MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

35th Annual

# AIME II

**American Invitational Mathematics Examination II**
**Wednesday, March 22, 2017**

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators, calculating devices, smart phones or watches, and computers are not permitted.**
4. A combination of the AIME and the American Mathematics Contest 12 are used to determine eligibility for participation in the USA Mathematical Olympiad (USAMO). A combination of the AIME and the American Mathematics Contest 10 are used to determine eligibility for participation in the USA Junior Mathematical Olympiad (USAJMO). The USAMO and USAJMO will be given on WEDNESDAY and THURSDAY, April 19 and 20, 2017.
5. Record all your answers, and identification information, on the AIME answer form. Only the answer form will be collected from you.

*The publication, reproduction, or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time during this period, via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

1. Find the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are subsets of neither $\{1, 2, 3, 4, 5\}$ nor $\{4, 5, 6, 7, 8\}$.

2. Teams $T_1$, $T_2$, $T_3$, and $T_4$ are in the playoffs. In the semifinal matches, $T_1$ plays $T_4$, and $T_2$ plays $T_3$. The winners of those two matches will play each other in the final match to determine the champion. When $T_i$ plays $T_j$, the probability that $T_i$ wins is $\frac{i}{i+j}$, and the outcomes of all the matches are independent. The probability that $T_4$ will be the champion is $\frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Find $p + q$.

3. A triangle has vertices $A(0, 0)$, $B(12, 0)$, and $C(8, 10)$. The probability that a randomly chosen point inside the triangle is closer to vertex $B$ than to either vertex $A$ or vertex $C$ can be written as $\frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Find $p + q$.

4. Find the number of positive integers less than or equal to 2017 whose base-three representation contains no digit equal to 0.

5. A set contains four numbers. The six pairwise sums of distinct elements of the set, in no particular order, are 189, 320, 287, 234, $x$, and $y$. Find the greatest possible value of $x + y$.

6. Find the sum of all positive integers $n$ such that $\sqrt{n^2 + 85n + 2017}$ is an integer.

7. Find the number of integer values of $k$ in the closed interval $[-500, 500]$ for which the equation $\log(kx) = 2\log(x + 2)$ has exactly one real solution.

8. Find the number of positive integers $n$ less than 2017 such that

$$1 + n + \frac{n^2}{2!} + \frac{n^3}{3!} + \frac{n^4}{4!} + \frac{n^5}{5!} + \frac{n^6}{6!}$$

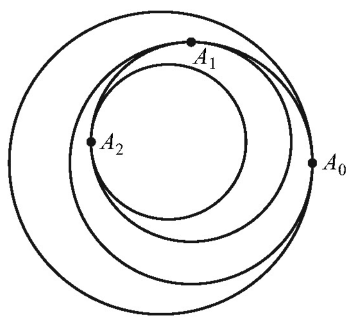
is an integer.

9. A special deck of cards contains 49 cards, each labeled with a number from 1 to 7 and colored with one of seven colors. Each number-color combination appears on exactly one card. Sharon will select a set of eight cards from the deck at random. Given that she gets at least one card of each color and at least one card with each number, the probability that Sharon can discard one of her cards and *still* have at least one card of each color and at least one card with each number is $\frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Find $p + q$.

10. Rectangle $ABCD$ has side lengths $AB = 84$ and $AD = 42$. Point $M$ is the midpoint of $\overline{AD}$, point $N$ is the trisection point of $\overline{AB}$ closer to $A$, and point $O$ is the intersection of $\overline{CM}$ and $\overline{DN}$. Point $P$ lies on the quadrilateral $BCON$, and $\overline{BP}$ bisects the area of $BCON$. Find the area of $\triangle CDP$.

11. Five towns are connected by a system of roads. There is exactly one road connecting each pair of towns. Find the number of ways there are to make all the roads one-way in such a way that it is still possible to get from any town to any other town using the roads (possibly passing through other towns on the way).

12. Circle $C_0$ has radius 1, and the point $A_0$ is a point on the circle. Circle $C_1$ has radius $r < 1$ and is internally tangent to $C_0$ at point $A_0$. Point $A_1$ lies on circle $C_1$ so that $A_1$ is located $90^\circ$ counterclockwise from $A_0$ on $C_1$. Circle $C_2$ has radius $r^2$ and is internally tangent to $C_1$ at point $A_1$. In this way a sequence of circles $C_1, C_2, C_3, \ldots$ and a sequence of points on the circles $A_1, A_2, A_3, \ldots$ are constructed, where circle $C_n$ has radius $r^n$ and is internally tangent to circle $C_{n-1}$ at point $A_{n-1}$, and point $A_n$ lies on $C_n$ $90^\circ$ counterclockwise from point $A_{n-1}$, as shown in the figure below. There is one point $B$ inside all of these circles. When $r = \frac{11}{60}$, the distance from the center of $C_0$ to $B$ is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.




13. For each integer $n \geq 3$, let $f(n)$ be the number of 3-element subsets of the vertices of a regular $n$-gon that are the vertices of an isosceles triangle (including equilateral triangles). Find the sum of all values of $n$ such that $f(n+1) = f(n) + 78$.

14. A $10 \times 10 \times 10$ grid of points consists of all points in space of the form $(i, j, k)$, where $i$, $j$, and $k$ are integers between 1 and 10, inclusive. Find the number of different lines that contain exactly 8 of these points.

15. Tetrahedron $ABCD$ has $AD = BC = 28$, $AC = BD = 44$, and $AB = CD = 52$. For any point $X$ in space, define $f(X) = AX + BX + CX + DX$. The least possible value of $f(X)$ can be expressed as $m\sqrt{n}$, where $m$ and $n$ are positive integers, and $n$ is not divisible by the square of any prime. Find $m + n$.

Your Exam Manager will receive a copy of the 2017 AIME Solution Pamphlet with the scores.

**CONTACT US –** *Correspondence about the problems and solutions for this AIME and orders for any of our publications should be addressed to:*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*The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:*


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**2017 USA(J)MO –** THE USA MATHEMATICAL OLYMPIAD (USAMO) and the USA MATHEMATICAL JUNIOR OLYMPIAD (USAJMO) are each a 6-question, 9-hour, essay-type examination. The best way to prepare for the USA(J)MO is to study previous years of these exams. Copies may be ordered from the web site indicated below.

**PUBLICATIONS –** For a complete listing of available publications please visit the MAA Bookstore or Competitions site at maa.org.


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