MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

35th Annual

AIME I

American Invitational Mathematics Examination I

Thursday, March 7, 2017

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Correspondence about the problems/solutions for this AIME and orders for any publications should be addressed to:

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The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

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1. **ANSWER (390):**

There are $\binom{15}{3} = 455$ ways to select 3 distinct vertices from among the 15 points. All of these selections give vertices of a triangle with positive area except for the selections consisting of 3 collinear points. There are $\binom{5}{3} = 10$ ways to select 3 points on side $\overline{AB}$, $\binom{6}{3} = 20$ ways to select 3 points on side $\overline{BC}$, and $\binom{7}{3} = 35$ ways to select 3 points on side $\overline{CA}$. Thus there are $455 - 10 - 20 - 35 = 390$ triangles with positive area.

2. **ANSWER (062):**

If the remainder is the same when each of 702, 787, and 855 is divided by m, then m must be a factor of $787 - 702 = 85$ and of $855 - 787 = 68$. The only common factor of 85 and 68 greater than 1 is 17, so m must be 17, and the common remainder is $r = 5$. Similarly, n must be a factor of $722 - 412 = 310$ and of $815 - 722 = 93$. The only common factor of 310 and 93 greater than 1 is 31, so n must be 31, and the common remainder is $9 \neq 5$. The requested sum is $17 + 31 + 5 + 9 = 62$.

3. **ANSWER (069):**

Any 20 consecutive positive integers have units digits whose sum is $0 + 1 + 2 + \cdots + 9 + 0 + 1 + 2 + \cdots + 9 = 2(1 + 9) + 2(2 + 8) + 2(3 + 7) + 2(4 + 6) + 2(5) = 90$. Therefore $d_{n+20} = d_n$ for all integers $n \geq 1$. Thus one only needs to calculate d_n for $1 \leq n \leq 20$, and

$$(d_1, d_2, \ldots, d_{20}) = (1, 3, 6, 0, 5, 1, 8, 6, 5, 5, 6, 8, 1, 5, 0, 6, 3, 1, 0, 0).$$

Therefore

$$\sum_{n=1}^{2017} d_n = 101 \sum_{n=1}^{20} d_n - \sum_{n=18}^{20} d_n = 101 \cdot 70 - 1 = 7069.$$

The requested remainder is 69.

4. **ANSWER (803):**

Let A, B, and C be the vertices of the base triangle so that $AB = AC = 20$ and $BC = 24$. Let D be the fourth vertex of the pyramid, P the foot of the altitude of the pyramid from D to its base, and M the midpoint of side $\overline{BC}$. By symmetry, P lies on $\overline{AM}$. By the Pythagorean Theorem, $AM = \sqrt{20^2 - 12^2} = 16$ and $DM^2 = 25^2 - 12^2 = 481$. Then $AP^2 + DP^2 = 25^2$ and $(16 - AP)^2 + DP^2 = DM^2 = 481$. Thus

$$AP^2 + DP^2 - \left((16 - AP)^2 + DP^2\right) = 625 - 481 = 144,$$

and so $32(AP) - 256 = 144$. Solving for AP yields $AP = \frac{25}{2}$. Therefore $DP = \sqrt{625 - \left(\frac{25}{2}\right)^2} = \frac{25}{2}\sqrt{3}$, and the volume of the pyramid is $\frac{1}{3} \cdot \frac{25}{2}\sqrt{3} \cdot \frac{1}{2} \cdot 24 \cdot 16 = 800\sqrt{3}$. The requested sum is $800 + 3 = 803$.

5. **ANSWER (321):**

The integer parts of the two representations must match, so $\underline{a}\,\underline{b}_{\text{eight}} = \underline{b}\,\underline{b}_{\text{twelve}}$. This implies $8a + b = 12b + b$, from which $a = \frac{3}{2}b$. Because both a and b must be positive integers less than 8, the only two possibilities for the ordered pair (b, a) are $(2, 3)$ and $(4, 6)$. For $b = 4$ and $a = 6$ the fractional part of the number equals $0.46_{\text{twelve}} = \frac{4}{12} + \frac{6}{144} = \frac{3}{8} = 0.30_{\text{eight}}$, so d would be 0. On the other hand if $b = 2$ and $a = 3$, then the fractional part is $0.23_{\text{twelve}} = \frac{2}{12} + \frac{3}{144} = \frac{3}{16} = 0.14_{\text{eight}}$, and $c = 1$ and $d = 4$. Indeed, $32.14_{\text{eight}} = 22.23_{\text{twelve}}$. The requested number is 321.

6. **ANSWER (048):**

The vertices of the triangle partition the circle into three arcs with degree measures $2x$, $2x$, and $360 - 4x$. The chord fails to intersect the triangle if and only if both of the chosen points are within the same arc. This occurs with probability

$$1 - \frac{14}{25} = \frac{11}{25} = \left(\frac{2x}{360}\right)^2 + \left(\frac{2x}{360}\right)^2 + \left(\frac{360 - 4x}{360}\right)^2.$$

Substituting $y = \frac{x}{180}$ transforms the equation into

$$2y^2 + (1 - 2y)^2 = \frac{11}{25}.$$

The solutions for y are $\frac{1}{5}$ and $\frac{7}{15}$, and the corresponding solutions for x are 36 and 84, respectively, so the requested difference is $84 - 36 = 48$.

7. **ANSWER (564):**

It follows from

$$\binom{6}{a+b} = \binom{6}{6-(a+b)}$$

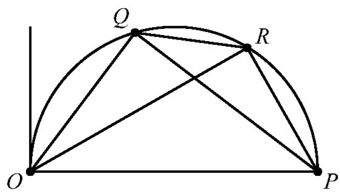
that

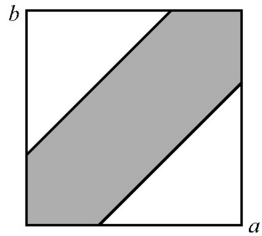
$$S = \sum_{a+b \le 6} \binom{6}{a}\binom{6}{b}\binom{6}{6-(a+b)} = \sum_{a+b+c=6} \binom{6}{a}\binom{6}{b}\binom{6}{c}.$$

For given values of a, b, and c, the term $\binom{6}{a}\binom{6}{b}\binom{6}{c}$ corresponds to the number of choices when selecting a elements from $\{1, 2, 3, 4, 5, 6\}$, b elements from $\{7, 8, 9, 10, 11, 12\}$, and c elements from $\{13, 14, 15, 16, 17, 18\}$. Thus S is equal to the number of 6-element subsets of $\{1, 2, 3, \ldots, 18\}$, namely $\binom{18}{6} = 18{,}564$. The requested remainder is 564.

8. **Answer (041):**

Because Q and R are on the same side of line OP and $\angle OQP = \angle ORP = 90^\circ$, points O, P, Q, and R lie on a semicircle with diameter $\overline{OP}$, as shown. Because the radius of the semicircle is 100, the length of chord $\overline{QR}$ is less than or equal to 100 if and only if the arc of the semicircle between Q and R has central angle of no more than 60°. This happens exactly when the inscribed angle $\angle QOR$ does not exceed 30°; that is $|a - b| \le 30$.





Each pair of values a and b corresponds to a unique point (a, b) in the ab-plane, with $0 < a < 75$ and $0 < b < 75$, and vice versa. The points in the ab-plane that meet the requirement are those within the square that satisfy the relation $-30 \le a - b \le 30$, that is, the shaded hexagonal region within the square, as shown. The two triangles inside the square and outside the hexagon are both isosceles right triangles with leg length $75 - 30 = 45$. Hence the requested probability is $1 - \frac{45^2}{75^2} = \frac{16}{25}$. The requested sum is $16 + 25 = 41$.

9. **Answer (045):**

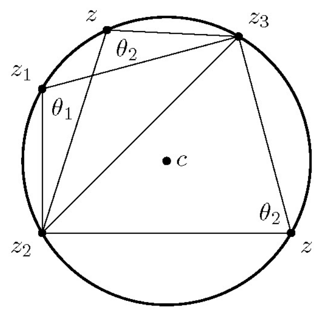
Because $100 \equiv 1 \pmod{99}$, $a_n \equiv a_{n-1} + n \pmod{99}$. Thus $a_n \equiv \sum_{k=10}^{n} k \pmod{99}$, which is equivalent to $\frac{(n+10)(n-9)}{2}$. Because $n + 10$ and $n - 9$ cannot both be multiples of 3, a_n is a multiple of 99 if and only if one of the following holds.

- $n - 9$ is a multiple of 99. The least $n > 10$ is 108.
- $n + 10$ is a multiple of 99. The least $n > 10$ is 89.
- $n + 10$ is a multiple of 9 while $n - 9$ is a multiple of 11. The least $n > 10$ is 53.
- $n + 10$ is a multiple of 11 while $n - 9$ is a multiple of 9. The least $n > 10$ is 45.

The requested minimum is 45.

10. **ANSWER (056):**

Let $\frac{z_3-z_1}{z_2-z_1} = r_1 \operatorname{cis}(\theta_1)$, where $0^\circ < \theta_1 < 180^\circ$.



If z is on or below the line through z_2 and z_3, then $\frac{z-z_2}{z-z_3} = r_2 \operatorname{cis}(\theta_2)$, where $0^\circ < \theta_2 < 180^\circ$. Because $r_1 \operatorname{cis}(\theta_1) \cdot r_2 \operatorname{cis}(\theta_2) = r_1 \cdot r_2 \cdot \operatorname{cis}(\theta_1 + \theta_2)$ is real, it follows that $\theta_1 + \theta_2 = 180^\circ$, meaning that z_1, z_2, z_3, and z lie on a circle. On the other hand, if z is above the line through z_2 and z_3, then $\frac{z-z_2}{z-z_3} = r_2 \operatorname{cis}(-\theta_2)$, where $0^\circ < \theta_2 < 180^\circ$. Because $r_1 \operatorname{cis}(\theta_1) \cdot r_2 \operatorname{cis}(\theta_2) = r_1 \cdot r_2 \cdot \operatorname{cis}(\theta_1 - \theta_2)$ is real, it follows that $\theta_1 = \theta_2$, meaning that z_1, z_2, z_3, and z lie on a circle. In either case, z must lie on the circumcircle of $\triangle z_1z_2z_3$, whose center is the intersection of the perpendicular bisectors of $\overline{z_1z_2}$ and $\overline{z_1z_3}$, namely, the lines $y = \frac{39+83}{2} = 61$ and $16(y-91) = -60(x-48)$. Thus the center of the circle is $c = 56 + 61i$. The imaginary part of z is maximal when z is at the top of the circle, and the real part of z is 56.

OR

Let $z = a + bi$, where a and b are real numbers. Then the given expression is

$$\begin{aligned}\frac{z_3 - z_1}{z_2 - z_1} \cdot \frac{z - z_2}{z - z_3} &= \frac{60 + 16i}{-44i} \cdot \frac{(a-18) + (b-39)i}{(a-78) + (b-99)i} \\ &= \frac{-4 + 15i}{11} \cdot \frac{\big((a-18) + (b-39)i\big)\big((a-78) - (b-99)i\big)}{(a-78)^2 + (b-99)^2}.\end{aligned}$$

This expression is real when the imaginary part of its numerator is 0, which means that

$$(-4 + 15i)\big((a-18) + (b-39)i\big)\big((a-78) - (b-99)i\big)$$

has imaginary part 0. That happens when

$$4\big((a-18)(b-99) - (a-78)(b-39)\big) + 15\big((a-18)(a-78) + (b-39)(b-99)\big) = 0,$$

which simplifies to $(a-56)^2 + (b-61)^2 = 1928$. Thus $a+bi$ lies on the circle centered at $56 + 61i$ with radius $\sqrt{1928}$. When b is maximal, z is at the top of the circle, and $a = 56$.

11. **ANSWER (360):**

In an array with $m = 5$ rename all entries less than 5 as L and all entries greater than 5 as G. Then one row of the array will contain the entries L5G in some order. The other two rows will either contain entries LLL and GGG or contain entries LLG and LGG. In the first case there are $3! = 6$ ways to permute the L5G entries, 1 way to permute each of the LLL and GGG entries, and $3! = 6$ ways to permute the three rows of the array. In the second case there are 6 ways to permute the L5G entries, 3 ways to permute each of the LLG and LGG entries, and 6 ways to permute the three rows of the array. Thus there is a total of $6 \cdot (3^2 + 1^2) \cdot 6 = 360$ ways to arrange the four L, four G, and one 5 entries in an array with $m = 5$. For any one such array there are $4! = 24$ ways to replace the four L entries with 1, 2, 3, and 4 and $4! = 24$ ways to replace the four G entries with 6, 7, 8, and 9, so $Q = 360 \cdot 24^2 = 207{,}360$. The requested remainder is 360.

12. **ANSWER (252):**

Let T be a product-free subset of $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. First note that $1 \notin T$ because $1 \cdot 1 = 1$. Assume T is nonempty and let t be the least element of T. Note that if $t \geq 4$, then the least possible product of any two elements of T is $4^2 = 16 > 10$; hence any subset of $\{4, 5, 6, 7, 8, 9, 10\}$ will be product-free. Counting the empty set, there are $2^7 = 128$ such subsets.

Next, suppose that $t = 2$. Then $4 \notin T$, and 7 and 8 can be in T (or not). The remaining elements of U to consider are the multiples of 3 and 5. The possible subsets of T that contain only multiples of 3 are $\emptyset, \{3\}, \{6\}, \{9\}$, and $\{6, 9\}$. Similarly, the possible subsets of T that contain only multiples of 5 are $\emptyset, \{5\}$, and $\{10\}$. Thus there are $2 \cdot 2 \cdot 5 \cdot 3 = 60$ product-free subsets of U that contain 2.

If $t = 3$, then 1, 2, and 9 cannot be in T. Any of the 6 values in $\{4, 5, 6, 7, 8, 10\}$ may or may not be in T, providing $2^6 = 64$ product-free subsets with 3 as the least element.

Therefore the requested number of subsets is $128 + 60 + 64 = 252$.

13. **ANSWER (059):**

Each of the sets $\{2, 3, 4, 5, 6, 7\}$, $\{9, 10, \ldots, 24\}$, and $\{32, 33, \ldots, 62\}$ has the form $\{n + 1, n + 2, \ldots, m \cdot n\}$ and contains no perfect cubes. Thus $Q(7) > 1$, $Q(3) > 8$, and $Q(2) > 31$.

For a given m let k be the greatest integer such that $k^3 \leq Q(m) - 1$ and $m \cdot (Q(m) - 1) < (k + 1)^3$. It follows that $mk^3 + 1 \leq (k + 1)^3$, so $(m - 1)k^2 - 3k - 3 \leq 0$. Solving for k and using the fact that k is an integer yields

$$k \leq \left\lfloor \frac{3 + \sqrt{12m - 3}}{2m - 2} \right\rfloor .$$

For $m = 2$, 3, 4, and 8, it follows that $k \le 3$, 2, 1, and 0, respectively. Because $m \cdot (Q(m) - 1) < (k+1)^3$, it follows that $Q(2) < 33$, $Q(3) < 10$, $Q(4) < 3$, and $Q(8) < \frac{9}{8}$.

By definition $Q(m) \ge Q(m+1) \ge 1$ for all m. Thus $Q(m) = 1$ for all $m \ge 8$. Also, $2 \le Q(7) \le Q(6) \le Q(5) \le Q(4) \le 2$, so $Q(4) = Q(5) = Q(6) = Q(7) = 2$. Finally, $Q(3) = 9$ and $Q(2) = 32$. Therefore

$$\sum_{m=2}^{2017} Q(m) = 32 + 9 + 4 \cdot 2 + 2010 \cdot 1 = 2059.$$

The requested remainder is 59.

14. **ANSWER (896):**

Simplify the first equation as follows.

$$\begin{aligned}
\log_a(\log_a 2) + \log_a 24 - 128 &= a^{128} \\
\log_a(24 \log_a 2) &= 128 + a^{128} \\
\log_a\left(2^{24}\right) &= a^{128} \cdot a^{a^{128}} \\
2^{24} &= a^{a^{128} \cdot a^{a^{128}}} \\
\left(2^3\right)^{\left(2^3\right)} &= \left(a^{a^{128}}\right)^{\left(a^{a^{128}}\right)}
\end{aligned}$$

Thus $a^{a^{128}} = 2^3$. Letting $a = 2^{\frac{b}{128}}$ shows that $\left(2^{\frac{b}{128}}\right)^{2^b} = 2^3$, which reduces to $3 \cdot 128 = b \cdot 2^b$. This implies that $b = 6$, so $a = 2^{\frac{3}{64}}$, and

$$x = a^{a^{256}} = \left(2^{\frac{3}{64}}\right)^{2^{\left(\frac{3}{64} \cdot 256\right)}} = 2^{192}.$$

Then $x \equiv 0 \pmod{8}$. Euler's Theorem shows that $2^{192} \equiv 2^{-8} \equiv 256^{-1} \pmod{125}$. Because $3 \cdot 42 = 126 \equiv 1 \pmod{125}$, it follows that 128 and 42 are inverses mod 125. Thus 256 and 21 are inverses mod 125, so $x \equiv 21 \pmod{125}$. Because $x \equiv 0 \pmod{8}$ and $x \equiv 21 \pmod{125}$, the Chinese Remainder Theorem implies that $x \equiv 896 \pmod{1000}$.

15. **ANSWER (145):**

Let the given triangle have vertices $(0,0)$, $(5,0)$, and $(0, 2\sqrt{3})$ in the coordinate plane. Then the hypotenuse of the triangle lies on the line $2\sqrt{3}x + 5y = 10\sqrt{3}$. Suppose the equilateral triangle has side length s with a side that has endpoints $(s\cos\theta, 0)$ and $(0, s\sin\theta)$ for some θ between 0 and $\frac{\pi}{2}$. The midpoint of that side is $\frac{s}{2}(\cos\theta, \sin\theta)$. The altitude of the equilateral triangle to this midpoint must have slope $\frac{\cos\theta}{\sin\theta}$ and have length $\frac{\sqrt{3}}{2}s$, so there is a vertex of the equilateral triangle at $\frac{s}{2}(\cos\theta+\sqrt{3}\sin\theta, \sin\theta+\sqrt{3}\cos\theta)$. Because this vertex lies on the line $2\sqrt{3}x+5y=10\sqrt{3}$, it follows that $\frac{s}{2}\cdot 2\sqrt{3}\left(\cos\theta+\sqrt{3}\sin\theta\right)+\frac{s}{2}\cdot 5\left(\sin\theta+\sqrt{3}\cos\theta\right)=10\sqrt{3}$, from which

$$s=\frac{20\sqrt{3}}{7\sqrt{3}\cos\theta+11\sin\theta}.$$

Let $A=\sqrt{(7\sqrt{3})^2+11^2}=2\sqrt{67}$. Then there is an angle α such that $\cos\alpha=\frac{7\sqrt{3}}{A}$ and

$$s=\frac{\frac{20\sqrt{3}}{A}}{\cos\alpha\cos\theta+\sin\alpha\sin\theta}=\frac{\frac{20\sqrt{3}}{A}}{\cos(\alpha-\theta)}.$$

Thus the minimum possible side length occurs when $\theta=\alpha$ and $s=\frac{20\sqrt{3}}{A}=\frac{10\sqrt{3}}{\sqrt{67}}$. The minimum area for the equilateral triangle is $\frac{\sqrt{3}}{4}s^2=\frac{75\sqrt{3}}{67}$. The requested sum is $75+67+3=145$.

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