MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

35th Annual

AIME II

American Invitational Mathematics Examination II

Wednesday, March 22, 2017

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

The publication, reproduction, or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time during this period, via copier, telephone, email, internet, or media of any type is a violation of the competition rules.

Correspondence about the problems/solutions for this AIME and orders for any publications should be addressed to:

MAA American Mathematics Competitions
Attn: Publications, PO Box 471, Annapolis Junction, MD 20701
Phone 800.527.3690 | Fax 240.396.5647 | amcinfo@maa.org

The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

Jonathan M. Kane
AIME Chair
kanej@uww.edu

1. **ANSWER (196):**

   There are $2^8 = 256$ subsets of $\{1,2,3,4,5,6,7,8\}$. Each of the sets $\{1,2,3,4,5\}$ and $\{4,5,6,7,8\}$ has $2^5 = 32$ subsets, and their intersection, $\{4,5\}$, has $2^2 = 4$ subsets. Thus the number of subsets of either $\{1,2,3,4,5\}$ or $\{4,5,6,7,8\}$ is $32 + 32 - 4 = 60$, and the number of subsets of $\{1,2,3,4,5,6,7,8\}$ with the required property is $256 - 60 = 196$.

2. **ANSWER (781):**

   Team T_4 will be the champion if and only if it wins its semifinal match (which it will do with probability $\frac{4}{4+1} = \frac{4}{5}$) and then beats whoever wins the other semifinal match. Considering the two possible outcomes of the other semifinal match gives the probability

   $$\frac{4}{5} \cdot \left(\frac{2}{5} \cdot \frac{4}{6} + \frac{3}{5} \cdot \frac{4}{7}\right) = \frac{4}{5} \cdot \frac{4}{5} \cdot \left(\frac{1}{3} + \frac{3}{7}\right) = \frac{16}{25} \cdot \frac{16}{21} = \frac{256}{525}.$$

   The requested sum is $256 + 525 = 781$.

3. **ANSWER (409):**

   Because the perpendicular bisector of a line segment $\overline{XY}$ is the set of points equidistant from X and Y, a point inside the triangle is closer to B than to either A or C if and only if it is to the right of the perpendicular bisector of $\overline{AB}$ and below the perpendicular bisector of $\overline{BC}$. The equation of the first perpendicular bisector is $x = 6$. The second perpendicular bisector passes through $D(10, 5)$ and has slope $\frac{2}{5}$, and thus it has the equation $y = \frac{2}{5}x + 1$. The two perpendicular bisectors intersect at $E\left(6, \frac{17}{5}\right)$. Let F be $(6, 0)$. The area of $BDEF$ is the sum of the area of $\triangle DEF$, which is $\frac{1}{2} \cdot \frac{17}{5} \cdot 4 = \frac{34}{5}$, and the area of $\triangle BDF$, which is $\frac{1}{2} \cdot 6 \cdot 5 = 15$. The probability that the randomly chosen point is closer to B than to either A or C is the area of quadrilateral $BDEF$ divided by the area of $\triangle ABC$, which is $\frac{1}{2} \cdot 12 \cdot 10 = 60$. Thus the required probability is $\frac{1}{60} \cdot \left(\frac{34}{5} + 15\right) = \frac{109}{300}$. The requested sum is $109 + 300 = 409$.

4. **ANSWER (222):**

   For $k = 1, 2, 3, 4, 5$, and 6 there are 2^k k-digit base-three numbers with digits of only 1 and 2. Because $2017 = 2202201_{\text{three}}$, all the 7-digit base-three numbers less than 2017 that do not contain a 0 begin with 11, 12, or 21. There are $3 \cdot 2^5 = 96$ of them. Thus there are $2 + 4 + 8 + 16 + 32 + 64 + 96 = 222$ positive integers less than or equal to 2017 whose base-three representations contain no 0s.

5. **ANSWER (791):**

   Let the elements of the set be a, b, c, and d. Because $(a+b)+(c+d)$, $(a+c)+(b+d)$, and $(a+d)+(b+c)$ have the same value, it must be possible to group the six values

189, 320, 287, 234, x, and y into pairs such that each of the three pairs has the same sum. There is no way to group 189, 320, 287, and 234 into such pairs, so x and y are not paired with each other, but each is instead paired with one of the known values, and the two remaining known values are paired with each other. If the sum of each pair is denoted by s, the total of all six values is $3s$. The total of all six values can also be seen to be $1030 + x + y$. Thus $x + y = 3s - 1030$, and $x + y$ will be maximized by maximizing s. Because s must be equal to the sum of two out of the four values 189, 320, 287, and 234, the value of s cannot exceed $320 + 287 = 607$, and the maximum value of $x+y$ is $3\cdot 607-1030 = 791$. Actual values for $a < b < c < d$ can be found by either requiring $a+b = 189$, $a+c = 234$, $b+c = 287$, and $a+d = 320$, which results in $\{a, b, c, d\} = \{68, 121, 166, 252\}$, or by requiring $a+b = 189$, $a+c = 234$, $a+d = 287$, and $b + c = 320$, which results in $\{a, b, c, d\} = \{51.5, 137.5, 182.5, 235.5\}$.

6. **ANSWER (195):**

For all positive integers n, $n^2 + 85n + 2017 > n^2 + 84n + 1764 = (n + 42)^2$, and $n^2+85n+2017 < n^2+90n+2025 = (n+45)^2$. Solving $n^2+85n+2017 = (n+43)^2$ gives $n = 168$, and solving $n^2+85n+2017 = (n+44)^2$ gives $n = 27$. The requested sum is $168 + 27 = 195$.

OR

Let $m^2 = n^2 + 85n + 2017$ so that $4m^2 = 4n^2 + 340n + 8068 = (2n + 85)^2 + 843$. Then $(2m - 2n - 85)(2m + 2n + 85) = 843 = 3 \cdot 281$, from which it follows that $2m - 2n - 85 = 1$ or $2m - 2n - 85 = 3$. Thus either $m = n + 43$ or $m = n + 44$ as above.

7. **ANSWER (501):**

The given equation is equivalent to the equation $kx = (x + 2)^2$, with the restrictions $kx > 0$ and $x > -2$. When $k > 0$, the restrictions require $x > 0$. Then Descartes' Rule of Signs shows that the related polynomial $x^2 + (4 - k)x + 4$ has either 0 or 2 positive roots unless the polynomial has a repeated positive root, as it does exactly when $k = 8$. When $k < 0$, the restrictions require that $-2 < x < 0$. On that interval kx decreases from $-2k$ to 0 while $(x + 2)^2$ increases from 0 to 4. This shows that the equation has exactly one real solution for each $k < 0$. Thus the equation has exactly one real solution for each k in the set $\{-500, -499, -498, \ldots, -1, 8\}$, which has 501 elements.

8. **ANSWER (134):**

The sum $\frac{n^2}{2!} + \frac{n^3}{3!} + \frac{n^4}{4!} + \frac{n^5}{5!} + \frac{n^6}{6!}$ is an integer if and only if multiplying the sum by 6! gives an integer $M = 360n^2 + 120n^3 + 30n^4 + 6n^5 + n^6$ that is divisible by 720.

This means that n^6 must be divisible by 6. Suppose $n = 6k$ for some positive integer k. Then $M = 360 \cdot 6^2k^2 + 120 \cdot 6^3k^3 + 30 \cdot 6^4k^4 + 6^6k^5 + 6^6k^6$, which is divisible by 720 if and only if $k^5 + k^6$ is divisible by 5. This happens exactly when either k or $k+1$ is a multiple of 5. Thus n must be either a multiple of 30 or 6 less than a multiple of 30. The possible values of n are $30m$, where $1 \le m \le 67$, and $30m - 6$, where $1 \le m \le 67$. There are 134 such values.

9. **ANSWER (013):**

In a set of eight cards that includes every color and every number, there will be exactly one repeated number and exactly one repeated color. If Sharon selects a set that includes the card with that number and color, Sharon can discard it. If the set does not include that card, Sharon cannot discard any card.

In the former case, there are 7! ways to choose one card of each number and color, and then 42 ways to choose the "extra" card. The total is $7! \cdot 42$.

In the latter case, there are 7 ways to choose which number is repeated, and $\binom{7}{2} = 21$ ways to choose which two cards of that number are used. There are then 5 ways to choose which color is repeated, and $\binom{6}{2} = 15$ ways to choose which cards of that color are used. There now remain four unused colors and four unused numbers, and there are $4! = 24$ ways to choose from those. The total is $7 \cdot 21 \cdot 5 \cdot 15 \cdot 24$.

The probability that a given set is of the first type is therefore $\frac{7! \cdot 42}{7! \cdot 42 + 7 \cdot 21 \cdot 5 \cdot 15 \cdot 24} = \frac{4}{9}$. The requested sum is $4 + 9 = 13$.

10. **ANSWER (546):**

Let Q be the projection of O onto $\overline{CD}$. Because $\triangle AND \sim \triangle QDO$, $\frac{OQ}{DQ} = \frac{42}{28} = \frac{3}{2}$, and because $\triangle QOC \sim \triangle DMC$, $\frac{OQ}{84-DQ} = \frac{21}{84} = \frac{1}{4}$. Solving this system of two equations yields $DQ = 12$ and $OQ = 18$. Then the area of $\triangle BON$ is $\frac{BN(42-OQ)}{2} = 672$, and the area of $\triangle BCO$ is $\frac{BC(84-DQ)}{2} = 1512$. Because $\triangle BON$ has a smaller area than $\triangle BCO$, point P must lie on $\overline{CO}$, and $\triangle BPC$ must have area $\frac{672+1512}{2} = 1092$. Thus the distance from P to $\overline{BC}$ is $\frac{2 \cdot 1092}{BC} = 52$, and the distance from P to $\overline{CD}$ is $52 \cdot \frac{DM}{CD} = 13$. The requested area of $\triangle CDP$ is $\frac{13 \cdot CD}{2} = 546$.

OR

Let R be the intersection of lines MC and AB, and let Q be the intersection of lines DN and BC. Let X, Y, Z, and W be the projections of point O onto line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$, respectively. Because $\triangle AND \sim \triangle BNQ$, $BQ = 84$. Because $\triangle RAM \sim \triangle CDM$, $RA = 84$. Because $\triangle QCO \sim \triangle DMO$, it follows from $\frac{OY}{OW} = \frac{CQ}{MD} = 6$ that $OY = 72$ and $OW = 12$. Because $\triangle RNO \sim \triangle CDO$, it follows from $\frac{OX}{OZ} = \frac{RN}{DC} = \frac{4}{3}$ that $OX = 24$ and $OZ = 18$. Then $[BCON] =$

$[BON] + [BCO] = \frac{OX \cdot BN}{2} + \frac{OY \cdot BC}{2} = 672 + 1512 = 2184$. Because $[BCO] > \frac{1}{2}[BCON]$, it must be that P lies on $\overline{OC}$. Then the result follows as above.

OR

Position the rectangle in the coordinate plane with vertices $A(0,0)$, $B(84,0)$, $C(84,42)$, and $D(0,42)$, midpoint $M(0,21)$, and trisection point $N(28,0)$. Then lines CM and DN have equations $y = \frac{1}{4}x + 21$ and $y = -\frac{3}{2}x + 42$, which intersect at $O(12,24)$. The Shoelace Formula gives $[BCON] = 2184$. Clearly P must lie on $\overline{OC}$, and $[BCP] = 1092$, from which it follows that the coordinates of P are $(32,29)$. The Shoelace Formula then gives the area of $\triangle CDP$ as 546.

11. **ANSWER (544):**

If an assignment of directions for the roads has a town where either all four of its roads are inbound or all four of its roads are outbound, then it is not possible to get from each town to every other town. Conversely, assume all towns have at least one inbound and one outbound road, and suppose it were not possible to get from town A to town B. Some road is inbound to B, so suppose that one can get from C to B. Some road is outbound from A which does not go to either B or C, so suppose it goes to D. Some road is outbound from D which does not go to either B or C, so it must go to the remaining town E. Thus no roads from A, B, D, or E can go to C, contradicting the assumption that C must have at least one inbound road. Therefore if all towns have at least one inbound and one outbound road, it is possible to get from each town to every other town.

It is left to count the number of ways of assigning directions to the $\binom{5}{2} = 10$ roads so that no town has roads that are only inbound or only outbound. There are $2^{10} = 1024$ ways to select directions for all 10 roads. Note that if there is a town where all the roads are outbound, then there is only one such town. To choose an assignment of directions so that one town has only outbound roads, choose one of the 5 towns to have all of its roads outbound, and then assign directions to the other $\binom{4}{2} = 6$ roads in one of $2^6 = 64$ ways for a total of $5 \cdot 64 = 320$ ways. Similarly, there are 320 ways to choose an assignment of directions so that one town has only inbound roads. However, some assignments have both a town with only inbound roads and a town with only outbound roads. Choose one town to have only outbound roads in one of 5 ways, choose a second town to have only inbound roads in one of 4 ways, and choose directions for the remaining 3 roads in one of $2^3 = 8$ ways, for a total of $5 \cdot 4 \cdot 8 = 160$ ways. It follows that there are $320 + 320 - 160 = 480$ assignments of directions that result in at least one town having either all outbound or all inbound roads. The requested number is then $1024 - 480 = 544$.

12. **ANSWER (110):**

Let C_0 be centered at 0 in the complex plane, and let $A_0 = 1$. Circle C_1 has radius r and center $1-r$. It follows that $A_1 = (1-r)+ir$, and because circle C_2 has radius r^2, C_2 has center $A_1 - ir^2 = (1-r)+ir(1-r)$. In general, if C_n has center $(1-r)+ir(1-r)+(ir)^2(1-r)+\cdots+(ir)^{n-1}(1-r)$ and radius r^n, then A_n will be in the direction rotated 90° counterclockwise from the direction of i^{n-1}, which is in the direction of i^n. Thus $A_n = (1-r)+ir(1-r)+(ir)^2(1-r)+\cdots+(ir)^{n-1}(1-r)+i^n r^n$, and the center of C_{n+1} will be $A_n - i^n r^{n+1} = (1-r)+ir(1-r)+(ir)^2(1-r)+\cdots+(ir)^{n-1}(1-r)+(ir)^n(1-r)$. Hence, by induction, it follows that the center of circle C_n is $(1-r)+ir(1-r)+(ir)^2(1-r)+\cdots+(ir)^{n-1}(1-r)$. The point common to all the circles is the limit of this sequence of center points, which is an infinite geometric series with sum $\frac{1-r}{1-ir}$. This number is a distance $\frac{1-r}{\sqrt{1+r^2}}$ from 0. So if $r=\frac{11}{60}$, the required distance is

$$\frac{1-\frac{11}{60}}{\sqrt{1+\left(\frac{11}{60}\right)^2}} = \frac{49}{61}.$$

The requested sum is $49+61=110$.

13. **ANSWER (245):**

Let O be the center of the regular polygon. Note that O is the circumcenter of any isosceles triangle whose vertices are vertices of the polygon. Moreover, if P is the vertex incident to the two congruent sides of the isosceles triangle, then the other two vertices are symmetric with respect to the line PO. For every vertex P of the polygon, there are exactly $\lfloor\frac{n-1}{2}\rfloor$ vertices of the polygon on one side of ray PO, and by symmetry, for every one of these vertices Q, there is a vertex Q' in the polygon on the other side of ray PO such that $\triangle PQQ'$ is isosceles. In this way every non-equilateral isosceles triangle is counted once, and for $n \equiv 0 \pmod 3$, every equilateral triangle is counted 3 times. Thus

$$f(n) = \begin{cases} n\lfloor\frac{n-1}{2}\rfloor - \frac{2n}{3} & \text{if } n \equiv 0 \pmod 3, \\ n\lfloor\frac{n-1}{2}\rfloor & \text{otherwise.} \end{cases}$$

Therefore

$$f(n+1)-f(n) = \begin{cases} 13k & \text{if } n = 6k, \\ 3k & \text{if } n = 6k+1, \\ 5k+1 & \text{if } n = 6k+2, \\ 7k+3 & \text{if } n = 6k+3, \\ 9k+6 & \text{if } n = 6k+4, \\ -(k+2) & \text{if } n = 6k+5. \end{cases}$$

The equations $5k+1=78$, $7k+3=78$, and $-(k+2)=78$ have no positive integer solutions. If $13k=78$, then $k=6$ and $n=6k=36$. If $3k=78$, then $k=26$ and

$n = 6k + 1 = 157$. If $9k + 6 = 78$, then $k = 8$ and $n = 6k + 4 = 52$. The sum of all values of n with the required property is $36 + 157 + 52 = 245$.

14. **ANSWER (168):**

Let the grid of points be referred to as the cube. Any line parallel to an edge of the cube and containing two of the grid points must contain 10 of the grid points. Thus no line parallel to an edge of the cube contains exactly 8 of the grid points.

There are 30 planes, each parallel to a face of the cube and each containing 100 of the grid points in a 10×10 square. The points of this grid determine 4 lines that each contain exactly 8 grid points. These lines are two units above and two units below each of the diagonals of each such square.

Now consider a line that is not parallel to an edge or a face of the cube and contains exactly 8 grid points. A vector parallel to such a line has the form $(a,\ b,\ c)$, where a, b, and c are nonzero integers that have no common factor greater than 1. The line has equation

$$(x,\ y,\ z) = (d,\ e,\ f) + t(a,\ b,\ c),$$

where d, e, and f are integers, and there are 8 consecutive integer values of t such that $x = at + d$, $y = bt + e$, and $z = ct + f$ are all integers between 1 and 10, inclusive. It follows that each of a, b, and c must be ± 1; without loss of generality, assume $a = 1$. Then there are four possible vectors that can be parallel to a line containing exactly 8 grid points: $(1,\ 1,\ 1)$, $(1,\ 1,\ -1)$, $(1,\ -1,\ 1)$, $(1,\ -1,\ -1)$. Each of these four vectors is parallel to one of the space diagonals of the cube. By symmetry, it suffices to find the number of lines parallel to the vector $(1,\ 1,\ 1)$, and then multiply this result by 4.

The lines parallel to $(1,\ 1,\ 1)$ must intersect the grid in a point on one of the planes $x = 1$, $y = 1$, or $z = 1$. Then any one of the 8-point lines has the form

$$(d,\ e,\ f) + t(1,\ 1,\ 1) = (d + t,\ e + t,\ f + t),$$

and the 8 grid points are realized for $t = 0, 1, 2, \ldots, 7$. Each of the coordinates $d + t$, $e + t$, and $f + t$ is between 1 and 10, inclusive, for these values of t, and at least one of these numbers is 11 when $t = 8$. Thus at least one of d, e, and f is 1, at least one is equal to 3, and the third is 1, 2, or 3. If two of d, e, and f are equal to 1, then $(d,\ e,\ f)$ is one of the triples $(1,\ 1,\ 3)$, $(1,\ 3,\ 1)$, or $(3,\ 1,\ 1)$. If two of d, e, and f are equal to 3, then $(d,\ e,\ f)$ is one of the triples $(1,\ 3,\ 3)$, $(3,\ 1,\ 3)$, or $(3,\ 3,\ 1)$. If d, e, and f are distinct, then there are $3! = 6$ possibilities for $(d,\ e,\ f)$. This accounts for 12 possibilities for $(d,\ e,\ f)$.

Thus the number of lines containing exactly 8 of the lattice points is $4 \cdot 30 + 4 \cdot 12 = 168$.

15. **ANSWER (682):**

Let M and N be midpoints of $\overline{AB}$ and $\overline{CD}$, respectively. The given conditions imply that $\triangle ABD \cong \triangle BAC$ and $\triangle CDA \cong \triangle DCB$, and therefore $MC = MD$ and $NA = NB$. It follows that M and N both lie on the common perpendicular bisector of $\overline{AB}$ and $\overline{CD}$, and thus line MN is that common perpendicular bisector. Points B and C are symmetric to A and D with respect to line MN. If X is a point in space and X' is the point symmetric to X with respect to line MN, then $BX = AX'$ and $CX = DX'$, so $f(X) = AX + AX' + DX + DX'$.

Let Q be the intersection of $\overline{XX'}$ and $\overline{MN}$. Then $AX + AX' \geq 2AQ$ and, similarly, $DX + DX' \geq 2DQ$, from which it follows that $f(X) \geq 2(AQ + DQ) = f(Q)$. It remains to minimize $f(Q)$ as Q moves along $\overline{MN}$.

Allow D to rotate around line MN to point D' in the plane AMN on the side of line MN opposite to A. Because N is on the axis of rotation, $D'N = DN$. It then follows that $f(Q) = 2(AQ + D'Q) \geq 2AD'$, and equality occurs when Q is the intersection of $\overline{AD'}$ and $\overline{MN}$. Thus $\min f(Q) = 2AD'$. Because $\overline{MD}$ is the median of $\triangle ADB$, the formula for the length of a median shows that $4MD^2 = 2AD^2 + 2BD^2 - AB^2 = 2 \cdot 28^2 + 2 \cdot 44^2 - 52^2$ and $MD^2 = 684$. By the Pythagorean Theorem $MN^2 = MD^2 - ND^2 = 8$.

Because $\angle AMN$ and $\angle D'NM$ are right angles, $(AD')^2 = (AM + D'N)^2 + MN^2 = (2AM)^2 + MN^2 = 52^2 + 8 = 4 \cdot 678$. It follows that $\min f(Q) = 2AD' = 4\sqrt{678}$. The requested sum is $4 + 678 = 682$.

Note that more is true. In any tetrahedron $ABCD$, the three lines passing through the pairs of midpoints of non-intersecting sides are concurrent. That is, the point Q above is the midpoint of $\overline{MN}$ as well as the midpoint of the segment connecting the midpoints of $\overline{AC}$ and $\overline{BD}$ and the midpoint of the segment connecting the midpoints of $\overline{AD}$ and $\overline{BC}$. An easy way to show this is to represent the sides of the tetrahedron using vectors.

The MAA American Invitational Mathematical Examination
A program of the Mathematical Association of America

Supported by major contributions from
Akamai Foundation
American Mathematical Society
Art of Problem Solving, Inc.
Dropbox
Jane Street Capital
MathWorks
Simons Foundation
The D.E. Shaw Group
Susquehanna International Group
Tudor Investment Corporation
Two Sigma