MATHEMATICAL ASSOCIATION OF AMERICA
CELEBRATING A CENTURY OF ADVANCING MATHEMATICS

Solutions Pamphlet

MAA American Mathematics Competitions

34th Annual

AIME II

American Invitational Mathematics Examination I

Wednesday, March 16, 2016

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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Correspondence about the problems/solutions for this AIME and orders for any publications should be addressed to:

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1. **Answer (108):**

   After the three people eat the peanuts, $444 - 5 - 9 - 25 = 405$ peanuts remain. Hence after eating the peanuts, for some positive integers b and d, Alex, Betty, and Charlie have $b - d$, b, and $b + d$ peanuts, respectively. Then $(b - d) + b + (b + d) = 3b = 405$, and $b = 135$. Thus Betty originally had 144 peanuts. Because the initial numbers of peanuts were in geometric progression, for some $r > 1$, Alex originally had $\frac{144}{r}$ peanuts and Charlie originally had $144r$ peanuts. Because there was originally a total of 444 peanuts, it follows that $\frac{144}{r} + 144 + 144r = 444$. The only solution greater than 1 is $r = \frac{4}{3}$. Alex initially had $\frac{144}{r} = 108$ peanuts.

2. **Answer (107):**

   Let x be the probability that it rains on Saturday and not on Sunday, y be the probability that it rains on Sunday and not on Saturday, and z be the probability that it rains on both days. Then the conditions of the problem imply that $x + z = \frac{2}{5}$, $y + z = \frac{3}{10}$, and the conditional probabilities satisfy $\frac{z}{x+z} = 2 \cdot \frac{y}{1-(x+z)}$ or $\frac{5}{2}z = \frac{10}{3}y$. Thus $z = \frac{4}{3}y$, so $y + \frac{4}{3}y = \frac{3}{10}$ which gives $y = \frac{9}{70}$. The required probability is $x + y + z = \frac{2}{5} + \frac{9}{70} = \frac{37}{70}$. The requested sum is $37 + 70 = 107$.

3. **Answer (265):**

   The system is equivalent to

$$\begin{aligned} xyz + \log_5 x &= 35 \\ xyz + \log_5 y &= 84 \\ xyz + \log_5 z &= 259. \end{aligned}$$

   Let $x = 5^{\alpha}$, $y = 5^{\beta}$, and $z = 5^{\gamma}$. Then

$$\begin{aligned} 5^{\alpha+\beta+\gamma} + \alpha &= 35 \\ 5^{\alpha+\beta+\gamma} + \beta &= 84 \\ 5^{\alpha+\beta+\gamma} + \gamma &= 259. \end{aligned}$$

   Adding these equations yields $3 \cdot 5^{\alpha+\beta+\gamma} + (\alpha + \beta + \gamma) = 378$. Let t be chosen so that $3t = \alpha + \beta + \gamma$. Then $3 \cdot 5^{3t} + 3t = 378$ and $125^t + t = 126$. Because $125^t + t$ is an increasing function of t, there is only one value of t satisfying this equation, and inspection shows that $t = 1$. Thus $\alpha + \beta + \gamma = 3$, implying that $5^3 + \alpha = 35$ and $\alpha = -90$; $5^3 + \beta = 84$ and $\beta = -41$; and $5^3 + \gamma = 259$ and $\gamma = 134$. The requested sum is $|\alpha| + |\beta| + |\gamma| = 90 + 41 + 134 = 265$.

4. **Answer (180):**

   Let the box contain r red cubes, g green cubes, and y yellow cubes. The given information implies that $r : g : y = 3 : 4 : 5$. Thus every $1 \times b \times c$ layer contains 15 yellow cubes, and each $a \times 1 \times c$ layer contains 15 red cubes. It follows that $bc = 36$ and $ac = 60$, so $abc^2 = 2160 = 2^4 \cdot 3^3 \cdot 5$. The value of abc is minimized when c is chosen to be as large as possible, that is, when $c = \gcd(36, 60) = 2^2 \cdot 3 = 12$. The corresponding values of a and b are 5 and 3, respectively, and the minimum volume of the box is 180. Note that this can be done if each of the five $1 \times 3 \times 12$ layers is colored in the pattern

$$\begin{matrix} R & R & R & G & G & G & G & Y & Y & Y & Y & Y \\ R & R & R & G & G & G & G & Y & Y & Y & Y & Y \\ R & R & R & G & G & G & G & Y & Y & Y & Y & Y \end{matrix}.$$

5. **Answer (182):**

Let $a = BC_0$, $b = AC_0$, and $c = AB$. Then $C_0C_1 = \frac{ab}{c}$, and $\triangle BC_0C_1$ is similar to $\triangle BAC_0$ with ratio of similarity $\frac{BC_0}{AB} = \frac{a}{c}$. Furthermore, for $n \geq 2$, $\triangle BC_{n-1}C_n$ is similar to $\triangle BC_{n-2}C_{n-1}$ with the same ratio, so $C_{n-1}C_n = \frac{ba^n}{c^n}$. The sum of all lengths C_nC_{n-1} is

$$\sum_{n=1}^{\infty} \frac{ba^n}{c^n} = \frac{\frac{ab}{c}}{1 - \frac{a}{c}} = \frac{ab}{c-a}.$$

This is $6p$, so $ab = 6(c-a)(a+b+c) = 6(c^2-a^2+bc-ab) = 6(b^2+bc-ab)$, from which $6c = 7a-6b$. Squaring both sides gives $36c^2 = 36(a^2+b^2) = 49a^2-84ab+36b^2$, which implies that $13a - 84b = 0$. Because a and b are relatively prime, it follows that $a = 84$ and $b = 13$. Thus $c = 85$, and $p = 84 + 13 + 85 = 182$.

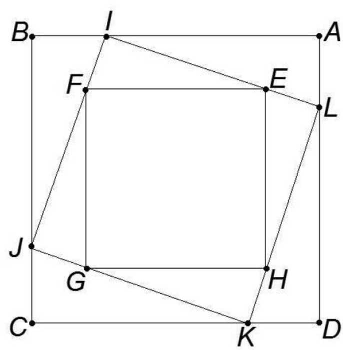
6. **Answer (275):**

The polynomial $P(-x) = 1 + \frac{1}{3}x + \frac{1}{6}x^2$ has nonnegative coefficients equal in absolute value to the coefficients of $P(x)$. The coefficients of $Q(-x) = P(-x)P(-x^3)P(-x^5)P(-x^7)P(-x^9)$ are nonnegative as well because $Q(-x)$ is a product of five polynomials with nonnegative coefficients. Thus the sum of the absolute values of the coefficients of $Q(x)$ is equal to the sum of the coefficients of $Q(-x)$, which is $Q(-1) = P(-1)^5 = \left(\frac{3}{2}\right)^5 = \frac{243}{32}$. The requested sum is $243 + 32 = 275$.

7. **Answer (840):**

Without loss of generality, let E and F be the vertices of $EFGH$ that are nearest A and B, respectively, and let I and J lie on $\overline{AB}$ and $\overline{BC}$, respectively. Because $\triangle EIF$ is similar to $\triangle JBI$, it follows that

$$\frac{EF}{IJ} = \frac{EF}{IF+FJ} = \frac{EF}{EI+IF} = \frac{IJ}{IB+BJ} = \frac{IJ}{AI+IB} = \frac{IJ}{AB},$$



implying that AB, IJ, and EF, in that order, form a decreasing geometric sequence. Hence the three squares have areas $AB^2 = 2016$, $IJ^2 = 2016r$, and $EF^2 = 2016r^2$ for some $0 < r < 1$. For all areas to be integers, r must be rational, and when r^2 is written as a fraction in lowest terms, its denominator must divide $2016 = 2^5 \cdot 3^2 \cdot 7$. Thus the only possible denominators for r written in lowest terms are 2, 3, 4, 6, and 12. Note that if $x = BI$, then

$$IJ^2 = x^2 + (AB - x)^2 = 2x^2 - 2x \cdot AB + AB^2 = 2\left(x - \frac{1}{2}AB\right)^2 + \frac{1}{2}AB^2 \geq \frac{1}{2}AB^2,$$

which implies that $IJ \geq \frac{1}{\sqrt{2}}AB$. Similarly, $EF \geq \frac{1}{\sqrt{2}}IJ$, implying that $EF \geq \frac{1}{2}AB$. Thus the only possible values of r are $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{7}{12}$, and $\frac{11}{12}$. Therefore the difference between the largest and smallest possible values of the area of $IJKL$ is $\left(\frac{11}{12} - \frac{1}{2}\right)(2016) = 840$.

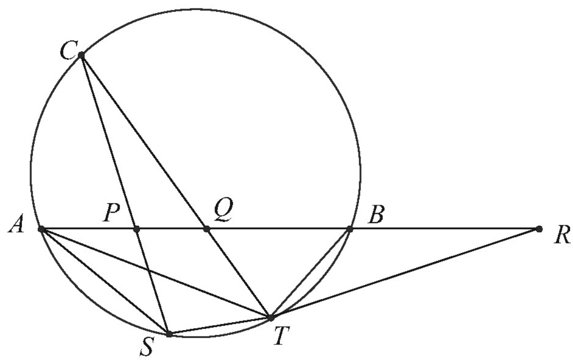
8. **Answer (728):**

It is easier to count the ordered triples (a, b, c) of positive integers with the product abc equal to the product $11 \cdot 21 \cdot 31 \cdot 41 \cdot 51 \cdot 61 = 3^2 \cdot 7 \cdot 11 \cdot 17 \cdot 31 \cdot 41 \cdot 61$. Exactly one of a, b, and c is divisible by each of 7, 11, 17, 31, 41, and 61. Either one of a, b, and c is divisible by 9 or exactly two of a, b, and c are divisible by 3. Hence there are $3^6 \cdot (3+3) = 2 \cdot 3^7$ such ordered triples (a, b, c). In three of these $2 \cdot 3^7$ ordered triples, two of a, b, and c equal 1. In three of these $2 \cdot 3^7$ ordered triples, two of a, b, and c equal 3. In these six cases a, b, and c are not distinct. In the remaining $2 \cdot 3^7 - 6$ ordered triples, a, b, and c are distinct. Each of the required unordered triples is represented by $3! = 6$ of the $2 \cdot 3^7 - 6$ ordered triples (a, b, c). Therefore the requested number of sets is $\frac{2 \cdot 3^7 - 6}{6} = 3^6 - 1 = 728$.

9. **Answer (262):**

Because $c_1 = 2 < 100$, it follows that $k - 1 \geq 2$, so $k \geq 3$. There are integers $d \geq 0$ and $r \geq 1$ such that $a_n = 1 + (n-1)d$ and $b_n = r^{n-1}$, so $100 = c_{k-1} = 1 + (k-2)d + r^{k-2}$ and $1000 = c_{k+1} = 1 + kd + r^k$. Subtracting these equations gives $900 = r^k - r^{k-2} + 2d = r^{k-3}(r-1)r(r+1) + 2d$. Because $(r-1)r(r+1)$ must be a multiple of 3, d is also a multiple of 3. Because $100 = 1 + (k-2)d + r^{k-2}$, it follows that r is also a multiple of 3. The restrictions $r^{k-2} \le 99$ and $r^k \le 999$ show that (r, k) must be one of $(3,3), (3,4), (3,5), (3,6), (6,3)$, or $(9,3)$. For the first five of these there is no integer value for d that satisfies all the required conditions, but if $r = 9$ and $k = 3$, then $d = 90$ does satisfy all the required conditions. In this case $c_k = 1 + (3-1)90 + 9^{3-1} = 262$.

10. **Answer (043):**



Extend $\overline{AB}$ through B to R so that $BR = 8$. Because $ACBT$ is cyclic, it follows by the Power of a Point Theorem that $CQ \cdot QT = AQ \cdot QB = 42$. Note that $PQ \cdot QR = 42 = CQ \cdot QT$. By the converse of the Power of a Point Theorem, it follows that $CPTR$ is cyclic. Because $CPTR$ and $ACTS$ are cyclic,

$$\angle BRT = \angle PRT = \angle PCT = \angle SCT = \angle SAT.$$

Because $ABTS$ is cyclic, it follows that $\angle AST = \angle RBT$. Hence $\triangle AST$ is similar to $\triangle RBT$, from which $\frac{AS}{ST} = \frac{RB}{BT}$ or $ST = AS \cdot \frac{BT}{RB} = \frac{35}{8}$. The requested sum is $35 + 8 = 43$.

11. **Answer (749):**

Consider numbers of the form p^m, where p is a prime and m is a nonnegative integer. Then $(p^m)^k$ has $km+1$ positive divisors. Thus all numbers N that are one more than a nonnegative multiple of k are k-nice. Conversely every k-nice number N must be one more than a nonnegative multiple of k. This is because if an integer a can be written in the form $p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i}$, where $p_1, p_2, \ldots, p_i$ are distinct primes and $m_1, m_2, \ldots, m_i$ are nonnegative integers, then a^k has $N = (km_1+1)(km_2+1)\cdots(km_i+1)$ positive divisors, and by considering each parenthesized factor of N modulo k, it is clear that $N \equiv 1 \pmod{k}$. It follows that there are $\lfloor \frac{999-1}{k} \rfloor + 1$ positive integers less than 1000 that are k-nice. Thus there are 143 positive integers less than 1000 that are 7-nice, and there are 125 positive integers less than 1000 that are 8-nice. Because 7 and 8 are relatively prime, there are $\lfloor \frac{998}{56} \rfloor + 1 = 18$ positive integers less than 1000 that are both 7-nice and 8-nice. Thus there are a total of $999 - (143 + 125 - 18) = 749$ positive integers less than 1000 that are neither 7-nice nor 8-nice.

12. **Answer (732):**

Consider the problem of painting k regions in a row rather than in a ring. Let A_k be the number of ways to paint k regions in a row so that no two adjacent regions receive the same color and the first and last regions are painted different colors. Let B_k be the number of ways to paint the k regions so that no two adjacent regions receive the same color and the first and last regions are painted the same color. Then $A_1 = 0$ and $B_1 = 4$. Note that for $k \geq 1$, $A_{k+1} = 2A_k + 3B_k$ and $B_{k+1} = A_k$. Thus $A_1 = 0, A_2 = 12$, and for $k > 1$, $A_{k+1} = 2A_k + 3A_{k-1}$, which allow the calculation of $A_3 = 24$, $A_4 = 84$, $A_5 = 240$, and $A_6 = 732$. The requested count is equal to $A_6 = 732$. It is easy to verify that $A_k = 3^k + 3(-1)^k$ satisfies the required recursion.

13. **Answer (371):**

There are $6! = 720$ configurations. The minimum possible score of 2 occurs when there is a rook on $(1,1)$, and the maximum possible score of 7 occurs when all the rooks are arranged on the squares $(k, 7-k)$ for $k = 1, 2, 3, 4, 5, 6$. Let a_n be the number of configurations whose score is exactly n, and let b_n be the number of configurations whose score is at least n. Then the total of all 720 scores is

$$\begin{aligned} 2a_2 + 3a_3 + \cdots + 7a_7 &= 2(a_2 + \cdots + a_7) + (a_3 + \cdots + a_7) + \cdots + (a_6 + a_7) + a_7 \\ &= 2b_2 + b_3 + b_4 + b_5 + b_6 + b_7. \end{aligned}$$

In each case the rooks are placed in row order; that is, a square is chosen in row 1, then a square in row 2, and so forth.

Because every configuration has a score of at least 2, conclude that $b_2 = 720$. The configurations counted by b_3 do not have a rook in $(1,1)$, so there are 5 choices for where to put the first row's rook, and $5!$ positions for the remaining rooks, showing that $b_3 = 5 \cdot 5! = 600$. The configurations counted by b_4 can have a rook in any of 4 positions in row 1 and a rook in any of 4 positions in row 2, and there are $4!$ positions for the remaining rooks, showing that $b_4 = 4 \cdot 4 \cdot 4! = 384$. Using a similar line of reasoning, $b_5 = 3 \cdot 3 \cdot 3 \cdot 3! = 162$, $b_6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2! = 32$, and $b_7 = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1! = 1$.

Then the total score is $2 \cdot 720 + 600 + 384 + 162 + 32 + 1 = 2619$. The average is $\frac{2619}{720} = \frac{291}{80}$. The requested sum is $291 + 80 = 371$.

14. **Answer (450):**

Let H be the foot of the perpendicular from P to the plane of $\triangle ABC$. Because $PA = PB = PC$, it follows that $HA = HB = HC$; that is, H is the centroid of the equilateral $\triangle ABC$. Likewise H is also the foot of the perpendicular from Q to the plane of $\triangle ABC$. Hence O is the midpoint of $\overline{PQ}$ and $PQ = 2d$.

Let D be the midpoint of side $\overline{AB}$. Hence $\overline{PD} \perp \overline{AB}$ and $\overline{QD} \perp \overline{AB}$, from which it follows that $\angle PDQ$ is the angle formed by planes ABP and ABQ, and so $\angle PDQ = 120^\circ$. Let $\angle PDH = x$ and $\angle QDH = y$. Then $\tan(x+y) = -\sqrt{3}$.

Set $AB = a$. Then $DC = \frac{\sqrt{3}a}{2}$, $DH = \frac{1}{3}DC = \frac{\sqrt{3}a}{6}$, and $PH = \tan x \cdot DH = \frac{\sqrt{3}a}{6}\tan x$. Likewise $QH = \frac{\sqrt{3}a}{6}\tan y$. Hence $2d = PQ = PH + HQ = \frac{\sqrt{3}a}{6}(\tan x + \tan y)$, or

$$\tan x + \tan y = \frac{4\sqrt{3}d}{a}.$$

Because $OP = OC = OQ$, conclude that O, the midpoint of $\overline{PQ}$, is the circumcenter of $\triangle PCQ$, from which it follows that $\angle PCQ = 90^\circ$. Then $\overline{CH}$ is the altitude to the hypotenuse of right $\triangle CPQ$, implying that $CH^2 = PH \cdot QH$. Hence

$$CH^2 = \left(\frac{2}{3}DC\right)^2 = \frac{a^2}{3} = PH \cdot HQ = \frac{a^2 \tan x \tan y}{12},$$

implying that $\tan x \tan y = 4$.

By the Tangent Angle Addition Formula, $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$ or

$$-\sqrt{3} = \frac{\frac{4\sqrt{3}d}{a}}{1-4},$$

implying that $d = \frac{3a}{4}$. Substituting $a = 600$ in the last equation yields $d = 450$.

15. **Answer (863):**

For $1 \le i \le 216$, let $b_i = \sqrt{1-a_i}$. Because $\sum_{i=1}^{216} a_i = 1$, it follows that $\sum_{i=1}^{216} b_i^2 = 215$. Also, if $\{x_i\}$ is a sequence of positive real numbers with $\sum_{i=1}^{216} x_i = 1$, then $\sum_{1\le i<j\le 216} 2x_ix_j = \left(\sum_{i=1}^{216} x_i\right)^2 - \sum_{i=1}^{216} x_i^2 = 1 - \sum_{i=1}^{216} x_i^2$.

Observe that for each i, $\left(\frac{x_i}{b_i}\right)^2 - \frac{2x_i}{215} + \left(\frac{b_i}{215}\right)^2 = \left(\frac{x_i}{b_i} - \frac{b_i}{215}\right)^2 \ge 0$, and thus summing over $1 \le i \le 216$ yields $\sum_{i=1}^{216}\left(\left(\frac{x_i}{b_i}\right)^2 - \frac{2x_i}{215} + \left(\frac{b_i}{215}\right)^2\right) \ge 0$. Because $\sum_{i=1}^{216} b_i^2 = 215$ and $\sum_{i=1}^{216} x_i = 1$, it follows that $\sum_{i=1}^{216}\left(\frac{x_i}{b_i}\right)^2 - \frac{2}{215} + \frac{1}{215^2}\cdot 215 \ge 0$, which is equivalent to $\frac{1}{215} \le \sum_{i=1}^{216}\left(\frac{x_i}{b_i}\right)^2 = \sum_{i=1}^{216}\frac{x_i^2}{1-a_i}$. Hence $\sum_{1\le i<j\le 216} 2x_ix_j = 1 - \sum_{i=1}^{216} x_i^2 \le \frac{214}{215} + \sum_{i=1}^{216}\frac{x_i^2}{1-a_i} - \sum_{i=1}^{216} x_i^2 = \frac{214}{215} + \sum_{i=1}^{216}\frac{a_ix_i^2}{1-a_i}$, so $\sum_{1\le i<j\le 216} x_ix_j \le \frac{107}{215} + \sum_{i=1}^{216}\frac{a_ix_i^2}{2(1-a_i)}$.

Equality occurs in this inequality if and only if for each i, $\frac{x_i}{b_i} - \frac{b_i}{215} = 0$ or $x_i = \frac{1-a_i}{215}$. Therefore such a sequence $\{x_i\}$ is unique and $x_2 = \frac{3}{860}$. The requested sum is $3 + 860 = 863$.

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