MATHEMATICAL ASSOCIATION OF AMERICA
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Solutions Pamphlet

MAA American Mathematics Competitions

34th Annual

AIME I

American Invitational Mathematics Examination I

Thursday, March 3, 2016

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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1. **Answer (336):**

$$\begin{aligned} S(a)+S(-a) &= \frac{12}{1-a}+\frac{12}{1+a}=\frac{2}{12}\cdot\frac{12^2}{1-a^2} \\ &= \frac{1}{6}S(a)S(-a)=\frac{1}{6}\cdot 2016=336 \end{aligned}$$

2. **Answer (071):**

   Because $1+2+3+4+5+6=21$, the probability of rolling a k with one of these dice is $\frac{k}{21}$. Then the probability of rolling a 7 with the pair of dice is

$$\frac{1\cdot 6+2\cdot 5+3\cdot 4+4\cdot 3+5\cdot 2+6\cdot 1}{21^2}=\frac{56}{21^2}=\frac{8}{63}.$$

   The requested sum is $8+63=71$.

3. **Answer (810):**

   There are 5 ways to start the path, then 9 ways to continue it on the upper pentagon (1, 2, 3, or 4 steps in either direction, or 0 steps). No matter where the path leaves the upper pentagon, there are 2 ways to move down to the lower pentagon, where it has 9 ways to continue horizontally before the final step down to the bottom vertex. There are $5\cdot 9\cdot 2\cdot 9=810$ such paths.

4. **Answer (108):**

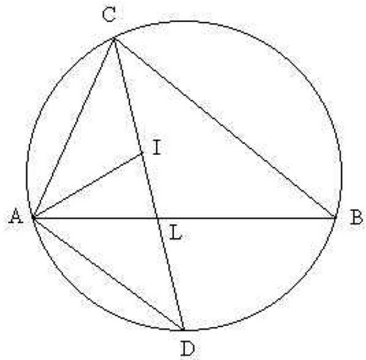
   Label the other vertices of the pyramid B, C, and D, where $\overline{AB}$ and $\overline{AC}$ are sides of a base of the prism. The dihedral angle of 60° is formed by faces $\triangle BAC$ and $\triangle BDC$. Let E be the midpoint of $\overline{BC}$. Then $\angle AED=60^\circ$. Because $\angle BAC=120^\circ$, it follows that $\angle EAB=60^\circ$ and $AE=AB\cos 60^\circ=6$. Therefore in right $\triangle DAE$, $h=AE\tan 60^\circ=6\sqrt{3}$. Hence $h^2=36\cdot 3=108$.

5. **Answer (053):**

   Let k represent the number of days it took Anh to finish the book. Then the total number pages read is $n+(n+1)+\cdots+(n+(k-1))=\frac{k}{2}(2n+k-1)$, and the total number of minutes taken is $t+(t+1)+\cdots+(t+(k-1))=\frac{k}{2}(2t+k-1)$. Thus $2\cdot 374=k(2n+k-1)$ and $2\cdot 319=k(2t+k-1)$. Note also that $55=374-319=\frac{k}{2}\cdot 2(n-t)=k(n-t)$. Then k must be a common factor of 748, 638, and 55, so k must be a factor of $\gcd(748,638,55)=11$. Because $k>1$, it follows that $k=11$. Then $2n+10=68$ and $2t+10=58$. Therefore $n=29$ and $t=24$. The requested sum is $29+24=53$.

6. **Answer (013):**

Because I is the incenter of $\triangle ABC$, $\overline{AI}$ is the angle bisector of $\angle BAC$ and $\angle LAI = \angle IAC$. It is also clear that $\angle BAD = \frac{1}{2}\overparen{BD} = \angle BCD$. Thus $\angle AID = \angle DCA + \angle IAC = \angle DCB + \angle LAI = \angle BAD + \angle LAI = \angle DAI$ and $DA = DI = 5$. Because $\angle LAD = \angle BAD = \angle BCD = \angle ACD$ and $\angle ADL = \angle ADC$, $\triangle DAL$ is similar to $\triangle DCA$. Hence $\frac{DA}{DL} = \frac{DC}{DA}$, or $\frac{5}{3} = \frac{5+IC}{5}$ and $IC = \frac{10}{3}$. The requested sum is $10 + 3 = 13$.



7. **Answer (103):**

The complex number is real if either $\frac{\sqrt{ab+2016}}{ab+100}$ is real and $\frac{\sqrt{|a+b|}}{ab+100} = 0$ or $\frac{\sqrt{ab+2016}}{ab+100}$ is imaginary and equals $\frac{\sqrt{|a+b|}}{ab+100}i$. In the former case, $a = -b$, and $\frac{\sqrt{ab+2016}}{ab+100}$ must be real. This will occur if and only if $ab+2016 \ge 0$ and $ab \ne -100$. So $a^2 \le 2016$, and there are $2 \cdot \lfloor\sqrt{2016}\rfloor + 1 - 2 = 87$ possible values for a (including 0, and excluding ± 10). In the latter case, $\sqrt{ab + 2016} = \sqrt{-|a + b|}$. If $a + b > 0$, then $ab+2016 = -(a+b)$, which can be rewritten as $(a+1)(b+1) = -2015 = -5{\cdot}13{\cdot}31$. There are 8 integer solutions to this equation with $a+b > 0$. Similarly if $a+b < 0$, then $ab+2016 = a+b$, which can be rewritten as $(a-1)(b-1) = -2015$. Again, there are 8 integer solutions to this equation with $a + b < 0$. This yields a total of $87 + 8 + 8 = 103$ possible ordered pairs.

8. **Answer (162):**

For any permutation p

$$s(p) = 100(a_1 + a_4 + a_7) + 10(a_2 + a_5 + a_8) + (a_3 + a_6 + a_9).$$

Because the units digit of m is 0, it follows that $a_3 + a_6 + a_9 = 10$ or 20.

If $a_3 + a_6 + a_9 = 10$, then $a_1 + a_4 + a_7 + a_2 + a_5 + a_8 = 45 - 10 = 35$, and $s(p) = 90(a_1 + a_4 + a_7) + 360 \ge 90(1 + 2 + 3) + 360 = 900$.

If $a_3+a_6+a_9 = 20$, then $s(p) = 90(a_1+a_4+a_7)+270 \ge 90\cdot 6+270 = 810$, with equality if and only if $\{a_1, a_4, a_7\} = \{1, 2, 3\}$. The set $\{a_3, a_6, a_9\}$ must equal $\{4, 7, 9\}$, $\{5, 6, 9\}$, or $\{5, 7, 8\}$.

For each $\{a_i, a_{i+3}, a_{i+6}\}$, there are 3! ways to permute these numbers. Hence there are $n = 3 \cdot 3! \cdot 3! \cdot 3! = 648$ permutations p with $s(p) = m = 810$ and $|m - n| = |810 - 648| = 162$.

9. **Answer (744):**

Let α, β, and γ denote the degree measures of $\angle BAC$, $\angle BAQ$, and $\angle CAS$, respectively, so that $\alpha + \beta + \gamma = 90^\circ$. Then $AS = AC\cos\gamma = 31\cos\gamma$ and $AQ = AB\cos\beta = 40\cos\beta$. The area of rectangle $AQRS$ is

$$\begin{aligned} AS \cdot AQ &= 31 \cdot 40 \cos\beta\cos\gamma \\ &= 1240 \cdot \frac{1}{2}\left(\cos(\beta+\gamma) + \cos(\beta-\gamma)\right) \\ &= 620\left(\cos(90^\circ - \alpha) + \cos(\beta-\gamma)\right) \\ &\le 620\,(\sin\alpha + 1). \end{aligned}$$

The extreme value is assumed when $\beta = \gamma = \frac{1}{2}(90^\circ - \sin^{-1}\frac{1}{5}) \approx 39.23^\circ$ giving the area $620(\frac{1}{5} + 1) = 744$.

10. **Answer (504):**

The ratio $\frac{a_2}{a_1}$ must be rational, so let $a_2 = \frac{ba_1}{a}$, where a and b are relatively prime positive integers and $a < b$. Because $a_3 = \frac{b^2 a_1}{a^2}$ is also an integer, there is an integer c such that $a_1 = ca^2$ and $a_2 = cab$. Thus the sequence begins ca^2, cab, cb^2, $cb(2b - a)$. Examining a few terms of the sequence suggests that for $k \ge 1$

$$\begin{aligned} a_{2k-1} &= c\big((k-1)b - (k-2)a\big)^2 \text{ and} \\ a_{2k} &= c\big((k-1)b - (k-2)a\big)\big(kb - (k-1)a\big). \end{aligned}$$

Then, because the sequence a_{2k-1}, a_{2k}, a_{2k+1} is geometric, it would follow that

$$a_{2k+1} = \frac{a_{2k}^2}{a_{2k-1}} = \frac{c^2\big((k-1)b - (k-2)a\big)^2\big(kb - (k-1)a\big)^2}{c\big((k-1)b - (k-2)a\big)^2} = c\big(kb - (k-1)a\big)^2.$$

It would then follow that for $k \ge 1$ that

$$\begin{aligned} a_{2k} &= c\big((k-1)b - (k-2)a\big)\big(kb - (k-1)a\big) \text{ and} \\ a_{2k+1} &= c\big(kb - (k-1)a\big)^2. \end{aligned}$$

Because a_{2k}, a_{2k+1}, a_{2k+2} is arithmetic, it would follow that

$$\begin{aligned} a_{2k+2} &= 2a_{2k+1} - a_{2k} \\ &= 2c\big(kb - (k-1)a\big)^2 - c\big((k-1)b - (k-2)a\big)\big(kb - (k-1)a\big) \end{aligned}$$

$$= c\,\big(kb - (k-1)a\big)\big((k+1)b - ka\big).$$

It can now be verified by mathematical induction that for all positive integers k

$$a_{2k} = c\,\big((k-1)b - (k-2)a\big)\big(kb - (k-1)a\big) \text{ and}$$
$$a_{2k+1} = c\,\big(kb - (k-1)a\big)^2.$$

In particular, $a_{13} = c(6b-5a)^2 = 2016 = 14 \cdot 12^2$. Therefore $6b - 5a$ is a factor of 12 and is also the seventh term in an arithmetic progression whose first two terms are a and b. Let $n = 6b - 5a$. Then $a < a + 6(b-a) = n$, and $6b = 5a + n \equiv n - a \pmod 6$, implying that $n - a$ is a multiple of 6. Thus $6 < a + 6 \le n \le 12$, and the only solution for (a, b, n) in positive integers is $(6, 7, 12)$. The corresponding value of c is 14, and $a_1 = 14 \cdot 6^2 = 504$.

11. **Answer (109):**

The given condition $(x-1)P(x+1) = (x+2)P(x)$ implies that $P(x)$ is divisible by $x - 1$. From this it follows that x divides $P(x+1)$, and the given condition then implies that x divides $P(x)$. Substituting $x - 1$ in place of x in the given condition yields $(x-2)P(x) = (x+1)P(x-1)$, which implies that $P(x)$ is divisible by $x + 1$. Thus there is a polynomial $L(x)$ such that $P(x) = x(x-1)(x+1)L(x)$. Substituting this for $P(x)$ in the given condition yields $(x-1)(x+1)x(x+2)L(x+1) = (x+2)x(x-1)(x+1)L(x)$ or $L(x+1) = L(x)$, implying that there is a constant c such that $L(x) = c$ for all x. It follows that $P(x) = c(x^3 - x)$. Now the condition $\big(P(2)\big)^2 = P(3)$ implies that $c = \frac{2}{3}$, so $P(\frac{7}{2}) = \frac{105}{4}$. The requested sum is $105 + 4 = 109$.

12. **Answer (132):**

Considering the expression $e(m) = m^2 - m + 11$ modulo 2, 3, 5, and 7 shows that none of these primes can be a factor of $e(m)$. Thus the smallest possible value for $e(m)$ with four prime factors is 11^4, but there is no integer m for which $e(m) = 11^4$. The next candidate for the smallest value for $e(m)$ with four prime factors is $11^3 \cdot 13$. If there actually is an integer m such that $m^2 - m + 11 = 11^3 \cdot 13$, then because $m^2 - m + 11 = m(m-1) + 11$ is a multiple of 11, there must be an integer k with either $m = 11k$ or $m = 11k + 1$. If $m = 11k$, the equation becomes $11k^2 - k + 1 = 11^2 \cdot 13$. It follows that $-k + 1$ must be divisible by 11. Clearly $k = 1$ does not work, but $k = 12$ satisfies the equation. If $m = 11k + 1$, there are no small values of k that satisfy the needed condition. Therefore the least positive integer m satisfying the needed condition is $m = 11 \cdot 12 = 132$.

13. **Answer (273):**

Let $T(h)$ be the expected number of jumps it will take Freddy to reach the river when he is a distance h from it. The problem asks for the value of $T(3)$. Note that $T(0) = 0$, and for each h with $1 \le h \le 23$ there is a probability of $\frac{1}{2}$ that

Freddy will stay the same distance from the river, a probability of $\frac{1}{4}$ that he will get one jump closer to the river, and a probability of $\frac{1}{4}$ that he will get one jump farther away. Thus

$$T(h) = 1 + \frac{1}{4}T(h-1) + \frac{1}{4}T(h+1) + \frac{1}{2}T(h),$$

which simplifies to $2T(h) = 4 + T(h-1) + T(h+1)$. For the special case $h = 24$,

$$T(24) = 1 + \frac{1}{3}T(23) + \frac{2}{3}T(24),$$

which simplifies to $T(24) = 3 + T(23)$. Summing the equations $2T(h) = 4 + T(h-1) + T(h+1)$ for $1 \le h \le 23$ yields

$$2\sum_{h=1}^{23} T(h) = 4 \cdot 23 + \sum_{h=0}^{22} T(h) + \sum_{h=2}^{24} T(h).$$

This simplifies to $T(1) + T(23) = 92 + T(24)$. Combining this with the equation $T(24) = 3 + T(23)$ yields $T(1) = 95$. From the recurrence $T(h+1) = 2T(h) - T(h-1) - 4$, it follows that $T(2) = 186$ and $T(3) = 273$.

14. **Answer (574):**

Let $A = (0,0)$ and $B = (1001, 429) = (143 \cdot 7, 143 \cdot 3)$. Between the points $(0,0)$ and $(7,3)$, $\overline{AB}$ can intersect the square at lattice point (m,n) only if it passes between the upper-left and lower-right corners of the square. That is,

$$n + \frac{1}{10} \ge \frac{3}{7}\left(m - \frac{1}{10}\right) \text{ and } n - \frac{1}{10} \le \frac{3}{7}\left(m + \frac{1}{10}\right).$$

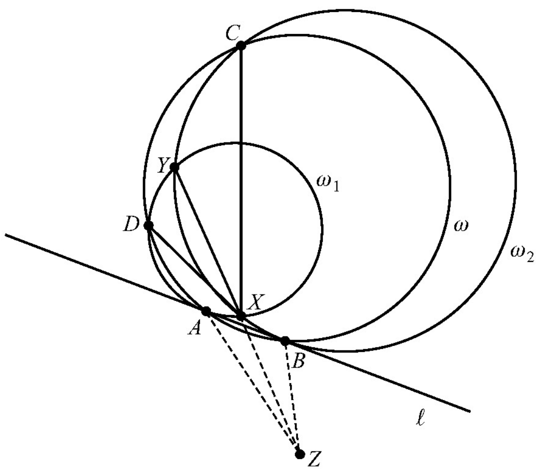
This implies $3m - 1 \le 7n \le 3m + 1$. The only lattice points (m,n) with $0 \le m \le 7$ which satisfy this requirement are $(0,0)$, $(2,1)$, $(5,2)$, and $(7,3)$. In the cases of $(0,0)$ and $(7,3)$, $\overline{AB}$ passes through the center of the square centered at that point, so it also intersects the circle centered at that point. In the cases of $(2,1)$ and $(5,2)$, $\overline{AB}$ passes through the lower-right and upper-left corners of the square centered at that point, respectively, so it does not interest the circle centered at that point. Altogether the segment joining $(0,0)$ to $(7,3)$ intersects 4 squares and 2 circles. The same conclusion applies to the segment joining $(7k-7, 3k-3)$ to $(7k, 3k)$ for each k with $1 \le k \le 143$. Because the points $(7k, 3k)$ belong to two of these segments for $1 \le k \le 142$, $\overline{AB}$ intersects $4 \cdot 143 - 142 = 430$ of the squares and $2 \cdot 143 - 142 = 144$ of the circles. The requested sum is $430 + 144 = 574$.

15. **Answer (270):**

Note that lines AD, XY, BC are the radical axes of pairs of circles ω and ω_1, ω_1 and ω_2, ω_2 and ω, respectively. Therefore lines AD, XY, and BC are either parallel to each other or concurrent. In the former case, ω_1 and ω_2 would be the same size, and by symmetry, $CX = DX$, contradicting the given condition. Hence it must be that lines AD, XY, and BC are concurrent at a point Z. Denote by M the intersection of segments $\overline{ZY}$ and $\overline{AB}$. By the Power of a Point Theorem it follows that $MA^2 = MX \cdot MY = MB^2$. In particular,

$$\begin{aligned} AB^2 &= 4MA^2 = 4MX \cdot MY = 4MX(MX + XY) \\ &= (2MX + XY)^2 - XY^2 = (MY + MX)^2 - XY^2. \end{aligned}$$

Claim: $(MX + MY)^2 = CX \cdot DX$. The claim implies that $AB^2 = CX \cdot DX - XY^2 = 37 \cdot 67 - 47^2 = 270$.



The proof of the claim is based on the following three observations: $ZAXB$ is cyclic; $\triangle XZC$ is similar to $\triangle XDZ$; and $ZAYB$ is a parallelogram.

Because $BCYX$ is cyclic, $\angle XBZ = \angle XYC$. Because $ADYX$ is cyclic, $\angle XAZ = \angle XYD$. Because C, Y, and D are collinear, $\angle XAZ + \angle XBZ = \angle XYD + \angle XYC = 180^\circ$, from which it follows that $ZAXB$ is cyclic, establishing the first observation. This is also a direct consequence of Miquel's Theorem.

Because $BCYX$ and $ZAXB$ are cyclic, $\angle XCB = \angle XYB$ and $\angle ABX = \angle AZX$. Because $\overline{AB}$ is tangent to ω_2 at B, $\angle ABX = \angle XYB$. Combining the three equations yields $\angle XCZ = \angle XCB = \angle XYB = \angle ABX = \angle AZX = \angle DZX$. Likewise, $\angle XZC = \angle XDZ$. Hence $\triangle XZC$ is similar to $\triangle XDZ$, establishing the second observation.

As in the previous paragraph, $\angle XYB = \angle AZX$ or $\overline{BY} \parallel \overline{AZ}$. Similarly, $\overline{AY} \parallel \overline{BZ}$. Thus $ZAYB$ is a parallelogram, establishing the third observation.

Because $ZAYB$ is a parallelogram, $MY = MZ$ and $MX + MY = XM + MZ = XZ$. Because $\triangle XZC$ and $\triangle XDZ$ are similar, $\frac{XZ}{XC} = \frac{XD}{XZ}$ or $XZ^2 = XC \cdot XD$. Combining the last two equations yields $(MX + MY)^2 = XZ^2 = XC \cdot XD$, establishing the claim.

The MAA American Invitational Mathematical Examination
A program of the Mathematical Association of America

Supported by major contributions from

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