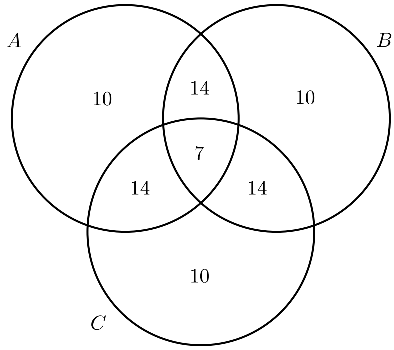
1. (Answer: 334)

   Abe paints at the rate of $\frac{1}{900}$ of the room per minute, Bea paints at the rate of $\frac{1}{900} \cdot \frac{3}{2} = \frac{1}{600}$ of the room per minute, and Coe paints at the rate of $\frac{1}{900} \cdot 2 = \frac{1}{450}$ of the room per minute. Thus together Abe and Bea paint at the rate of $\frac{1}{900} + \frac{1}{600} = \frac{1}{360}$ of the room per minute, and all three of them together paint at the rate of $\frac{1}{360} + \frac{1}{450} = \frac{1}{200}$ of the room per minute. Then in the first hour and a half Abe paints $\frac{1}{900} \cdot 90 = \frac{1}{10}$ of the room. So together, Abe and Bea paint another $\frac{4}{10} = \frac{2}{5}$ of the room in $\frac{2}{5} \div \frac{1}{360} = 144$ minutes. Finally, Abe, Bea, and Coe paint together to paint half the room in $\frac{1}{2} \div \frac{1}{200} = 100$ minutes. The total time for painting the room is $90 + 144 + 100 = 334$ minutes.

2. (Answer: 076)

   For simplicity, assume without loss of generality that the population is 100 men, and sketch a Venn diagram displaying their risk factors. Each set showing the three risk factors by themselves must contain 10 men, while each set showing the intersections of exactly two risk factors must contain 14 men. To make the intersection of all three sets represent $\frac{1}{3}$ of the entire intersection of A and B, that intersection must contain 7 men. Adding up all the numbers in the Venn diagram so far shows that the union of the three sets contains 79 men. That leaves 21 men who have none of the risk factors. Because there are 55 men who do not have risk factor A, the required probability is $\frac{21}{55}$. The requested sum is $21 + 55 = 76$.

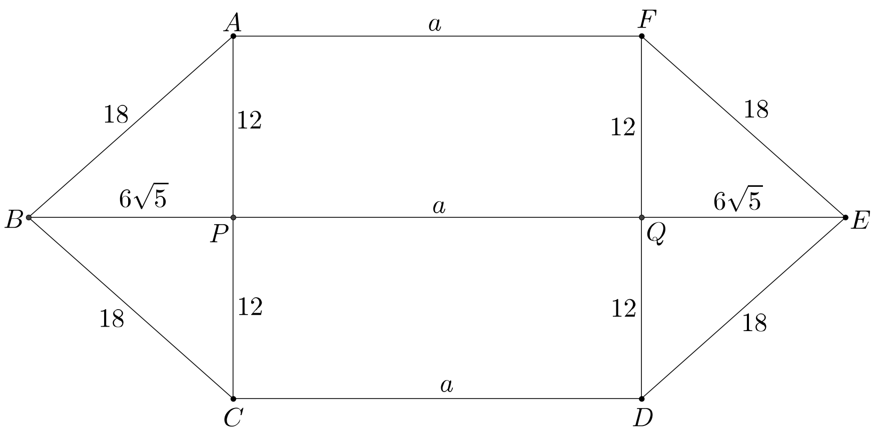


3. (Answer : 720)

   Let the vertices of the hexagon be labeled A, B, C, D, E, F with $AB = BC = DE = EF = 18$, and let the intersections of $\overline{AC}$ and $\overline{DF}$ with $\overline{BE}$ be P and Q, respectively. See the figure below. With $AC = 24$, the hexagon has the same area as the rectangle, which is $36a$. It follows by the Pythagorean Theorem that $BP = 6\sqrt{5}$. The area of the hexagon can be calculated by finding twice the area of trapezoid $ABEF$, so

$$2 \cdot 12 \cdot \frac{a + (a + 2 \cdot 6\sqrt{5})}{2} = 36a.$$

Solving yields $a = 12\sqrt{5}$, so $a^2 = 720$.



4. (Answer: 447)

Let $x = 0.abab\overline{ab}$ and $y = 0.abcabc\overline{abc}$. Then $x = \frac{ab}{99}$ and $y = \frac{abc}{999}$, so

$$\frac{33}{37} = \frac{ab}{99} + \frac{abc}{999} = \frac{111ab + 11abc}{3^3 \cdot 11 \cdot 37}.$$

Because this fraction must reduce to $\frac{33}{37}$, it must be the case that the numerator $111ab + 11abc$ is a multiple of 11. Thus $111ab$ must be a multiple of 11, and because 111 is relatively prime to 11, it follows that ab is a multiple of 11 and $a = b$. Thus

$$\frac{27 \cdot 33}{3^3 \cdot 37} = \frac{33}{37} = \frac{111aa + 11aac}{3^3 \cdot 11 \cdot 37} = \frac{111a + aac}{3^3 \cdot 37} = \frac{221a + c}{3^3 \cdot 37}.$$

Thus, $891 = 33 \cdot 27 = 221a + c$, and it follows that $a = 4$ and $c = 7$. Therefore the three-digit number abc is 447.

5. (Answer: 420)

Because the coefficient of x^2 in both $p(x)$ and $q(x)$ is 0, the remaining root of $p(x)$ is $t = -r - s$, and the remaining root of $q(x)$ is $t - 1$. The coefficients of x in $p(x)$ and $q(x)$ are both equal to a, and equating the two coefficients gives

$$rs + st + tr = (r + 4)(s - 3) + (s - 3)(t - 1) + (t - 1)(r + 4),$$

from which $t = 4r - 3s + 13$. Furthermore, $b = -rst$, so

$$b + 240 = -rst + 240 = -(r + 4)(s - 3)(t - 1),$$

from which $rs - 4st + 3tr - 3r + 4s + 12t - 252 = 0$. Substituting $t = 4r - 3s + 13$ gives

$$12r^2 - 24rs + 12s^2 + 84r - 84s - 96 = 0,$$

which is equivalent to $(r - s)^2 + 7(r - s) - 8 = 0$, and the solutions for $r - s$ are 1 and -8. If $r - s = 1$, then the roots of $p(x)$ are r, $s = r - 1$, and $t = 4r - 3s + 13 = r + 16$. Because the sum of the roots is 0, $r = -5$. In this case the roots are $-5, -6$, and 11, and $b = -rst = -330$. If $r - s = -8$, then the roots of $p(x)$ are r, $s = r + 8$, and $t = 4r - 3s + 13 = r - 11$. In this case the roots are $1, 9$, and -10, and $b = -rst = 90$. Therefore the requested sum is $|-330| + |90| = 420$.

6. (Answer: 167)

The conditional probability that the third roll will be a six given that the first two rolls are sixes is the conditional probability that Charles rolls three sixes given that his first two rolls are sixes. This is

$$\frac{\frac{1}{2}\left(\frac{2}{3}\right)^3 + \frac{1}{2}\left(\frac{1}{6}\right)^3}{\frac{1}{2}\left(\frac{2}{3}\right)^2 + \frac{1}{2}\left(\frac{1}{6}\right)^2} = \frac{\frac{65}{432}}{\frac{17}{72}} = \frac{65}{102}.$$

The requested sum is $65 + 102 = 167$.

7. (Answer: 021)

Note that

$$f(k) = [(k + 1)(k + 2)]^{(-1)^k} = \begin{cases} (k + 1)(k + 2) & \text{if } k \text{ is even,} \\ \frac{1}{(k+1)(k+2)} & \text{if } k \text{ is odd.} \end{cases}$$

Therefore

$$\sum_{k=1}^{n} \log_{10} f(k) = \log_{10}\left(\prod_{k=1}^{n} f(k)\right) =$$

$$\begin{cases} \log_{10}\left(\frac{3\cdot4\cdot5\cdots(n+2)}{2\cdot3\cdot4\cdots(n+1)}\right) = \log_{10}\left(\frac{n+2}{2}\right) & \text{if } n \text{ is even,} \\ \log_{10}\left(\frac{3\cdot4\cdot5\cdots(n+1)}{2\cdot3\cdot4\cdots(n+2)}\right) = \log_{10}\left(\frac{1}{2(n+2)}\right) = -\log_{10}(2n+4) & \text{if } n \text{ is odd.} \end{cases}$$

For $|\sum_{k=1}^{n} \log_{10} f(k)|$ to be 1, either $\frac{n+2}{2} = 10$ with n even or $2n + 4 = 10$ with n odd, so $n = 18$ or $n = 3$. Thus the requested sum is $18 + 3 = 21$.

8. (Answer: 254)

Let circles C, D, and E have centers C, D, and E, respectively. Let circle E be tangent to $\overline{AB}$ at F. Let circle E have radius s, and circle D have radius $b = 3s$. Then $CE = 2 - s$, $DE = b + s = 4s$, $EF = s$, and $DC = 2 - 3s$. The Pythagorean Theorem applied to $\triangle CEF$ gives $CF = \sqrt{(2-s)^2 - s^2} = \sqrt{4-4s}$ and to $\triangle DEF$ gives $DF = \sqrt{(4s)^2 - s^2} = s\sqrt{15}$. Because $DF = DC + CF$, it follows that $s\sqrt{15} = (2-3s) + \sqrt{4-4s}$. Squaring and simplifying twice reduces this equation to $9s^2 + 84s - 44 = 0$, which has solutions $s = \frac{-14\pm\sqrt{240}}{3}$. Thus $b = 3s = \sqrt{240} - 14$, and the requested sum is $240 + 14 = 254$.

9. (Answer: 581)

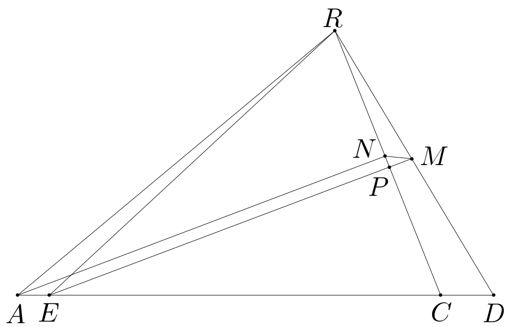
There is one subset of the chairs that contains all ten chairs. If a subset of the chairs does not contain all ten chairs and contains at least three adjacent chairs, then there is a sequence of four adjacent chairs where the first chair (counting clockwise) is not in the subset and the other three chairs are in the subset. There are ten possible places for this sequence of four chairs, and $2^{10-4} = 64$ ways to determine which of the other $10 - 4$ chairs are in the subset. This double counts the subsets that contain two disconnected sequences of three or more adjacent chairs. The number of subsets containing two disconnected sequences of three or more adjacent chairs can be counted by noting that there are $\frac{10\cdot3}{2} = 15$ ways of selecting the two sequences of four chairs (10 ways to select a first sequence of 4 chairs times 3 ways of selecting a second sequence of 4 chairs from the remaining 6 chairs divided by 2 because each pair of 4 chairs gets counted twice) and $2^2 = 4$ ways to decide which of the other two chairs are in the subset. It follows that the number of subsets of chairs containing at least three adjacent chairs is $1 + 10 \cdot 64 - 15 \cdot 4 = 581$.

10. (Answer: 147)

Suppose that w satisfies the equation. Then $z^2 + zw + w^2 = 0$. Multiply both sides of the equation by $z - w$ to get $z^3 - w^3 = 0$. Thus, w is z times a cube root of 1. This means that P is an equilateral triangle inscribed in a circle of radius

2014. The area of such a triangle is $\frac{3(2014)^2\sqrt{3}}{4} = 3(1007)^2\sqrt{3}$. So the requested remainder is $3 \cdot 7^2 = 147$.

11. (Answer: 056)



Let N be the midpoint of $\overline{CR}$ and P be the intersection of $\overline{EM}$ and $\overline{CR}$. In isosceles triangle ARC, median $\overline{AN}$ is perpendicular to the base $\overline{CR}$, implying that $\overline{AN} \parallel \overline{EM}$. Note that $\overline{MN}$ is a midline of $\triangle RCD$, from which it follows that $\overline{NM} \parallel \overline{CD}$ and $CD = 2MN$. Therefore $MNAE$ is a parallelogram, and $CD = 2MN = 2AE$.

Angle $EDR = 180^\circ - (75^\circ + 45^\circ) = 60^\circ$. Let $\angle DEM = x$. Then $\angle REM = 45^\circ - x$, $\angle EMR = 60^\circ + x$, $\angle CRD = 90^\circ - \angle EMR = 30^\circ - x$, and, because $\triangle ECP$ is a right triangle, $\angle ACR = 90^\circ - x$. Applying the Law of Sines in $\triangle REM$ and $\triangle MED$ gives

$$\frac{RM}{EM} = \frac{\sin(45^\circ - x)}{\sin 75^\circ} \quad \text{and} \quad \frac{EM}{MD} = \frac{\sin 60^\circ}{\sin x}.$$

Multiplying the two equations together yields

$$1 = \frac{\sin(45^\circ - x)\sin 60^\circ}{\sin 75^\circ \sin x} \quad \text{or} \quad \frac{\sin 75^\circ}{\sin 60^\circ} = \frac{\sin(45^\circ - x)}{\sin x}.$$

Then

$$\frac{\sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ}{\sin 60^\circ} = \frac{\sin 45^\circ \cos x - \cos 45^\circ \sin x}{\sin x},$$

and

$$\frac{\sqrt{3}+1}{\sqrt{3}} = \cot x - 1,$$

from which it follows that

$$\cot x = \frac{1+2\sqrt{3}}{\sqrt{3}} \quad \text{or} \quad \tan x = \frac{\sqrt{3}}{1+2\sqrt{3}} = \frac{6-\sqrt{3}}{11}.$$

Applying the Law of Sines to $\triangle CDR$ gives

$$CD = \frac{CD}{RD} = \frac{\sin(30^\circ - x)}{\sin(90^\circ + x)} = \frac{\sin 30^\circ \cos x - \cos 30^\circ \sin x}{\cos x} =$$
$$\frac{1 - \sqrt{3}\tan x}{2} = \frac{14 - 6\sqrt{3}}{22},$$

and because $RD = 1$ it follows that

$$AE = \frac{CD}{2} = \frac{14 - 6\sqrt{3}}{44} = \frac{7 - \sqrt{27}}{22}.$$

The requested sum is $7 + 27 + 22 = 56$.

12. (Answer : 399)

The condition $\cos(3A) + \cos(3B) + \cos(3C) = 1$ implies

$$\begin{aligned}
0 &= 1 - \cos 3A - (\cos 3B + \cos 3C) \\
&= 2\sin^2\left(\frac{3}{2}A\right) - 2\cos\left(\frac{3}{2}(B+C)\right)\cos\left(\frac{3}{2}(B-C)\right) \\
&= 2\sin^2\left(\frac{3}{2}A\right) + 2\sin\left(\frac{3}{2}A\right)\cos\left(\frac{3}{2}(B-C)\right) \\
&= 2\sin\left(\frac{3}{2}A\right)\left(\sin\left(\frac{3}{2}A\right) + \cos\left(\frac{3}{2}(B-C)\right)\right) \\
&= 2\sin\left(\frac{3}{2}A\right)\left(-\cos\left(\frac{3}{2}(B+C)\right) + \cos\left(\frac{3}{2}(B-C)\right)\right) \\
&= 4\sin\left(\frac{3}{2}A\right)\sin\left(\frac{3}{2}B\right)\sin\left(\frac{3}{2}C\right).
\end{aligned}$$

Therefore one of $\angle A$, $\angle B$, or $\angle C$ must be 120°. The largest value of the remaining side of $\triangle ABC$ is obtained when the 120° angle is between the sides of lengths 10 and 13. In this case the Law of Cosines implies that the third side has length $\sqrt{10^2 + 13^2 + 10 \cdot 13} = \sqrt{399}$.

13. (Answer: 028)

Let R_j and L_j represent the right and left shoes, respectively, from the jth adult. Call a set S of $k > 0$ pairs made by the child *good* if S contains the shoes from exactly k adults. The condition to be satisfied is that no good set contains fewer than 5 pairs. Note that if a set of k pairs is good, then the complementary set of $10 - k$ pairs is also good. Therefore the required condition can be met in only one of two ways:

- The set of all 10 pairs is the only good set. In this case L_1 can be paired with any of 9 right shoes (L_1 cannot be paired with R_1). Relabeling if necessary, it may be assumed that L_1 is paired with R_2. Then L_2 can be paired with any of 8 right shoes (L_2 cannot be paired with R_1 or R_2). Again by relabeling, it may be assumed that L_2 is paired with R_3, and L_3 can be paired with any of 7 right shoes. Continuing, it is seen that there are 9! pairings for which the set of all 10 pairs is the only good set.

- There are 2 good sets of 5 pairs each. The set of 10 left shoes can be partitioned into 2 sets of 5 left shoes in $\frac{1}{2} \cdot \binom{10}{5}$ ways. For each such partition of the left shoes, the reasoning of the preceding case can be used to establish that for each of the sets of 5 left shoes, there are 4! possible arrangements of right shoes that result in a good set of 5 pairs. Thus there are $\frac{1}{2} \cdot \binom{10}{5} \cdot (4!)^2 = \frac{10!}{50} = \frac{1}{5} \cdot 9!$ pairings for which there are 2 good sets of 5 pairs each.

The total number of possible pairings is 10!, so the required probability is $\frac{9!+\frac{1}{5}\cdot 9!}{10!} = \frac{3}{25}$. The requested sum is $3 + 25 = 28$.

OR

There is a permutation π such that the pairs made by the child are of the form $\{L_j, R_{\pi(j)}\}$. There are 10! equally likely permutations. The permutation π will factor into cycles of various lengths. For the conditions of the problem to be met, the permutation can have no cycle of length less than 5. This can happen if the permutation is a single cycle of length 10 where no subset of fewer than all 10 pairs of shoes can all be properly matched such as $(1, 3, 8, 6, 2, 9, 7, 5, 4, 10)$. It can also happen if the permutation is the product of two cycles of length 5 where the pairs of shoes partition into two groups of size 5, and no proper subset of either group can be properly matched such as $(1, 3, 8, 6, 2)(9, 7, 5, 4, 10)$. Counting the number of cycles of length 10 and products of two cycles of length 5 is then the same as the counting in the two cases of the previous solution.

14. (Answer: 077)

Let ω be the circumcircle of $\triangle ABC$, and let E be the intersection of ray AD and ω. Because $\angle BAE = \angle CAE$, E is the midpoint of arc BC, and so $\overline{EM} \perp \overline{BC}$. The projection of three collinear points A, P, and E on line BC are H, N, and M, respectively, with N the midpoint of segment $\overline{HM}$. Thus P is the midpoint of segment $\overline{AE}$. Because they subtend equal arcs, $\angle CBE = \angle EAB$. By the Law of Sines

$$\frac{AE}{AB} = \frac{\sin(\angle ABE)}{\sin(\angle AEB)} = \frac{\sin(\angle ABC + \angle CBE)}{\sin(\angle ACB)}$$
$$= \frac{\sin(\angle ABC + \angle EAB)}{\sin(\angle ACB)} = \frac{\sin 120^\circ}{\sin 45^\circ} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2},$$

implying that $AE = 10 \cdot \frac{\sqrt{6}}{2} = 5\sqrt{6}$. Thus $AP = \frac{1}{2}AE = \frac{5}{2}\sqrt{6}$, and $AP^2 = \frac{75}{2}$. The requested sum is $75 + 2 = 77$.

15. (Answer : 149)

Number the primes as follows: $\rho_0 = 2$, $\rho_1 = 3$, $\rho_2 = 5$, ... It turns out that each x_n can be expressed in terms of the digits of the binary representation of n. That is

* If the binary representation of $n = \overline{d_m d_{m-1} \ldots d_1 d_0} = \sum_{i=0}^{m} d_i 2^i$ $(d_i \in \{0, 1\})$, then $x_n = \prod_{i=0}^{m} \rho_i^{d_i}$.

The claim can be proved by mathematical induction on n as follows.

For $n = 1$ (*) holds true, because $x_1 = \frac{x_0 p(x_0)}{X(x_0)} = \frac{p(1)}{X(1)} = 2$. Proceeding by induction, assume that (*) holds true for some integer $n \geq 1$. If n is an even integer, then $d_0 = 0$, and so $n = \sum_{i=1}^{m} d_i 2^i$ and $x_n = \prod_{i=1}^{m} \rho_i^{d_i}$. Therefore $p(x_n) = 2$, $X(x_n) = 1$, and by the definition of $\{x_n\}$, $x_{n+1} = 2x_n$. Because $n + 1 = 2^0 + \sum_{i=1}^{m} d_i 2^i$, it follows that $x_{n+1} = \rho_0 \prod_{i=1}^{m} \rho_i d^i$ proving that (*) holds true for $n + 1$ if n is even.

Assume now that n is odd, so $d_0 = 1$. Consider two cases: $n \neq 2^{m+1} - 1$ or $n = 2^{m+1} - 1$ for some positive integer m. In the first case the binary representation of n contains at least one digit 0. Let j be the smallest number for which $d_j = 0$, that is $n = \overline{d_m d_{m-1} \ldots d_{j+1} 011 \ldots 11} = \sum_{i=0}^{j-1} 2^i + \sum_{i=j+1}^{m} d_i 2^i$. By the induction hypothesis $x_n = \prod_{i=0}^{j-1} \rho_i \cdot \prod_{i=j+1}^{m} \rho_i^{d_i}$. Thus $p(x_n) = \rho_j$, and $X(x_n) = \prod_{i=0}^{j-1} \rho_i$. It follows that $n + 1 = \overline{d_m d_{m-1} \ldots d_{j+1} 100 \ldots 00} = 2^j + \sum_{i=j+1}^{m} d_i 2^i$, and the recurrence implies $x_{n+1} = \frac{x_n \rho_j}{X(x_n)} = \rho_j \prod_{i=j+1}^{m} \rho_i^{d_i}$, proving (*) in this case.

Consider now the case $n = 2^{m+1} - 1$, so the binary representation of n contains only ones, so $n = \overline{11 \ldots 11} = \sum_{i=0}^{m} 2^i$. In this case $p(x_n) = \rho_{m+1}$, and $X(x_n) = \prod_{i=0}^{m} \rho_i = x_n$. Therefore $x_{n+1} = \frac{x_n \rho_{m+1}}{X(x_n)} = \rho_{m+1}$, and $n + 1 = 2^{m+1}$, proving (*) in this case and completing the proof by induction.

Note that as a consequence of this claim each x_n is a unique square-free integer.

Because $2090 = 2 \cdot 5 \cdot 11 \cdot 19 = \rho_0 \cdot \rho_2 \cdot \rho_4 \cdot \rho_7$, the requested value of $t = 2^0 + 2^2 + 2^4 + 2^7 = 149$.

The problems and solutions in this contest were proposed by Steve Blasberg, Steve Dunbar, Jacek Fabrykowski, Zuming Feng, Elgin Johnston, Jonathan Kane, Cap Khoury, Matthew McMullen, Tamas Szabo, Dave Wells, Ronald Yannone.