MAA AMC

American Mathematics Competitions

Solutions Pamphlet

MAA American Mathematics Competitions

36th Annual

AIME II

American Invitational Mathematics Examination II

Wednesday, March 21, 2018

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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Correspondence about the problems/solutions for this AIME and orders for any publications should be addressed to:

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The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

Jonathan M. Kane
AIME Chair

1. **ANSWER (800):**

   Let x be the number of meters from A to B. Then the distance from B to C is $1800 - x$ meters. Because Paul runs 4 times as fast as Eve, he covers $\frac{4}{5}$ of the distance from B to C while Eve covers $\frac{1}{5}$ of the distance, so Paul has run $\frac{4}{5}(1800 - x)$ meters when he meets Eve. When Paul meets Ina he has run $\frac{8}{5}(1800 - x)$ meters, and Ina has run x meters. Because Paul runs twice as fast as Ina, he must have run $2x$ meters in the time that he ran $\frac{8}{5}(1800 - x)$ meters. Thus $2x = \frac{8}{5}(1800 - x)$, and $x = 800$.

2. **ANSWER (112):**

   Calculating the first few values of a_k shows that they repeat with period 10 and $a_8 = 7$. It follows that $a_{2018} \cdot a_{2020} \cdot a_{2022} = a_8 \cdot a_0 \cdot a_2 = 7 \cdot 2 \cdot 8 = 112$.

3. **ANSWER (371):**

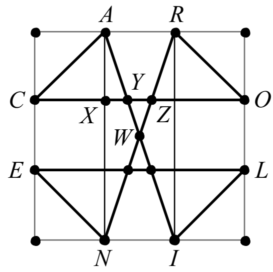
   The integer b must satisfy the equations $3b + 6 = j^2$ and $2b + 7 = k^3$ for positive integers j and k. Because $3b + 6$ is a perfect square and is divisible by 3, it must be divisible by 9, implying that $b \equiv 1 \pmod 3$. The perfect cubes modulo 9 are 0, 1, and 8, so $2b + 7 \equiv 0, 1, \text{ or } 8 \pmod 9$, from which $b \equiv 1$, 6, or $5 \pmod 9$. Taken together, the conditions imply that $b \equiv 1 \pmod 9$. Therefore $k^3 = 2b + 7$ is an odd multiple of 9, so k is an odd multiple of 3. Only if $k = 3$ or 9 can b have a value less than 1000. If $k = 3$, then $b = 10$, and in this case $3b + 6 = 36$, which is a perfect square. If $k = 9$, then $b = 361$, and $3b + 6 = 1089$, which is also a perfect square. Thus the requested sum is $10 + 361 = 371$.

   OR

   If $b < 1000$, then $27_b < 2007$. The number $27_b = 2b + 7$ must be odd, and the odd cubes up to 2007 are $1^3 = 1$, $3^3 = 27$, $5^3 = 125$, $7^3 = 343$, $9^3 = 729$, and $11^3 = 1331$. The corresponding values of b are -3, 10, 59, 168, 361, and 662, making $3b + 6$ equal to -3, 36, 183, 510, 1089, and 1992, respectively. Only 36 (when $b = 10$) and 1089 (when $b = 361$) are perfect squares. The requested sum is $10 + 361 = 371$.

4. **ANSWER (023):**

The interior angles of $CAROLINE$ are equal to $135°$. Lines AR, OL, IN, and EC enclose a square, and $CAROLINE$ is inscribed in the square, with $\overline{CO} \parallel \overline{AR} \parallel \overline{NI} \parallel \overline{EL}$ and $\overline{AN} \parallel \overline{CE} \parallel \overline{OL} \parallel \overline{RI}$, as shown.



In particular, $CARO$ is a trapezoid with bases 1 and 3 and height 1, and $ARIN$ is a 1×3 rectangle. Segment $\overline{CO}$ intersects segments $\overline{AN}$, $\overline{AI}$, and $\overline{RN}$ at X, Y, and Z, respectively, and segments $\overline{AI}$ and $\overline{RN}$ intersect at W. Then $CX = AX = 1$. Because $\triangle AYX \sim \triangle AIN$, it follows that $\frac{XY}{AX} = \frac{NI}{AN} = \frac{1}{3}$, so $XY = \frac{1}{3}$ and $CY = \frac{4}{3}$. For any region $\mathcal{R}$, let $[\mathcal{R}]$ denote the area of $\mathcal{R}$. Then $[CAY] = \frac{2}{3}$. By symmetry, $OZ = CY = \frac{4}{3}$ and $YZ = CO - CY - OZ = 3 - \frac{8}{3} = \frac{1}{3}$. The distance from W to $\overline{YZ}$ is $\frac{1}{2}$, so $[YZW] = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{12}$. By symmetry, $[CORNELIA] = 4[CAY] + 2[YZW] = \frac{8}{3} + \frac{1}{6} = \frac{17}{6}$. The requested sum is $17 + 6 = 23$.

5. **ANSWER (074):**

Multipling these equations together yields

$$\begin{aligned} x^2y^2z^2 &= (xy)(yz)(zx) \\ &= (-80 - 320i)(60)(-96 + 24i) \\ &= 80 \cdot 60 \cdot 24(-1 - 4i)(-4 + i) \\ &= 240^2 \cdot 2(-1 - 4i)(-4 + i) \\ &= 240^2(16 + 30i) \\ &= 240^2(5 + 3i)^2. \end{aligned}$$

Thus $xyz = \pm 240(5 + 3i)$. Dividing this equation by each of the three given equations yields $x = \pm(20 + 12i)$, $y = \pm(-10 - 10i)$, and $z = \pm(-3 + 3i)$. Hence $x + y + z = \pm(7 + 5i)$ and $(a, b) = \pm(7, 5)$. Thus $a^2 + b^2 = 7^2 + 5^2 = 74$.

6. **ANSWER (037):**

Let the polynomial be P. Note that

$$\begin{aligned} P(x) &= (x^4 - 2x^2 + 3x - 2) + 2a(x^3 + x^2 - 2x) \\ &= (x-1)(x+2)(x^2 - x + 1) + 2ax(x-1)(x+2) \\ &= (x-1)(x+2)\big(x^2 + (2a-1)x + 1\big). \end{aligned}$$

The roots of $P(x)$ will be real precisely when the roots of $x^2 + (2a-1)x + 1$ are real, which happens if and only if the discriminant $(2a-1)^2 - 4 \geq 0$. This is equivalent to $a \leq -\frac{1}{2}$ or $a \geq \frac{3}{2}$. The desired probability is therefore the probability that a randomly chosen real number from $[-20, 18]$ does not lie in $\left(-\frac{1}{2}, \frac{3}{2}\right)$, which is $\frac{18-(-20)-2}{18-(-20)} = \frac{36}{38} = \frac{18}{19}$. The requested sum is $18 + 19 = 37$.

7. **ANSWER (020):**

For $1 \leq k \leq 2450$, $[AP_kQ_k] = \frac{k}{2450}[ABC]$, where the brackets denote area. Because the ratio of the areas of similar figures is the square of the ratio of the corresponding side lengths,

$$P_kQ_k = (BC)\sqrt{\frac{k}{2450}} = 5\sqrt{3}\sqrt{\frac{k}{2450}} = 5\sqrt{\frac{3k}{2 \cdot 35^2}} = \frac{1}{7}\sqrt{\frac{3k}{2}}.$$

This last expression is rational if and only if $k = 6j^2$ for some positive integer j. Because $k \leq 2450$, this is satisfied by $j = 1, 2, 3, \ldots, 20$, giving 20 possible values of k.

8. **ANSWER (556):**

Let `r` and `R` denote jumps right from (x, y) to $(x+1, y)$ and $(x+2, y)$, respectively, and let `u` and `U` denote jumps upward from (x, y) to $(x, y+1)$ and $(x, y+2)$, respectively. Then a jump sequence $\mathcal{S}$ from $(0, 0)$ to $(4, 4)$ must contain the jumps

- exactly one of `rrR` (or any of its permutations), `rrrr`, or `RR`;
- exactly one of `uuU` (or any of its permutations), `uuuu`, or `UU`.

Let $\mathcal{N}(\mathcal{S}_1, \mathcal{S}_2)$ denote the number of jump sequences whose respective subsequences of rightward and upward jumps are $\mathcal{S}_1$ (or some permutation thereof) and $\mathcal{S}_2$ (or some permutation thereof). The following table counts all the possible sequences of jumps.

$(\mathcal{S}_1, \mathcal{S}_2)$	$\mathcal{N}(\mathcal{S}_1, \mathcal{S}_2)$
$(\texttt{rrrr}, \texttt{uuuu})$	$\binom{8}{4} = 70$
$(\texttt{RR}, \texttt{UU})$	$\binom{4}{2} = 6$
$(\texttt{RR}, \texttt{uuuu})$ $(\texttt{rrrr}, \texttt{UU})$	$\binom{6}{2} = 15$
$(\texttt{rrR}, \texttt{uuuu})$ $(\texttt{rrrr}, \texttt{uuU})$	$\binom{7}{4,2,1} = 105$
$(\texttt{rrR}, \texttt{UU})$ $(\texttt{RR}, \texttt{uuU})$	$\binom{5}{2,2,1} = 30$
$(\texttt{rrR}, \texttt{uuU})$	$\binom{6}{2,2,1,1} = 180$

The total number of jump sequences is obtained by adding the numbers in the second column that correspond to each row represented in the first column of the table. The requested number of jump sequences is therefore $70 + 6 + 2 \cdot 15 + 2 \cdot 105 + 2 \cdot 30 + 180 = 556$.

OR

For integers a and b, let $N(a, b)$ be the number of jump sequences that begin at $(0, 0)$ and end at (a, b). Then $N(0, 0) = 1$ and $N(a, b) = 0$ if $a < 0$ or $b < 0$. For nonnegative integers a and b with $a + b \geq 1$, the given conditions imply the following recursion:

$$N(a, b) = N(a-1, b) + N(a, b-1) + N(a-2, b) + N(a, b-2).$$

Using this recursion to complete the following table shows that the requested number of jump sequences is 556.

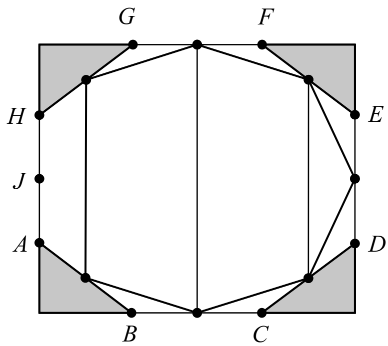
	5	20	71	207	556	$(4, 4)$
	3	10	32	84	207	
	2	5	14	32	71	
	1	2	5	10	20	
$(0, 0)$	1	1	2	3	5	

9. **ANSWER (184):**

Form the convex heptagon S whose vertices are the midpoints of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FG}$, and $\overline{GH}$. The heptagon S can be partitioned into two trapezoids with bases of lengths 17 and 23 and height $\frac{19}{2}$, and one triangle with base length 17 and height 4. Thus S has area

$$2 \cdot \frac{17 + 23}{2} \cdot \frac{19}{2} + \frac{17 \cdot 4}{2} = 414.$$

The 7 centroids of the 7 triangles in the problem all lie on the line segments from the point J to these 7 midpoints, and each centroid is $\frac{2}{3}$ the distance from J to its corresponding midpoint. As a result, the required heptagon R whose vertices are these 7 centroids is the image of S under a dilation centered at J with ratio $\frac{2}{3}$. The requested area of R is therefore $414 \cdot \left(\frac{2}{3}\right)^2 = 184$.



10. **ANSWER (756):**

Suppose $f(x)$ has the desired property with i fixed points and j additional elements that each maps to a fixed point. Then each of the remaining $5 - i - j$ elements maps to one of the j elements that maps to a fixed point. For a given i and j, there are $\binom{5}{i}$ ways to select the i fixed points, $\binom{5-i}{j}$ ways to select the j elements to map to these fixed points, i^j ways to define a mapping of those j points to the fixed points, and j^{5-i-j} ways to map the remaining points. Thus the required number of functions is

$$\sum_{i+j\le 5} \binom{5}{i}\binom{5-i}{j} i^j j^{5-i-j}.$$

Note that there are no functions with $i = 0$ or with $j = 0$ and $i < 5$. The identity function is the one function with $j = 0$ and $i = 5$. The sum contains 21 terms, of which 11 are nonzero. The sum equals $20 + 120 + 60 + 5 + 60 + 240 + 80 + 60 + 90 + 20 + 1 = 756$.

11. **ANSWER (461):**

For positive integers n and $k \le n$, call a permutation of $1, 2, 3, \ldots, n$ *k-stable* if its first k terms are a permutation of $1, 2, 3, \ldots, k$. Let $a_{n,k}$ be the number of permutations of $1, 2, 3, \ldots, n$ such that k is the least positive integer for which the permutation is *k-stable*. The quantity to be found is $a_{6,6}$.

Note that $\sum_{k=1}^{n} a_{n,k} = n!$. Because every permutation of $1, 2, 3, \ldots, k$ can be extended to a permutation of $1, 2, 3, \ldots, n$ in $(n-k)!$ ways, it follows that $a_{n,k} = (n-k)!a_{k,k}$. Thus the sequence $(a_{n,n})$ satisfies the recursion $a_{n,n} = n! - \sum_{k=1}^{n-1}(n-k)!a_{k,k}$. It is easily verified that $a_{1,1} = a_{2,2} = 1$ and $a_{3,3} = 3$. Therefore

$$\begin{aligned} a_{4,4} &= 24 - (6 \cdot 1 + 2 \cdot 1 + 1 \cdot 3) = 13, \\ a_{5,5} &= 120 - (24 \cdot 1 + 6 \cdot 1 + 2 \cdot 3 + 1 \cdot 13) = 71, \text{ and} \\ a_{6,6} &= 720 - (120 \cdot 1 + 24 \cdot 1 + 6 \cdot 3 + 2 \cdot 13 + 1 \cdot 71) = 461. \end{aligned}$$

12. **ANSWER (112):**

Let a, b, c, and d denote AP, BP, CP, and DP, respectively, and let $\theta = \angle CPD$. The statement about equal areas says that $\frac{1}{2}(ab + cd)\sin\theta = \frac{1}{2}(ad + bc)\sin(\pi - \theta)$, which implies $(a-c)(d-b) = 0$. The cases where $a = c$ and $b = d$ are similar, so assume that $a = c$. By the Law of Cosines

$$\begin{aligned} a^2 + b^2 + 2ab\cos\theta &= 196 \text{ and} \\ a^2 + b^2 - 2ab\cos\theta &= 100, \end{aligned}$$

so $a^2 + b^2 = 148$ and $ab\cos\theta = 24$. Also

$$\begin{aligned} a^2 + d^2 + 2ad\cos\theta &= 260 \text{ and} \\ a^2 + d^2 - 2ad\cos\theta &= 100, \end{aligned}$$

so $a^2 + d^2 = 180$ and $ad\cos\theta = 40$. Thus $d^2 - b^2 = 32$ and $\frac{d}{b} = \frac{5}{3}$, which yields $d = 5\sqrt{2}$, $b = 3\sqrt{2}$, $a = \sqrt{130}$, and $\cos^2\theta = \frac{16}{65}$ and $\sin^2\theta = \frac{49}{65}$. The requested area is

$$\frac{1}{2}(a+c)(b+d)\sin\theta = a(b+d)\sin\theta = \sqrt{130} \cdot 8\sqrt{2} \cdot \frac{7}{\sqrt{65}} = 112.$$

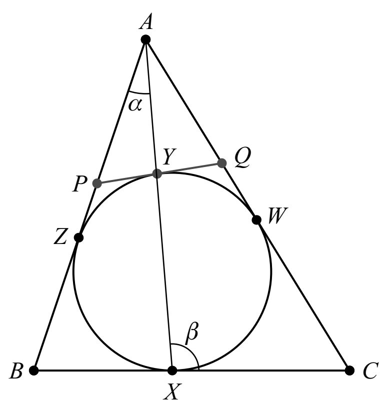
13. **ANSWER (647):**

Call a sequence of dice rolls an *even sequence* if 1-2-3 first occurs as the last three rolls of a sequence of an even number of rolls, and an *odd sequence* if it first occurs as the last three rolls of a sequence of an odd number of rolls. Let a be the probability that Misha rolls an odd sequence. Let b be the conditional probability that Misha rolls an odd sequence given that her first roll is a 1, and let c be the conditional probability that Misha rolls an odd sequence given that her first two rolls are 1-2. Misha can roll an odd sequence either by rolling a 1 on her first roll with probability $\frac{1}{6}$ and then completing an odd sequence with probability b, or by rolling something other than a 1 on her first roll with probability $\frac{5}{6}$ followed by rolling an even sequence with probability $1-a$. Similarly, given that Misha's first roll is a 1, she can roll a 1 on her second roll and then, ignoring the first roll, complete an even sequence with probability $1-b$; she can roll a 2 on her second roll and then complete an odd sequence with probability c; or she can roll another number followed by an odd sequence with probability a. Finally, given that Misha's first two rolls are 1-2, she can roll a 1 on her third roll and then, ignoring the first two rolls, complete an odd sequence with probability b; roll a 3; or roll another number followed by an even sequence with probability $1-a$. Thus

$$\begin{aligned} a &= \frac{1}{6}b + \frac{5}{6}(1-a) \\ b &= \frac{1}{6}(1-b) + \frac{1}{6}c + \frac{4}{6}a \\ c &= \frac{1}{6}b + \frac{1}{6} + \frac{4}{6}(1-a). \end{aligned}$$

This system can be solved by substitution to get $a = \frac{216}{431}$, $b = \frac{221}{431}$, and $c = \frac{252}{431}$. The requested sum is $216 + 431 = 647$.

14. **ANSWER (227):**



Let sides $\overline{AB}$ and $\overline{AC}$ be tangent to ω at Z and W, respectively. Let $\alpha = \angle BAX$ and $\beta = \angle AXC$. Because $\overline{PQ}$ and $\overline{BC}$ are both tangent to ω and $\angle YXC$ and $\angle QYX$ subtend the same arc of ω, it follows that $\angle AYP = \angle QYX = \angle YXC = \beta$. By equal tangents, $PZ = PY$. Applying the Law of Sines to $\triangle APY$ yields

$$\frac{AZ}{AP} = 1 + \frac{ZP}{AP} = 1 + \frac{PY}{AP} = 1 + \frac{\sin\alpha}{\sin\beta}.$$

Similarly, applying the Law of Sines to $\triangle ABX$ gives

$$\frac{AZ}{AB} = 1 - \frac{BZ}{AB} = 1 - \frac{BX}{AB} = 1 - \frac{\sin\alpha}{\sin\beta}.$$

It follows that

$$2 = \frac{AZ}{AP} + \frac{AZ}{AB} = \frac{AZ}{3} + \frac{AZ}{7},$$

implying $AZ = \frac{21}{5}$. Applying the same argument to $\triangle AQY$ yields

$$2 = \frac{AW}{AQ} + \frac{AW}{AC} = \frac{AZ}{AQ} + \frac{AZ}{AC} = \frac{21}{5}\left(\frac{1}{AQ} + \frac{1}{8}\right),$$

from which $AQ = \frac{168}{59}$. The requested sum is $168 + 59 = 227$.

15. **ANSWER (185):**

Because $f(n)$ and $f(n+1)$ can differ by at most 3 for $n = 0, 1, 2, 3, 4, 5$, if f decreases k times, then $12 = f(6) \le -1 \cdot k + 3(6-k) = 18 - 4k$. This implies that f can decrease at most $k = 1$ time.

If f never decreases, then $f(n+1) - f(n) \in \{1, 2, 3\}$ for all n. Let a, b, and c denote the number of times this difference is 1, 2, and 3, respectively. Then $a + b + c = 6$ and $a + 2b + 3c = 12$. Subtracting the first equation from the second yields $b + 2c = 6$, so $(b, c) = (6, 0)$, $(4, 1)$, $(2, 2)$, or $(0, 3)$. These yield $a = 0$, 1, 2, or 3, respectively, so the number of possibilities in this case is

$$\binom{6}{0,6,0} + \binom{6}{1,4,1} + \binom{6}{2,2,2} + \binom{6}{3,0,3} = 1 + 30 + 90 + 20 = 141.$$

If f decreases from $f(0)$ to $f(1)$ or from $f(5)$ to $f(6)$, then $f(2)$ or $f(4)$, respectively, is determined. The only solutions to $a + b + c = 4$ and $a + 2b + 3c = 10$ are $(a, b, c) = (1, 0, 3)$ and $(a, b, c) = (0, 2, 2)$, so the number of functions is

$$2\left[\binom{4}{1,0,3} + \binom{4}{0,2,2}\right] = 20.$$

Finally, suppose that $f(n+1) < f(n)$ for some $n = 1, 2, 3, 4$. Note that the condition $|(n+1) - (n-1)| \le |f(n+1) - f(n-1)|$ implies that $f(n+1) - f(n-1) \ge 2$, so it must be that $f(n) - f(n+1) = 1$ and

$$f(n+2) - f(n+1) = f(n) - f(n-1) = 3.$$

This means that $f(n-1)$ and $f(n+2)$ are uniquely determined by the value of $f(n)$, and, in particular, that $f(n+2) - f(n-1) = 5$. As a result, there are three more values of f to determine, and they must provide a total increase of 7. The only ways to do this are either to have two differences of 3 and one difference of 1, which can be arranged in 3 ways, or to have one difference of 3 and two differences of 2, which can be arranged in 3 ways. Thus for each of the 4 possibilities for n, there are 6 ways to arrange the increases, giving a total of 24 ways.

The total number of functions is $141 + 20 + 24 = 185$.

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