#### 1. Answer (722):

Observe

$$\begin{split} B - A &= (1 - 39) + (3 \times 2 - 1 \times 2) + (5 \times 4 - 3 \times 4) + (7 \times 6 - 5 \times 6) + \cdots \\ &+ (39 \times 38 - 37 \times 38) \\ &= -38 + (2 \times 2) + (2 \times 4) + (2 \times 6) + \cdots + (2 \times 38) \\ &= -38 + 4 \times (1 + 2 + 3 + \cdots + 19) \\ &= -38 + 4 \times \frac{19 \cdot 20}{2} = 722. \end{split}$$

### 2. Answer (139):

There are  $\binom{9}{3} = 84$  equally likely ways to select the three delegates who fall asleep. There are  $\binom{4}{2}(2+3) = 30$  ways to select them so that exactly two delegates are from the United States,  $\binom{3}{2}(2+4) = 18$  ways to select them so that exactly two delegates are from Canada, and  $\binom{2}{2}(3+4) = 7$  ways to select them so that exactly two delegates are from Mexico. The desired probability is therefore  $\frac{30+18+7}{84} = \frac{55}{84}$ . The requested sum is 55 + 84 = 139.

### 3. Answer (307):

Let  $16p + 1 = n^3$  for some positive integer n. Because 16p + 1 is odd, both n and  $n^2 + n + 1$  are also odd. It follows that  $16p = n^3 - 1 = (n - 1)(n^2 + n + 1)$ , so 16 = n - 1 and  $p = n^2 + n + 1$ . Thus  $p = 17^2 + 17 + 1 = 307$ , which is prime.

### 4. Answer (507):

Place the triangles on a coordinate grid so that A is at (-16, 0), B is at (0, 0), and C is at (4, 0). Then D is at  $(-8, 8\sqrt{3})$ , E is at  $(2, 2\sqrt{3})$ , M is at  $(-7, \sqrt{3})$ , and N is at  $(-2, 4\sqrt{3})$ . Because  $BM = MN = NB = \sqrt{52}$ ,  $\triangle BMN$  is equilateral with area  $\frac{\sqrt{3}}{4}(BM)^2 = 13\sqrt{3} = x$ . Thus  $x^2 = 169 \cdot 3 = 507$ .

### 5. Answer (341):

The probability of a match on Monday is  $\frac{1}{9}$ , so the probability of a mismatch is  $\frac{8}{9}$ . A mismatch on Tuesday can occur in one of three ways:

- (a) On Tuesday Sandy selects both of the mismatching colors selected Monday. The probability of this is  $\frac{2}{8} \cdot \frac{1}{7} = \frac{1}{28}$ . In this case Sandy has three colors from which to select a pair on Wednesday.
- (b) On Tuesday Sandy selects one of the colors selected Monday and a new color. The probability of this is  $\frac{2}{8} \cdot \frac{6}{7} + \frac{6}{8} \cdot \frac{2}{7} = \frac{3}{7}$ . In this case Sandy has two colors from which to select a pair on Wednesday.
- (c) On Tuesday Sandy selects two new colors. The probability of this is  $\frac{6}{8} \cdot \frac{4}{7} = \frac{3}{7}$ . In this case Sandy has only one color from which to select a pair on Wednesday.

The probability of achieving the first match on Wednesday is therefore  $\frac{8}{9}\left(\frac{1}{28} \cdot \frac{6}{6} \cdot \frac{1}{5} + \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{1}{5} + \frac{3}{7} \cdot \frac{2}{6} \cdot \frac{1}{5}\right) = \frac{8}{9} \cdot \frac{13}{140} = \frac{26}{315}$ . The requested sum is 26 + 315 = 341.

## $\mathbf{OR}$

The probability of Sandy getting a mismatch on Monday, a mismatch on Tuesday, and a match on Wednesday is the same as the probability of Sandy getting a match on Monday, a mismatch on Tuesday, and a mismatch on Wednesday. The latter probability is  $\frac{1}{9} \cdot \frac{6}{7} \cdot \frac{13}{15} = \frac{26}{315}$ .

### 6. Answer (058):

Because A, I, H, G, F, and E are equally spaced, let  $\alpha = \angle ECF = \angle FCG = \angle GCH = \angle HCI = \angle ICA$ . It follows that  $\angle ACE = \angle ABE = \angle ADE = 5\alpha$ . Also,  $\widehat{AHG} = 3\alpha$  so  $\angle AHG = \frac{360^{\circ} - 3\alpha}{2}$ . Because  $\angle ACE = 5\alpha$ ,  $\widehat{ACE} = 360^{\circ} - 10\alpha$ , and  $\widehat{ABD} = 270^{\circ} - \frac{15\alpha}{2}$ . Thus  $\angle ABD = \frac{360^{\circ} - (270^{\circ} - \frac{15\alpha}{2})}{2} = 45^{\circ} + \frac{15\alpha}{4}$ . Then  $\angle ABD - \angle AHG = (45^{\circ} + \frac{15\alpha}{4}) - (180^{\circ} - \frac{3\alpha}{2}) = \frac{21\alpha}{4} - 135^{\circ} = 12^{\circ}$ . Hence  $\alpha = 28^{\circ}$ .

Now  $\widehat{EFG} = 2\alpha$ , so  $\angle EAG = \alpha = 28^{\circ}$ . From above  $\widehat{ABD} = 270^{\circ} - \frac{15\alpha}{2} = 60^{\circ} = \widehat{BCE}$ , so  $\angle BAE = \frac{60^{\circ}}{2} = 30^{\circ}$ . Finally  $\angle BAG = \angle BAE + \angle EAG = 30^{\circ} + 28^{\circ} = 58^{\circ}$ .

### 7. Answer (539):

Let AE = s, so CD = 2s, and  $CE = \sqrt{5}s$ . Note that  $\triangle CDE$ ,  $\triangle JFC$ ,  $\triangle HBJ$ ,  $\triangle NKH$ , and  $\triangle MAN$  are similar to each other. Let x = FG and y = KL. Then  $2s = BC = BJ + JC = \frac{x}{\sqrt{5}} + \frac{x\sqrt{5}}{2}$ , so  $x = \frac{2s}{\frac{1}{\sqrt{5}} + \frac{\sqrt{5}}{2}} = \frac{4\sqrt{5}s}{7}$ . Then  $AH = 2s - HB = 2s - \frac{2x}{\sqrt{5}} = \frac{6s}{7}$ . Hence  $\frac{6s}{7} = AH = AN + NH = \frac{y}{\sqrt{5}} + \frac{y\sqrt{5}}{2}$ , so  $y = \frac{\frac{6s}{7}}{\frac{1}{\sqrt{5}} + \frac{\sqrt{5}}{2}} = \frac{12\sqrt{5}s}{49}$ . The ratio of the areas of squares FGHJ and KLMN is  $\left(\frac{x}{y}\right)^2 = \left(\frac{\frac{4\sqrt{5}s}{7}}{\frac{12\sqrt{5}s}{49}}\right)^2 = \frac{49}{9}$ . Thus if square KLMN has area 99, square FGHJ has area  $99 \cdot \frac{49}{9} = 539$ .

OR

With x and y defined as above, note that AN + NH + HB = BJ + JC so

$$\frac{1}{\sqrt{5}}y + \frac{\sqrt{5}}{2}y + \frac{2}{\sqrt{5}}x = \frac{1}{\sqrt{5}}x + \frac{\sqrt{5}}{2}x.$$

Thus

$$\left(\frac{1}{\sqrt{5}} + \frac{\sqrt{5}}{2}\right)y = \left(\frac{1}{\sqrt{5}} + \frac{\sqrt{5}}{2} - \frac{2}{\sqrt{5}}\right)x$$

Then the ratio of the areas of squares FGHJ and KLMN is

$$\left(\frac{x}{y}\right)^2 = \left(\frac{\frac{1}{\sqrt{5}} + \frac{\sqrt{5}}{2}}{\frac{\sqrt{5}}{2} - \frac{1}{\sqrt{5}}}\right)^2 = \left(\frac{2+5}{5-2}\right)^2 = \frac{49}{9},$$

and the result is as above.

### 8. Answer (695):

When adding two numbers, a carry operation replaces 10 in the kth position with 1 in the (k + 1)st position, thus reducing the sum of these values by 10 - 1 = 9. Therefore s(n + 864) = s(n) + s(864) - 9c, where c is the number of carries performed when adding the two numbers.

To have s(n) = 20 and 20 = s(n + 864) = s(n) + s(864) - 9c = 20 + 18 - 9c, there must be precisely c = 2 carries when performing the addition n + 864. Assuming that n < 1000, represent the three-digit number n as  $\underline{t} \underline{u} \underline{v}$ . Because  $u + v \le 9 + 9 = 18$ , it follows that  $t \ge 2$ , and thus the hundreds position t + 8must carry. Thus either the tens or the ones position carries, and the other does not.

If the ones carry and the tens do not, then the middle digit of the sum must be  $6 + u + 1 \le 9$ , so  $u \le 2$ . The only possibility is then n = 929, which indeed satisfies the conditions: s(929) = 20 = s(929 + 864) = s(1793).

In the other case, the tens carry and the ones do not, so  $v + 4 \leq 9$  and  $v \leq 5$ . It follows that  $t = 20 - u - v \geq 20 - 9 - 5 = 6$ , so  $n \geq 695$ . Indeed s(695) = 20 = s(695 + 864) = s(1559). The smallest positive integer for which s(n) = s(n + 864) = 20 is 695.

### $\mathbf{OR}$

Note that the least n for which s(n) = 20 is n = 299. For n with s(n) = 20, let n have, in order, the digits a, b, and 20 - (a + b), so  $a \ge 2$  and  $a + b \ge 11$ . If  $a + b \le 14$ , and b < 3, then n + 864 has digits 1, a - 2, b + 7, and 14 - (a + b), implying that s(n + 864) = 20. The least such n occurs when a = 9 and b = 2, so n = 929. If  $a + b \le 14$ , and  $b \ge 3$ , then n + 864 has digits 1, a - 1, b - 3, and 14 - (a + b), implying that s(n + 864) = 11.

If  $a + b \ge 15$  or b < 4, then n + 864 has digits 1, a - 2, b + 6, or 24 - (a + b), implying that s(n + 864) = 29. If  $a + b \ge 15$  and  $b \ge 4$ , then n + 864 has digits 1, a - 1, b - 4, and 24 - (a + b), implying that s(n + 864) = 20. The least such n occurs when a = 6 and b = 9, so n = 695.

### 9. Answer (494):

Note that if  $a_{k-1} = a_k$ , then  $a_{k+2} = 0$ , and if  $|a_k - a_{k-1}| = 1$ , then  $a_{k+2} = a_{k+1}$ , and  $a_{k+4} = 0$ . Therefore the sequence contains a term of 0 if  $(a_1, a_2, a_3)$  has one of the forms (j, j, k), (j, k, k), (j, j+1, k), (j, k, k+1), (j, j-1, k), or (j, k, k-1). There are 100 triples with each of the first two forms and 90 triples with each of the other four forms, for a total of  $2 \cdot 100 + 4 \cdot 90 = 560$ . However, the 10 triples of the form (j, j, j), the 9 triples of each of the forms (j, j, j+1), (j, j+1, j), (j, j+1, j+1), (j, j, j-1), (j, j-1, j), and (j, j-1, j-1), and the 8 triples of each of the forms (j, j+1, j+2) and (j, j-1, j-2) have each been counted twice, so there are  $560 - 10 - 6 \cdot 9 - 2 \cdot 8 = 480$  triples of one of these forms.

In addition, if  $(a_1, a_2, a_3) = (j, j+2, 1)$  or (j, j-2, 1), then  $a_4 = 2$ ,  $a_4 - a_3 = 1$ , and  $a_8 = 0$ . There are 8 triples of each of these forms, but (3, 1, 1) and (4, 2, 1) have already been counted. Thus there are at least 480 + 8 + 6 = 494 sequences that contain a term of 0.

To see that there are no other possibilities, note first that if  $|a_2 - a_1| \ge 2$ ,  $|a_3 - a_2| \ge 2$ , and  $a_3 \ge 2$ , then  $a_4 \ge 2a_3 > a_3$ , and  $|a_4 - a_3| \ge a_3 \ge 2$ . The same argument then establishes that  $a_k > a_{k-1}$  for  $k \ge 4$ , so  $a_k \ne 0$  for all k. Further, if  $|a_2 - a_1| = m \ge 3$ ,  $|a_3 - a_2| \ge 2$ , and  $a_3 = 1$ , then  $a_4 = m \ge 3$  and  $|a_4 - a_3| = m - 1 \ge 2$ . As above,  $a_k > a_{k-1}$  for  $k \ge 4$ . Therefore the number of sequences that contain a term of 0 is 494.

### 10. Answer (072):

Without loss of generality assume that f(x) has a positive leading coefficient. Polynomial f(x) has degree 3, so each of f(x) + 12 and f(x) - 12 has at most three distinct roots. Because 1,2,3,5,6,7 are among these combined roots, each polynomial has precisely three of these as roots. Because f has a local maximum at a point a and a local minimum at a point b where a < b, f increases on the interval  $(-\infty, a)$ , decreases on the interval (a, b), and increases on the interval  $(b, \infty)$ . This shows that f(x) must be equal to -12 at x = 1, 5, and 6 and equal to 12 at x = 2, 3, and 7. Thus if f(x) has leading coefficient c, then f(x)-12 = c(x-2)(x-3)(x-7), so f(x) = c(x-2)(x-3)(x-7)+12. Similarly f(x) = c(x-1)(x-5)(x-6)-12. Then f(0) = -42c+12 = -30c-12 implying that c = 2 and |f(0)| = 72.

# 11. Answer (108):

Let M denote the midpoint of side BC. Note that A, I, and M are collinear. Set a = AB, and b = BM. Note that a > BI = 8. Then

$$\cos(\angle ABM) = \frac{BM}{AB} = \frac{b}{a}$$
 and  $\cos(\angle IBM) = \frac{BM}{BI} = \frac{b}{8}$ .

The double-angle formula for cosine yields

$$\frac{b}{a} = 2 \cdot \frac{b^2}{8^2} - 1$$
 or  $ab^2 - 32b - 32a = 0$ .

Solving for a yields  $a = \frac{32b}{b^2-32}$ . Because BC = 2b must be an integer, let c = 2b so that  $a = \frac{64c}{c^2-128}$ . This shows that  $c^2 > 128$ , so c > 11, and c = 2BM < 2BI = 16. Testing c = 12, 13, 14, and 15, it follows that a is an integer only when c = 12 and a = 48, and the perimeter of  $\triangle ABC$  is 48 + 48 + 12 = 108.

### 12. Answer (431):

Consider the number of 1000-element subsets of  $\{1, 2, 3, ..., 2015\}$  with least element  $j, 1 \leq j \leq 1016$ . Every such subset has 999 elements greater than j, and thus the number of such subsets is  $\binom{2015-j}{999}$ . Therefore the sum of all least elements of the subsets under consideration is  $\sum_{j=1}^{1016} j \binom{2015-j}{999}$ . Thus the arithmetic mean in question is equal to  $\frac{\sum_{j=1}^{1016} j \binom{2015-j}{999}}{\binom{2015}{1000}}$ .

Note that the number of 1001-element subsets of  $\{0, 1, 2, \ldots, 2015\}$  where j is the second smallest element is  $j\binom{2015-j}{999}$  because there are j choices for the least element of the subset and  $\binom{2015-j}{999}$  choices for the largest 999 elements. Thus  $\sum_{j=1}^{1016} j\binom{2015-j}{999} = \binom{2016}{1001}$ .

Hence the required arithmetic mean is  $\frac{\sum_{j=1}^{1016} j\binom{2015-j}{999}}{\binom{2015}{1000}} = \frac{\binom{2016}{1001}}{\binom{2015}{1000}} = \frac{2016}{1001} = \frac{288}{143}$ The requested sum is 288 + 143 = 431.

### OR

Another way to simplify  $\sum_{j=1}^{1016} j \binom{2015-j}{999}$  is to apply the Hockey Stick Theorem,  $\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}$ , to get  $\sum_{j=1}^{1016} j \binom{2015-j}{999} = \sum_{j=0}^{1015} (1016-j) \binom{999+j}{j} = \sum_{k=0}^{1015} \sum_{j=0}^{k} \binom{999+j}{j} = \sum_{k=0}^{1015} \binom{1000+k}{k} = \binom{2016}{1015} = \binom{2016}{1001}.$ 

#### 13. Answer (091):

Let

 $P = \sin 1^{\circ} \sin 3^{\circ} \sin 5^{\circ} \cdots \sin 89^{\circ},$ 

and let

 $Q = \sin 2^{\circ} \sin 4^{\circ} \sin 6^{\circ} \cdots \sin 88^{\circ}.$ 

Then

 $PQ = \sin 1^{\circ} \sin 2^{\circ} \sin 3^{\circ} \cdots \sin 89^{\circ}.$ 

But also

 $PQ = \sin 89^{\circ} \sin 88^{\circ} \sin 87^{\circ} \cdots \sin 1^{\circ}.$ 

Therefore

$$P^{2}Q^{2} = \sin 1^{\circ} \sin 89^{\circ} \sin 2^{\circ} \sin 88^{\circ} \sin 3^{\circ} \sin 87^{\circ} \cdots \sin 89^{\circ} \sin 1^{\circ}$$
  
= sin 1° cos 1° sin 2° cos 2° sin 3° cos 3° · · · sin 89° cos 89°.

Then

$$2^{89}P^2Q^2 = (2\sin 1^{\circ}\cos 1^{\circ})(2\sin 2^{\circ}\cos 2^{\circ})(2\sin 3^{\circ}\cos 3^{\circ})\cdots(2\sin 89^{\circ}\cos 89^{\circ})$$
  
= sin 2° sin 4° sin 6° · · · sin 178°  
= (sin 2° sin 4° sin 6° · · · sin 88°)(sin 92° sin 94° sin 96° · · · sin 178°)  
= (sin 2° sin 4° sin 6° · · · sin 88°)(sin 88° sin 86° sin 84° · · · sin 2°)  
= Q^2.

Thus  $2^{89}P^2Q^2 = Q^2$ . Because  $Q^2 \neq 0$ , the requested product of cosecants equals  $\frac{1}{P^2} = 2^{89}$ . Because 89 is prime, this representation with integers greater than 1 is unique. The requested sum is 2 + 89 = 91.

### 14. Answer (483):

For positive integer k, if  $1 \le k^2 \le x < (k+1)^2$ , then  $x\lfloor\sqrt{x}\rfloor = kx$ . The inequalities  $n \le x < n+1$  and  $0 \le y \le x\lfloor\sqrt{x}\rfloor$  define a trapezoid with height 1 and average of the bases  $\frac{(2n+1)k}{2}$ . The area of this trapezoid, which is A(n+1) - A(n), is an integer if k is even and a half-integer if k is odd. Hence for even values of k, A(n+1) is an integer if and only if A(n) is an integer, and for odd values of k, A(n+1) is an integer if and only if A(n) is not an integer.

For  $k \geq 1$ , let  $I_k$  be the set of the 2k+1 integers n such that  $k^2 < n \leq (k+1)^2$ . If k is even, the values of A(n) for  $n \in I_k$  are either all integers or all non-integers, according to whether  $A(k^2)$  is or is not an integer. Furthermore, if k is odd, the values of A(n) for  $n \in I_k$  alternate between integers and non-integers, beginning with an integer if  $A(k^2)$  is a non-integer and vice versa. Because A(2) is not an integer, the number of integer values of A(n) for elements of each set  $I_k$  can be calculated by considering k modulo 4: k = 4j - 3, 4j - 2, 4j - 1, 4j.

Because  $A((4j-3)^2)$  is an integer, the values of A(n) for  $n \in I_{4j-3}$  alternate between integers and non-integers, beginning and ending with a non-integer. Thus there are 4j-3 integer values of A(n) for  $n \in I_{4j-3}$ .

Because  $A((4j-2)^2)$  is not an integer, there are no integer values of A(n) for  $n \in I_{4j-2}$ .

Because  $A((4j-1)^2)$  is not an integer, the values of A(n) for  $n \in I_{4j-1}$  alternate between integers and non-integers, beginning and ending with an integer. Thus there are 4j integer values of A(n) for  $n \in I_{4j-1}$ .

Because  $A((4j)^2)$  is an integer, all 8j+1 values of A(n) for  $n \in I_{4j}$  are integers.

$$\sum_{j=1}^{7} (16j-2) = 16\left(\frac{7\cdot 8}{2}\right) - 7\cdot 2 = 434.$$

There are additionally 29 integer values of A(n) for  $29^2 < n \le 30^2$ , none for  $30^2 < n \le 31^2$ , and 20 for  $31^2 < n \le 1000$ , for a total of 434 + 29 + 20 = 483 integer values of A(n).

### 15. Answer (053):

Orient the (uncut) block so that the circular face containing A and B rests on the floor. Let O be the center of the cylinder, let M be the midpoint of segment  $\overline{AB}$ , and let R be the region whose area must be computed.

Project the diagram into the plane of the bottom face; point O projects to O', the center of the bottom face, and region R projects to region R' as shown. The circular segment S enclosed by chord  $\overline{AB}$  and minor arc  $\overline{AB}$  may be created by removing  $\triangle AO'B$  from sector AO'B, so S has area  $\frac{1}{3} \cdot \pi \cdot 6^2 - \frac{1}{2} \cdot 6 \cdot 6 \cdot \sin 120^\circ = 12\pi - 9\sqrt{3}$ . Thus region R' has area  $\pi \cdot 6^2 - 2(12\pi - 9\sqrt{3}) = 12\pi + 18\sqrt{3}$ .

Because  $\triangle O'MA$  is a  $30-60-90^{\circ}$  right triangle, O'M = 3. Because OO' = 4 and  $\triangle OO'M$  is a right triangle, OM = 5. Because R' is the orthogonal projection of R, the two are related by a stretch in the direction perpendicular to the floor: stretching region R' by a factor of  $\frac{OM}{O'M} = \frac{5}{3}$  in the  $\overline{O'M}$  direction (and by a factor of 1 in the direction parallel to  $\overline{AB}$ ) results in a shape congruent to region R. Thus the area of R is  $\frac{5}{3}$  the area of R', so the area of R is  $20\pi + 30\sqrt{3}$ . The requested sum is 20 + 30 + 3 = 53.



The problems and solutions for this AIME were contributed by Zachary Abel, Steve Dunbar, Jacek Fabrykowski, Zuming Feng, Peter Gilchrist, Ellina Grigorieva, Jerry Grossman, David Hankin, Elgin Johnston, Jonathan Kane, Tamas Szabo, Alan Vraspir and David Wells.