BO_2P$ are isosceles and similar to each other, and $\overline{PO_2} \parallel \overline{AO}$ and $\overline{PO_1} \parallel \overline{BO}$, and therefore PO_1OO_2 is a parallelogram. In particular, O and P lie on opposite sides of line O_1O_2. Also note that P and Q lie on opposite sides of line O_1O_2.



Because PO_1OO_2 is a parallelogram, $OO_2 = O_1P = O_1Q$ and $OO_1 = O_2P = O_2Q$. It follows from the last two equations that $\triangle OO_1O_2$ is congruent to $\triangle QO_2O_1$ by SSS. Then O_1OQO_2 is a trapezoid with $\overline{OQ} \parallel \overline{O_1O_2}$. Because $\overline{PQ}$ is the common chord of ω_1 and ω_2, $\overline{O_1O_2} \perp \overline{PQ}$. Thus $\overline{OQ} \perp \overline{PQ}$, and therefore Q is the midpoint of $\overline{XY}$ and $QX = QY = \frac{11}{2}$. By the Power of a Point Theorem,

$$15 = AP \cdot PB = PX \cdot PY = (QX - PQ)(PQ + QY) = \frac{121}{4} - PQ^2,$$

so $PQ^2 = \frac{121}{4} - 15 = \frac{61}{4}$. The requested sum is $61 + 4 = 65$.

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