1. **Answer (131):**

   Let a and b be integers such that $N = 0.78a = 1.16b$. Then $50N = 39a$ and $25N = 29b$. Therefore N must be a common multiple of 29 and 39. Their least common multiple, $29 \cdot 39 = 1131$, satisfies the requirements, with $a = 50 \cdot 29 = 1450$ and $b = 25 \cdot 39 = 975$. The requested remainder is 131.

2. **Answer (025):**

   Without loss of generality it may be assumed that there are 100 students in the school. Then the students taking Latin consist of 40 freshman, $30(0.8) = 24$ sophomores, $20(0.5) = 10$ juniors, and $10(0.2) = 2$ seniors. The required probability is the number of sophomores taking Latin divided by the number of students taking Latin or $\frac{24}{40+24+10+2} = \frac{24}{76} = \frac{6}{19}$. The requested sum is $6 + 19 = 25$.

3. **Answer (476):**

   Assume that there is such an m less than 1000, and let $m = 100a+10b+c$ where a, b, and c are the digits of m. According to the required properties, there is an integer n such that $100a + 10b + c = 17n$ and $a + b + c = 17$. Subtracting the second equation from the first gives $99a + 9b = 9(11a + b) = 17(n - 1)$. Thus $n-1$ is divisible by 9. If $n-1 = 9$ or $n-1 = 18$, then $17n = 170$ or $17n = 323$, respectively, and neither of these has digits that sum to 17. If $n - 1 = 27$, then $17n = 476$, whose digits indeed sum to 17. Thus the requested integer is $m = 476$.

4. **Answer (018):**

   Let F be one of the vertices of the smaller base, let H be the foot of the altitude from F to the larger base, and let G be the vertex of the larger base closer to H. Because the trapezoid is isosceles, it follows that $GH = \frac{1}{2}(\log 192 - \log 3) = \frac{1}{2}(\log \frac{192}{3}) = \frac{1}{2}\log 64 = \frac{1}{2}\log 2^6 = 3\log 2$. Note that $FH = \log 2^4 = 4\log 2$; hence right $\triangle FGH$ has sides in the ratio of $3 : 4 : 5$, and thus $FG = 5\log 2$. The perimeter of the trapezoid is therefore $\log 3 + \log 192 + 10\log 2 = 2\log 3 + 16\log 2 = \log 2^{16}3^2$. The requested sum is $16 + 2 = 18$.

5. **Answer (090):**

   In each row of an $n \times n$ grid of squares, there are $n - 1$ pairs of adjacent squares. Thus there are $n(n - 1)$ pairs of horizontally adjacent squares in the grid. Similarly there are $n(n-1)$ pairs of vertically adjacent squares in the grid. Out of the $\binom{n^2}{2}$ equally likely ways to select two squares in the grid, there are $2n(n - 1)$ ways to select the two squares so that they are adjacent. Hence the required condition is $\frac{1}{2015} > \frac{2n(n-1)}{\binom{n^2}{2}} = \frac{2n(n-1)\cdot 2}{n^2(n^2-1)} = \frac{4}{n^2+n}$, which simplifies to $n^2 + n > 8060$. The least positive integer satisfying this is $n = 90$.

6. **Answer (440):**

Let the roots be r, s, and t, with $r \le s \le t$. Then $r+s+t=a$, and $rs+st+tr=\frac{a^2-81}{2}$, so $r^2+s^2+t^2=(r+s+t)^2-2(rs+st+tr)=81$. The positive integer solutions for (r,s,t) are $(1,4,8)$, $(4,4,7)$, and $(3,6,6)$. The corresponding values of a are 13, 15, and 15, respectively. Because there are two possible values of $c=2rst$, it follows that $a=15$, and the two possible values of c are $2\cdot 4\cdot 4\cdot 7=224$ and $2\cdot 3\cdot 6\cdot 6=216$. The requested sum is $224+216=440$.

7. **Answer (161):**

By Heron's formula, the area of $\triangle ABC$ is 90. Then the altitude from A has length $h=\frac{2\cdot 90}{25}$. The altitude from A in $\triangle APQ$ has length $\frac{PQ}{BC}h=\frac{w}{25}h$. It follows that $PS=h-\frac{w}{25}h$, so

$$\text{Area}(PQRS)=PQ\cdot PS=w\left(h-\frac{w}{25}h\right)=hw-\frac{h}{25}w^2=hw-\frac{2\cdot 90}{25^2}w^2$$

and $\beta=\frac{2\cdot 90}{25^2}=\frac{36}{125}$. The requested sum is $36+125=161$.

OR

Let $f(w)$ denote the area of the rectangle of side w. Because $f(0)=f(25)=0$,

$$f(w)=\alpha w-\beta w^2=\beta w(25-w).$$

It is easy to check that if $w=\frac{25}{2}$, then $\text{Area}(PQRS)=\frac{1}{2}\cdot 90=45$. Therefore

$$45=f\left(\frac{25}{2}\right)=\beta\cdot\frac{25}{2}\left(25-\frac{25}{2}\right)=\frac{25^2}{4}\beta.$$

Hence

$$\beta=\frac{4\cdot 45}{25^2}=\frac{180}{625}=\frac{36}{125}.$$

OR

The Law of Cosines can be used to calculate $\cos(\angle ABC)=\frac{4}{5}$ and $\cos(\angle ACB)=\frac{77}{85}$. Then $\tan(\angle ABC)=\frac{3}{4}$ and $\tan(\angle ACB)=\frac{36}{77}$. Let $h=PS$. Then $25=BC=\frac{h}{\tan(\angle ABC)}+w+\frac{h}{\tan(\angle ACB)}$, from which $h=\frac{36(25-w)}{125}$. Then the area of the rectangle is $wh=\frac{36}{5}w-\frac{36}{125}w^2$.

8. **Answer (036):**

First observe that if $a = 1$ or $b = 1$, then $\frac{a^3b^3+1}{a^3+b^3} = 1$. Assume that $a \geq 2$ and $b \geq 2$. The inequality $\frac{ab+1}{a+b} < \frac{3}{2}$ implies that $2ab + 2 < 3a + 3b$, and hence $3b - 2 > a(2b - 3)$ giving $3b - 2 > 4b - 6$ which implies that $b < 4$; by symmetry $a < 4$. The pair $(a, b) = (3, 3)$ does not satisfy $\frac{ab+1}{a+b} < \frac{3}{2}$, but checking the pairs $(a, b) = (2, 2)$ and $(a, b) = (2, 3)$, it is seen that the maximum value of $\frac{a^3b^3+1}{a^3+b^3}$ is $\frac{31}{5}$, which occurs at $(a, b) = (2, 3)$. The requested sum is $31 + 5 = 36$.

OR

From $2ab + 2 < 3a + 3b$ it follows that $2ab - 3a - 3b + 2 < 0$ implying $4ab - 6a - 6b + 4 + 5 < 5$ and $(2a - 3)(2b - 3) < 5$. Thus $2a - 3$ and $2b - 3$ are odd integers whose product is less than 5. This shows that either a or b is 1, or $\{a, b\} \subseteq \{2, 3\}$, and the analysis proceeds as above.

9. **Answer (384):**

The region inside the cube sitting inside the barrel is a right triangular pyramid with an equilateral triangle for a base and three other faces that are congruent right isosceles triangles. The center of the equilateral triangular base is the center of the circle at the top of the barrel. Because the barrel has radius 4, the equilateral triangle has side length $4\sqrt{3}$ and altitude 6. It follows that the legs of the isosceles right triangular faces of the pyramid have length $\frac{4\sqrt{3}}{\sqrt{2}} = 2\sqrt{6}$. The volume of the displaced water is the volume of the pyramid. Reorienting the pyramid so that its base is a right isosceles triangle with legs of length $2\sqrt{6}$, and its height is $2\sqrt{6}$ shows that the volume is $\frac{1}{3}\left(\frac{1}{2}(2\sqrt{6})^2\right)(2\sqrt{6}) = 8\sqrt{6}$. The requested value is $(8\sqrt{6})^2 = 384$.

10. **Answer (486):**

Let S_n be the number of quasi-increasing permutations of $1, 2, \ldots, n$. It is easy to check that $S_1 = 1$, $S_2 = 2$, and $S_3 = 6$. For $n \geq 3$, $S_n = 3S_{n-1}$ is proved as follows.

First note that if $a_1, a_2, \ldots, a_{n-1}$ is a quasi-increasing permutation of $1, 2, \ldots, n-1$, then one can construct a quasi-increasing permutation of $1, 2, \ldots, n$ by placing the n immediately in front of $n - 1$, immediately in front of $n - 2$, or after a_{n-1}. If n is placed in any other position, then it will be followed by an integer k with $1 \leq k \leq n - 3$ so $n \leq k + 2$ will not be true. Thus every quasi-increasing permutation of $1, 2, \ldots, n-1$ leads to 3 quasi-increasing permutations of $1, 2, \ldots, n$.

Conversely, suppose that we have a quasi-increasing permutation $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$. If $a_n = n$ or $a_1 = n$, then removing a_n results in a quasi-increasing permutation of $1, 2, \ldots, n-1$. If $n = a_k$, $k \neq 1, n$, then $a_{k+1} = n - 1$

or $a_{k+1} = n - 2$. In either case

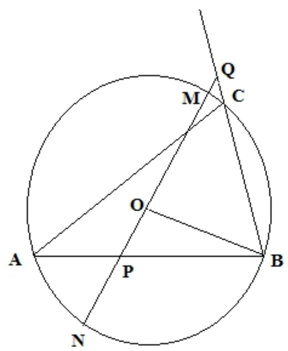
$$a_{k-1} < n \leq a_{k+1} + 2,$$

so removing n again results in a quasi-increasing permutation of $1, 2, \ldots, n-1$. Furthermore, as the previous paragraph showed, for each quasi-increasing permutation π of $1, 2, \ldots, n-1$, there are exactly 3 quasi-increasing permutations of $1, 2, \ldots, n$ that result in π when n is removed. This completes the proof that $S_n = 3S_{n-1}$ for $n \geq 3$.

Hence $S_n = 3^{n-2}S_2 = 2 \cdot 3^{n-2}$ for $n \geq 3$. In particular, $S_7 = 2 \cdot 3^5 = 486$.

11. **Answer (023):**

Let line PQ intersect the circumcircle at points M and N as shown in the figure. Because $\overline{MN}$ is a diameter and $\overline{OB}$ is perpendicular to $\overline{MN}$, it follows that $\widehat{BN} = \widehat{BM}$. Thus $\angle QPB = \frac{\widehat{BM}+\widehat{AN}}{2} = \frac{\widehat{BN}+\widehat{AN}}{2} = \frac{\widehat{ANB}}{2} = \angle ACB$. Hence $\triangle ABC \sim \triangle QBP$, and $\frac{BP}{BC} = \frac{QB}{AB}$. It follows that $BP = \frac{4(4.5)}{5} = \frac{18}{5}$. The requested sum is $18 + 5 = 23$.



OR

Let M and N be defined as above, and let $x = BP$, so $PA = 5 - x$. The Power ofa Point Theorem applied to point P shows $x(5-x) = BP \cdot PA = PM \cdot PN = BO^2 - OP^2$, and applied to point Q shows $\frac{1}{2} \cdot \frac{9}{2} = QC \cdot QB = QM \cdot QN = QO^2 - BO^2$. Then $\frac{9}{4} + 5x - x^2 = QO^2 - OP^2 = (BQ^2 - BO^2) - (BP^2 - BO^2) = BQ^2 - BP^2 = \left(\frac{9}{2}\right)^2 - x^2$. Thus $5x = \frac{81}{4} - \frac{9}{4} = 18$ and $x = \frac{18}{5}$.

12. **Answer (548):**

Let a_k, b_k, and c_k be the number of acceptable strings of length k that begin with exactly 1, 2, or 3 of the same letter, respectively. For $k \geq 3$, $b_{k+1} = a_k$, $c_{k+1} = b_k$, and $a_{k+1} = a_k + b_k + c_k = a_k + a_{k-1} + a_{k-2}$. Using the fact that

$a_1 = 2$, $a_2 = 2$, and $a_3 = 4$, the recursion can be used to find the first 11 terms of the sequence a_n to be 2, 2, 4, 8, 14, 26, 48, 88, 162, 298, and 548. The number of strings of length 10 that satisfy the requirement is $a_{10}+b_{10}+c_{10} = a_{11} = 548$.

13. **Answer (628):**

First notice that

$$\begin{aligned} a_n &= \sum_{k=1}^{n} \frac{\sin(\frac{1}{2})\sin(k)}{\sin(\frac{1}{2})} \\ &= \sum_{k=1}^{n} \frac{\cos(k-\frac{1}{2}) - \cos(k+\frac{1}{2})}{2\sin(\frac{1}{2})} \\ &= \frac{\cos(\frac{1}{2}) - \cos(n+\frac{1}{2})}{2\sin(\frac{1}{2})}. \end{aligned}$$

Therefore $a_n < 0$ if and only if $\cos(\frac{1}{2}) < \cos(n+\frac{1}{2})$. Because the cosine function has period 2π, and $\cos x = \cos(2\pi - x)$, this inequality holds if and only if n is between $2\pi m - 1$ and $2\pi m$ for some positive integer m. In other words, the index of the mth negative term in the given sequence is the greatest integer less than $2\pi m$. Because $3.14 < \pi < 3.145$, it follows that $628 < 200\pi < 629$. Thus the index of the 100th negative term is 628.

14. **Answer (089):**

Note that neither x nor y can equal zero, as otherwise, the left-hand sides of the two given equations would both equal 0. Therefore let $y = kx$ for some nonzero value of k. The given equations then become

$$x^9k^5 + x^9k^4 = 810 \qquad \text{and} \qquad x^9k^6 + x^9k^3 = 945.$$

Note that the left-hand sides of the above equations have a common factor of $x^9k^3(k+1)$. Furthermore, k cannot equal -1, as otherwise, the left-hand sides of the above two equations would both equal 0. Thus

$$\frac{945}{810} = \frac{7}{6} = \frac{x^9k^6 + x^9k^3}{x^9k^5 + x^9k^4} = \frac{x^9k^3(k+1)(k^2-k+1)}{x^9k^4(k+1)}$$

which simplifies to $6k^2 - 13k + 6 = 0$. The solutions of this quadratic equation are $k = \frac{2}{3}$ and $\frac{3}{2}$. Because k is positive, it follows that x and y must also be positive. When $k = \frac{2}{3}$, $x^9 \cdot \left(\frac{32}{243} + \frac{16}{81}\right) = 810$, so $x^9 = \frac{810 \cdot 243}{80} = \frac{3^9}{2^3}$. Then $x^3 = \frac{27}{2}$ and $y^3 = \left(\frac{2}{3}\right)^3 \cdot \frac{27}{2} = 4$. Similarly, if $k = \frac{3}{2}$, then $x^3 = 4$ and $y^3 = \frac{27}{2}$. In either case, $2x^3 + (xy)^3 + 2y^3 = 2 \cdot \frac{27}{2} + \frac{27}{2} \cdot 4 + 2 \cdot 4 = 89$.

15. **Answer (129):**

Let P and Q be the centers of the circles $\mathcal{P}$ and $\mathcal{Q}$, respectively. Let F be on $\overline{CQ}$ so that $CBPF$ is a rectangle. Note that in right $\triangle PFQ$, $PQ = 1 + 4 = 5$ and $QF = 4 - 1 = 3$, so $BC = PF = 4$.

Let G and H be on ℓ so that $\overline{BG}$ and $\overline{CH}$ are altitudes of $\triangle ABD$ and $\triangle ACE$, respectively, as shown, and let ℓ intersect line BC at I. Because the sectors of the two circles cut off by ℓ are similar with a $1:4$ ratio, it follows that $AE = 4AD$. Because $\triangle ABD$ and $\triangle ACE$ have the same areas, it follows that $BG = 4CH$. Because $\triangle IGB$ is similar to $\triangle IHC$, it follows that $4IC = IB = IC + 4$ and $IC = \frac{4}{3}$.

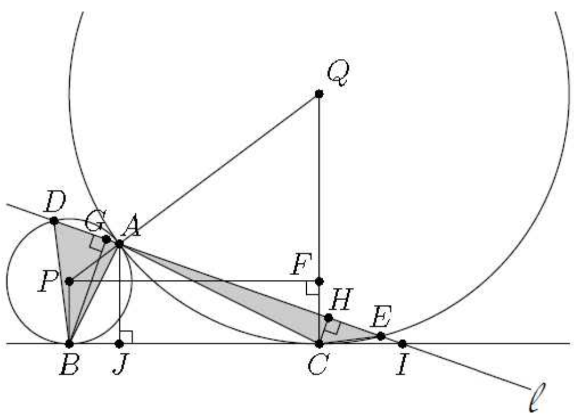
Calculate AI by letting J be the projection of A onto line BC. Because $PA = \frac{1}{5}PQ$, it follows that $AJ = \frac{4}{5}PB + \frac{1}{5}QC = \frac{8}{5}$ and $BJ = \frac{1}{5}BC = \frac{4}{5}$. Then

$$AI = \sqrt{AJ^2 + IJ^2} = \sqrt{\left(\frac{8}{5}\right)^2 + \left(\frac{68}{15}\right)^2} = \frac{4}{3}\sqrt{13}.$$

Now calculate the area of $\triangle DBA$ by finding BG and AD. For the former, by similarity $\triangle BGI \sim \triangle AJI$, it follows that $\frac{BG}{BI} = \frac{AJ}{AI}$, giving $BG = \frac{32}{65}\sqrt{13}$. For the latter, the Power of a Point Theorem gives $IA \cdot ID = IB^2$, so $ID = \frac{64}{39}\sqrt{13}$ and $AD = ID - IA = \frac{4}{13}\sqrt{13}$. So the area of $\triangle DBA$ is

$$\frac{1}{2}AD \cdot BG = \frac{1}{2} \cdot \frac{4}{13}\sqrt{13} \cdot \frac{32}{65}\sqrt{13} = \frac{64}{65}.$$

The requested sum is $64 + 65 = 129$.



The problems and solutions for this AIME were contributed by Zachary Abel, Steve Dunbar, Jacek Fabrykowski, Zuming Feng, Peter Gilchrist, Ellina Grigorieva, Jerry Grossman, Chris Jeuell, Elgin Johnston, Jonathan Kane, Matthew McMullen, Tamas Szabo, Alan Vraspir and David Wells.