MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

35th Annual

AIME I

American Invitational Mathematics Examination I
Tuesday, March 7, 2017

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators, calculating devices, smart phones or watches, and computers are not permitted.**
4. A combination of the AIME and the American Mathematics Contest 12 are used to determine eligibility for participation in the USA Mathematical Olympiad (USAMO). A combination of the AIME and the American Mathematics Contest 10 are used to determine eligibility for participation in the USA Junior Mathematical Olympiad (USAJMO). The USAMO and USAJMO will be given on WEDNESDAY and THURSDAY, April 19 and 20, 2017.
5. Record all your answers, and identification information, on the AIME answer form. Only the answer form will be collected from you.

The publication, reproduction, or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time during this period, via copier, telephone, email, internet, or media of any type is a violation of the competition rules.

1. Fifteen distinct points are designated on $\triangle ABC$: the 3 vertices A, B, and C; 3 other points on side $\overline{AB}$; 4 other points on side $\overline{BC}$; and 5 other points on side $\overline{CA}$. Find the number of triangles with positive area whose vertices are among these 15 points.

2. When each of 702, 787, and 855 is divided by the positive integer m, the remainder is always the positive integer r. When each of 412, 722, and 815 is divided by the positive integer n, the remainder is always the positive integer $s \neq r$. Find $m + n + r + s$.

3. For a positive integer n, let d_n be the units digit of $1 + 2 + 3 + \cdots + n$. Find the remainder when
$$\sum_{n=1}^{2017} d_n$$
is divided by 1000.

4. A pyramid has a triangular base with side lengths 20, 20, and 24. The three edges of the pyramid from the three corners of the base to the fourth vertex of the pyramid all have length 25. The volume of the pyramid is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

5. A rational number written in base eight is $\underline{a}\,\underline{b}\,.\,\underline{c}\,\underline{d}$, where all digits are nonzero. The same number in base twelve is $\underline{b}\,\underline{b}\,.\,\underline{b}\,\underline{a}$. Find the base-ten number $\underline{a}\,\underline{b}\,\underline{c}$.

6. A circle is circumscribed around an isosceles triangle whose two congruent angles have degree measure x. Two points are chosen independently and uniformly at random on the circle, and a chord is drawn between them. The probability that the chord intersects the triangle is $\frac{14}{25}$. Find the difference between the largest and smallest possible values of x.

7. For nonnegative integers a and b with $a + b \leq 6$, let $T(a, b) = \binom{6}{a}\binom{6}{b}\binom{6}{a+b}$. Let S denote the sum of all $T(a, b)$, where a and b are nonnegative integers with $a+b \leq 6$. Find the remainder when S is divided by 1000.

8. Two real numbers a and b are chosen independently and uniformly at random from the interval $(0, 75)$. Let O and P be two points in the plane with $OP = 200$. Let Q and R be points on the same side of line OP such that the degree measures of $\angle POQ$ and $\angle POR$ are a and b, respectively, and $\angle OQP$ and $\angle ORP$ are both right angles. The probability that $QR \leq 100$ is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

9. Let $a_{10} = 10$, and for each integer $n > 10$ let $a_n = 100a_{n-1} + n$. Find the least $n > 10$ such that a_n is a multiple of 99.

10. Let $z_1 = 18 + 83i$, $z_2 = 18 + 39i$, and $z_3 = 78 + 99i$, where $i = \sqrt{-1}$. Let z be the unique complex number with the properties that $\frac{z_3 - z_1}{z_2 - z_1} \cdot \frac{z - z_2}{z - z_3}$ is a real number and the imaginary part of z is the greatest possible. Find the real part of z.

11. Consider arrangements of the 9 numbers $1, 2, 3, \ldots, 9$ in a 3×3 array. For each such arrangement, let a_1, a_2, and a_3 be the medians of the numbers in rows 1, 2, and 3, respectively, and then let m be the median of $\{a_1, a_2, a_3\}$. Let Q be the number of arrangements for which $m = 5$. Find the remainder when Q is divided by 1000.

12. Call a set S *product-free* if there do not exist $a, b, c \in S$ (not necessarily distinct) such that $ab = c$. For example, the empty set and the set $\{16, 20\}$ are product-free, whereas the sets $\{4, 16\}$ and $\{2, 8, 16\}$ are not product-free. Find the number of product-free subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

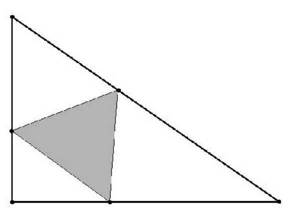
13. For every $m \geq 2$, let $Q(m)$ be the least positive integer with the following property: For every $n \geq Q(m)$, there is always a perfect cube k^3 in the range $n < k^3 \leq m \cdot n$. Find the remainder when

$$\sum_{m=2}^{2017} Q(m)$$

is divided by 1000.

14. Let $a > 1$ and $x > 1$ satisfy $\log_a\left(\log_a(\log_a 2) + \log_a 24 - 128\right) = 128$ and $\log_a(\log_a x) = 256$. Find the remainder when x is divided by 1000.

15. The area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle with side lengths $2\sqrt{3}$, 5, and $\sqrt{37}$, as shown, is $\frac{m\sqrt{p}}{n}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m + n + p$.



Your Exam Manager will receive a copy of the 2017 AIME Solution Pamphlet with the scores.

CONTACT US – *Correspondence about the problems and solutions for this AIME and orders for any of our publications should be addressed to:*

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2017 USA(J)MO – THE USA MATHEMATICAL OLYMPIAD (USAMO) and the USA MATHEMATICAL JUNIOR OLYMPIAD (USAJMO) are each a 6-question, 9-hour, essay-type examination. The best way to prepare for the USA(J)MO is to study previous years of these exams. Copies may be ordered from the web site indicated below.

PUBLICATIONS – For a complete listing of available publications please visit the MAA Bookstore or Competitions site at maa.org.

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