MAA AMC
American Mathematics Competitions

American Mathematics Competitions

36th Annual

AIME II

American Invitational Mathematics Examination II
Wednesday, March 21, 2018

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR COMPETITION MANAGER GIVES THE SIGNAL TO BEGIN.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor penalties for wrong answers.
3. Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on the exam will require the use of a calculator.
4. A combination of your AIME score and your American Mathematics Contest 12 score are used to determine eligibility for participation in the USA Mathematical Olympiad (USAMO). A combination of your AIME score and your American Mathematics Contest 10 score are used to determine eligibility for participation in the USA Junior Mathematical Olympiad (USAJMO). The USAMO and USAJMO will be given on WEDNESDAY and THURSDAY, April 18 and 19, 2018.
5. Record all your answers, and identification information, on the AIME answer sheet. Only the answer sheet will be collected from you.

The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

1. Points A, B, and C lie in that order along a straight path where the distance from A to C is 1800 meters. Ina runs twice as fast as Eve, and Paul runs twice as fast as Ina. The three runners start running at the same time with Ina starting at A and running toward C, Paul starting at B and running toward C, and Eve starting at C and running toward A. When Paul meets Eve, he turns around and runs toward A. Paul and Ina both arrive at B at the same time. Find the number of meters from A to B.

2. Let $a_0 = 2$, $a_1 = 5$, and $a_2 = 8$, and for $n > 2$ define a_n recursively to be the remainder when $4(a_{n-1} + a_{n-2} + a_{n-3})$ is divided by 11. Find $a_{2018} \cdot a_{2020} \cdot a_{2022}$.

3. Find the sum of all positive integers $b < 1000$ such that the base-b integer 36_b is a perfect square and the base-b integer 27_b is a perfect cube.

4. In equiangular octagon $CAROLINE$, $CA = RO = LI = NE = \sqrt{2}$ and $AR = OL = IN = EC = 1$. The self-intersecting octagon $CORNELIA$ encloses six non-overlapping triangular regions. Let K be the area enclosed by $CORNELIA$, that is, the total area of the six triangular regions. Then $K = \frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

5. Suppose that x, y, and z are complex numbers such that $xy = -80 - 320i$, $yz = 60$, and $zx = -96 + 24i$, where $i = \sqrt{-1}$. Then there are real numbers a and b such that $x + y + z = a + bi$. Find $a^2 + b^2$.

6. A real number a is chosen randomly and uniformly from the interval $[-20, 18]$. The probability that the roots of the polynomial

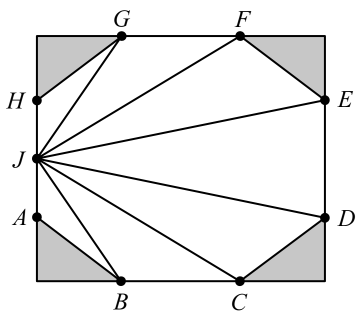
$$x^4 + 2ax^3 + (2a - 2)x^2 + (-4a + 3)x - 2$$

are all real can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

7. Triangle ABC has side lengths $AB = 9$, $BC = 5\sqrt{3}$, and $AC = 12$. Points $A = P_0, P_1, P_2, \ldots, P_{2450} = B$ are on segment $\overline{AB}$ with P_k between P_{k-1} and P_{k+1} for $k = 1, 2, \ldots, 2449$, and points $A = Q_0, Q_1, Q_2, \ldots, Q_{2450} = C$ are on segment $\overline{AC}$ with Q_k between Q_{k-1} and Q_{k+1} for $k = 1, 2, \ldots, 2449$. Furthermore, each segment $\overline{P_k Q_k}$, $k = 1, 2, \ldots, 2449$, is parallel to $\overline{BC}$. The segments cut the triangle into 2450 regions, consisting of 2449 trapezoids and 1 triangle. Each of the 2450 regions has the same area. Find the number of segments $\overline{P_k Q_k}$, $k = 1, 2, \ldots, 2450$, that have rational length.

8. A frog is positioned at the origin in the coordinate plane. From the point (x, y), the frog can jump to any of the points $(x + 1, y)$, $(x + 2, y)$, $(x, y + 1)$, or $(x, y + 2)$. Find the number of distinct sequences of jumps in which the frog begins at $(0, 0)$ and ends at $(4, 4)$.

9. Octagon $ABCDEFGH$ with side lengths $AB = CD = EF = GH = 10$ and $BC = DE = FG = HA = 11$ is formed by removing four $6-8-10$ triangles from the corners of a 23×27 rectangle with side $\overline{AH}$ on a short side of the rectangle, as shown. Let J be the midpoint of $\overline{HA}$, and partition the octagon into 7 triangles by drawing segments $\overline{JB}$, $\overline{JC}$, $\overline{JD}$, $\overline{JE}$, $\overline{JF}$, and $\overline{JG}$. Find the area of the convex polygon whose vertices are the centroids of these 7 triangles.



10. Find the number of functions $f(x)$ from $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3, 4, 5\}$ that satisfy $f(f(x)) = f(f(f(x)))$ for all x in $\{1, 2, 3, 4, 5\}$.

11. Find the number of permutations of $1, 2, 3, 4, 5, 6$ such that for each k with $1 \le k \le 5$, at least one of the first k terms of the permutation is greater than k.

12. Let $ABCD$ be a convex quadrilateral with $AB = CD = 10$, $BC = 14$, and $AD = 2\sqrt{65}$. Assume that the diagonals of $ABCD$ intersect at point P, and that the sum of the areas of $\triangle APB$ and $\triangle CPD$ equals the sum of the areas of $\triangle BPC$ and $\triangle APD$. Find the area of quadrilateral $ABCD$.

13. Misha rolls a standard, fair six-sided die until she rolls 1-2-3 in that order on three consecutive rolls. The probability that she will roll the die an odd number of times is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

14. The incircle ω of $\triangle ABC$ is tangent to $\overline{BC}$ at X. Let $Y \neq X$ be the other intersection of $\overline{AX}$ and ω. Points P and Q lie on $\overline{AB}$ and $\overline{AC}$, respectively, so that $\overline{PQ}$ is tangent to ω at Y. Assume that $AP = 3$, $PB = 4$, $AC = 8$, and $AQ = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

15. Find the number of functions f from $\{0, 1, 2, 3, 4, 5, 6\}$ to the integers such that $f(0) = 0$, $f(6) = 12$, and

$$|x - y| \le \big|f(x) - f(y)\big| \le 3|x - y|$$

for all x and y in $\{0, 1, 2, 3, 4, 5, 6\}$.

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Your competition manager will receive a copy of the
2018 AIME Solution Pamphlet with the scores.

Questions and comments about problems
and solutions for this exam should be sent to:

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Send questions and comments about administrative arrangements to:

amcinfo@maa.org

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