1. (Answer: 150)

   Let r be the rate that Tom bicycles in minutes per mile. Then Tom runs at the rate of $2r$ and swims at the rate of $10r$. Because four and a quarter hours is 255 minutes,

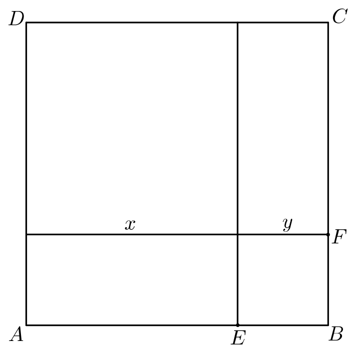
$$\frac{1}{2}\cdot 10r + 30r + 8\cdot 2r = 255.$$

   This simplifies to $51r = 255$, so $r = 5$. Tom bicycles at the rate of 5 minutes per mile, so the number of minutes he spends bicycling is $30\cdot 5 = 150$.

2. (Answer: 200)

   If a number satisfies the given conditions, then because its first and last digits are equal, and the number is a multiple of 5, its first and last digits must be 5. The sum of the middle three digits must also be a multiple of 5. If one chooses any two values for the first two of those three digits, there will be two possible choices for the third digit that will make the sum equal to a multiple of 5. Thus there are $10\cdot 10\cdot 2 = 200$ choices for such a number.

3. (Answer: 018)



   Let the sides of the two smaller squares have lengths x and y so that the square $ABCD$ has side length $x+y$. It is given that $x^2+y^2 = \frac{9}{10}(x+y)^2$. Then $10(x^2+y^2) = 9(x^2+y^2)+18xy$, and $x^2+y^2 = 18xy$. The requested sum is $\frac{x}{y}+\frac{y}{x} = \frac{x^2+y^2}{xy} = 18$.

4. (Answer: 429)

   The colors of the 13 cells in the array can be chosen in $\binom{13}{5} = 1287$ ways. For an array to satisfy the given symmetry condition, the center square must be blue, and each of the three-square "L" shapes must contain 2 red squares and 1 blue square. Once one of the L's is colored, the other three L's must each be a rotation of the first L. Because 2 red squares and 1 blue square can be placed in one of the L shapes in 3 ways, there are only 3 such arrangements. Therefore the probability that an array satisfies the given symmetry condition is $\frac{3}{1287} = \frac{1}{429}$. The requested denominator is 429.

5. (Answer: 098)

   The equation $8x^3 - 3x^2 - 3x - 1 = 0$ is equivalent to $x^3 + 3x^2 + 3x + 1 = 9x^3$. Thus $(x+1)^3 = 9x^3$, so $x = \frac{1}{\sqrt[3]{9}-1}$. Multiplying numerator and denominator by $\left(\sqrt[3]{9}\right)^2 + \sqrt[3]{9} + 1$ yields $x = \frac{\sqrt[3]{81}+\sqrt[3]{9}+1}{8}$. The requested sum is $81 + 9 + 8 = 98$.

6. (Answer: 047)

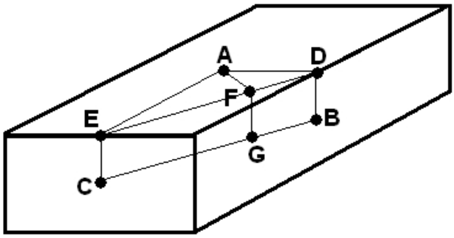
   The probability that all three mathematics textbooks end up in the first box is the probability that the three mathematics textbooks are selected out of the $\binom{12}{3}$ equally likely ways to select three textbooks. That is $\frac{1}{\binom{12}{3}} = \frac{1}{220}$. The probability that all three mathematics textbooks end up in the second box is the probability that three mathematics textbooks and one other of the nine remaining textbooks are selected for the second box out of the $\binom{12}{4}$ equally likely ways to select four textbooks. That is $\frac{9}{\binom{12}{4}} = \frac{1}{55}$. Finally, the probability that all three mathematics textbooks end up in the last box is $\frac{\binom{9}{2}}{\binom{12}{5}} = \frac{1}{22}$. The probability that all three mathematics textbooks end up in the same box is $\frac{1}{220} + \frac{1}{55} + \frac{1}{22} = \frac{1+4+10}{220} = \frac{15}{220} = \frac{3}{44}$. The requested sum is $3 + 44 = 47$.

   OR

   Let $\binom{n}{p,q,r} = \frac{n!}{p!q!r!}$. There are $\binom{12}{3,4,5}$ ways for Melinda to place her books in the boxes. There are $\binom{9}{0,4,5}$ ways with all the math books in the first box, $\binom{9}{3,1,5}$ ways with all of them in the second box, and $\binom{9}{3,4,2}$ ways with all of them in the third box. Thus the probability is

   $$\frac{\binom{9}{0,4,5} + \binom{9}{3,1,5} + \binom{9}{3,4,2}}{\binom{12}{3,4,5}} = \frac{3}{44}.$$

7. (Answer: 041)



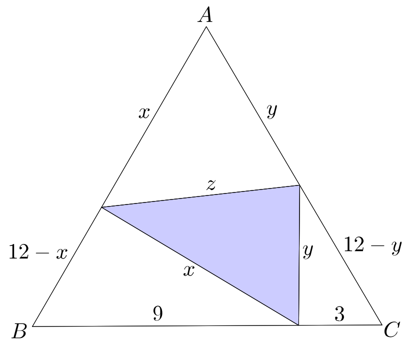
Let the box have height h. Let A be the center of the 12×16 face, B be the center of the $16 \times h$ face, and C be the center of the $12 \times h$ face. Let D be the midpoint of the edge of the box common to the face containing A and the face containing B, and let E be the midpoint of the edge of the box common to the faces containing A and C. Let F be the foot of the altitude to $\overline{DE}$ of $\triangle AED$, and G be the foot of the altitude to $\overline{BC}$ of $\triangle ABC$. Because $AD = 6$ and $AE = 8$, $DE = 10$. The area $[ADE] = \frac{6 \cdot 8}{2} = \frac{10 \cdot AF}{2}$, so $AF = \frac{24}{5}$. Similarly, $[ABC] = 30 = \frac{10 \cdot AG}{2}$, so $AG = 6$. Note that the projection of $\overline{AG}$ onto the face of A is $\overline{AF}$, so $\triangle AFG$ is a right triangle. It follows that $AF^2 + FG^2 = AG^2$ or $\left(\frac{24}{5}\right)^2 + \left(\frac{h}{2}\right)^2 = 6^2$. Solving for h gives $h = \frac{36}{5}$. The requested sum is $36 + 5 = 41$.

8. (Answer: 371)

The domain of $f(x)$ is the solution of the inequality $-1 \le \log_m(nx) \le 1$, or equivalently, $\frac{1}{mn} \le x \le \frac{m}{n}$. Thus the domain is a closed interval of length $\frac{1}{2013} = \frac{m}{n} - \frac{1}{mn} = \frac{m^2-1}{mn}$. Hence $n = \frac{2013(m^2-1)}{m}$. Because m and $m^2 - 1$ are relatively prime, m must be a factor of $2013 = 3 \cdot 11 \cdot 61$. The smallest possible value of n is $\frac{2013(3^2-1)}{3} = 5368$, and the smallest possible sum $m + n$ is $5368 + 3 = 5371$. The requested remainder is 371.

9. (Answer: 113)

Let the fold line intersect sides $\overline{AB}$ and $\overline{AC}$ at distances x and y from point A, respectively, as shown. Let the fold line have length z. Then the Law of Cosines gives $9^2 + (12 - x)^2 - 2 \cdot 9 \cdot (12 - x) \cdot \cos B = x^2$, which simplifies to $x = \frac{39}{5}$. Similarly, $3^2 + (12 - y)^2 - 2 \cdot 3 \cdot (12 - y) \cdot \cos C = y^2$, which simplifies to $y = \frac{39}{7}$. Finally, $z^2 = \left(\frac{39}{5}\right)^2 + \left(\frac{39}{7}\right)^2 - 2 \cdot \frac{39}{5} \cdot \frac{39}{7} \cdot \cos A$, which simplifies to $z = \frac{39\sqrt{39}}{35}$. The requested sum is $39 + 39 + 35 = 113$.



10. (Answer: 080)

Because $P(x)$ has real coefficients, the number $r - si$ must also be a zero of $P(x)$, and the third zero must be a real number q. The sum of the zeros is $a = q + 2r$, so q is an integer. The product of the zeros is $65 = q(r^2 + s^2)$, so $r^2 + s^2$ is a factor of 65, and therefore must be 1, 5, 13, or 65. Note that $5 = 1^2 + 2^2, 13 = 2^2 + 3^2$, and $65 = 1^2 + 8^2 = 4^2 + 7^2$. Therefore $\{|r|, |s|\}$ is one of the sets $\{1, 2\}, \{2, 3\}, \{1, 8\}, \{4, 7\}$, and each set corresponds to 4 distinct polynomials $P(x)$. For the set $\{1, 2\}$, $q = \frac{65}{5} = 13$, for $\{2, 3\}$, $q = \frac{65}{13} = 5$, and for the last two sets $q = \frac{65}{65} = 1$. The requested sum is then

$$\sum_{a,b} p_{a,b} = 4 \cdot 13 + 4 \cdot 5 + 8 \cdot 1 = 80.$$

11. (Answer: 148)

Because N is divisible by 16, 15, and 14, it must be a multiple of $\text{lcm}(14, 15, 16) = 1680$. Let $N = 1680m$, for some positive integer m.

The number of students present must be a whole number no greater than 16. Of these, only 9, 11, and 13 are not factors of $1680 = 2^4 \cdot 3^1 \cdot 5^1 \cdot 7^1$. Therefore $x = 9, y = 11$, and $z = 13$. Note that $1680 \equiv 3 \pmod{13}$. Because 1680 and 13 are relatively prime, $1680m$ will have a remainder of 3 when divided by 13 if and only if $m \equiv 1 \pmod{13}$. In turn, $1680 \equiv 8 \pmod{11}$, 11 and 1680 are relatively prime, and the least residues mod 11 of the multiples of 1680 are 8, 5, 2, 10, 7, 4, 1, 9, 6, 3, and 0, repeating. Therefore $1680m$ will have a remainder of 3 when divided by 11 if and only if $m \equiv 10 \pmod{11}$. Lastly, $1680 \equiv 6 \pmod{9}$, and the least residues mod 9 of the multiples of 1680 are 6, 3, and 0, repeating, so $1680m$ will have a remainder of 3 when divided by 9 if and only if $m \equiv 2 \pmod{3}$.

The Chinese Remainder Theorem shows that the three congruences above have a unique solution. To find it, let $m = 1 + 13i$, for some nonnegative integer i. Then

$13i+1 \equiv 10 \pmod{11}$, $13i \equiv 9 \equiv -13 \pmod{11}$, $i \equiv -1 \pmod{11}$, $i = -1+11j$ for some nonnegative integer j, and $m = 1+13(-1+11j) = -12+143j$.
In turn, $143j - 12 \equiv 2 \pmod 3$, $143j \equiv 14 \equiv 143 \pmod 3$, $j \equiv 1 \pmod 3$, $j = 1+3k$ for some nonnegative integer k, and $m = -12+143(1+3k) = 131+429k$.

Therefore N is of the form $1680(131+429k)$ for some nonnegative integer k, so the least possible value of $N = 1680 \cdot 131 = 2^4 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 131$. The requested sum is $2+3+5+7+131 = 148$.

OR

Because $1680m$ is congruent to 3 mod 9, mod 11, and mod 13, the product $9 \cdot 11 \cdot 13 = 1287$ must divide $1680m - 3$ or, for some integer n, $1680m = 1287n+3$ or $560m = 429n+1$. The Euclidean Algorithm can now be used to show $m = 131$ and $n = 171$, and the result follows as above.

12. (Answer: 021)

Note that $AB = BQ = QC = CD = 1$ and $\angle RDE = 60^\circ$. If E lies on $\overline{RP}$, then $\angle REF = 75^\circ + 120^\circ = 195^\circ$ and F lies outside the triangle. Thus F must lie on $\overline{RP}$, point E is in the interior of $\triangle PQR$, and $\triangle RFD$ is a $45-45-90^\circ$ triangle. Because $FD = \sqrt{3} = DR$, the length of $\overline{RQ}$ is $2+\sqrt{3}$. Let the altitude from P to $\overline{QR}$ have length x. Then RQ can be expressed as $x + \frac{x}{\sqrt{3}} = 2+\sqrt{3}$. Solving for x yields $x = \frac{3+\sqrt{3}}{2}$, and the area of $\triangle PQR$ is $\frac{1}{2} \cdot \frac{3+\sqrt{3}}{2} \cdot (2+\sqrt{3}) = \frac{9+5\sqrt{3}}{4}$. The requested sum is $9+5+3+4 = 21$.

13. (Answer: 961)

By Heron's Formula, the area of $\triangle AB_0C_0$ is 90. Triangle $B_0C_1C_0$ is similar to $\triangle AB_0C_0$ with ratio of similarity $r = \frac{B_0C_0}{AC_0} = \frac{17}{25}$, and $\triangle AB_1C_1$ is similar to $\triangle AB_0C_0$ with ratio of similarity $\frac{AC_1}{AC_0} = \frac{25-17r}{25} = 1-r^2$. Therefore the area of $\triangle B_0C_1B_1$ is $90\left(1-r^2-\left(1-r^2\right)^2\right)$. For $n > 1$, the ratio of similarity between $\triangle AB_nC_n$ and $\triangle AB_{n-1}C_{n-1}$ is also $1-r^2$, so the ratio of their areas is $(1-r^2)^2$. The ratio of the area of $\triangle B_{n-1}C_nB_n$ to that of $\triangle B_{n-2}C_{n-1}B_{n-1}$ is also $(1-r^2)^2$. Hence the areas of the triangles $B_{n-1}C_nB_n$ for $n \ge 1$ form a geometric sequence with initial term $90r^2\left(1-r^2\right)$ and ratio $\left(1-r^2\right)^2$. Because the triangles have disjoint interiors, the area of their union is the sum of the series, which is

$$\frac{90r^2\left(1-r^2\right)}{1-\left(1-r^2\right)^2} = 90\left(1 - \frac{1}{2-r^2}\right) = 90\left(1 - \frac{625}{961}\right) = \frac{90 \cdot 336}{961}.$$

Thus $q = 961$.

14. (Answer: 036)

Notice that

$$Q+iP = 1+\frac{1}{2}(-\sin\theta+i\cos\theta)+\frac{1}{4}(-\cos 2\theta-i\sin 2\theta)+\frac{1}{8}(\sin 3\theta-i\cos 3\theta)+\cdots$$

$$= 1+\frac{1}{2}ie^{i\theta}+\frac{1}{4}i^2e^{2i\theta}+\frac{1}{8}i^3e^{3i\theta}+\cdots$$

$$= \frac{2}{2-ie^{i\theta}} = \frac{2\,(2+\sin\theta+i\cos\theta)}{5+4\sin\theta}.$$

Thus $P = \frac{2\cos\theta}{5+4\sin\theta}$ and $Q = \frac{4+2\sin\theta}{5+4\sin\theta}$. So

$$\frac{2\sqrt{2}}{7} = \frac{P}{Q} = \frac{\cos\theta}{2+\sin\theta},$$

$$32+32\sin\theta+8\sin^2\theta = 49-49\sin^2\theta,$$

$$57\sin^2\theta+32\sin\theta-17 = (3\sin\theta-1)(19\sin\theta+17) = 0,$$

and

$$\sin\theta = -\frac{17}{19} \quad \text{or} \quad \frac{1}{3}.$$

The requested sum for the negative solution is $17+19 = 36$.

15. (Answer: 272)

Let $d = a-b = c-a$. Then there are integers m and n so that $np-d = B-A = C-B = mp+2d$, implying that $3d = (n-m)p$. Note that $2d < p$, so 3 cannot divide $n-m$. It follows that $p = 3$, and $(b,a,c) = (0,1,2)$. Therefore the ordered triples that meet the conditions are precisely those of the form $(1+3j, 3+3j+3k, 5+3j+6k)$, where $j \geq 0$, $k \geq 0$, and $j+2k \leq 31$. Thus

$$N = \sum_{k=0}^{15}(32-2k) = 16\cdot 32 - 2\left(\frac{15\cdot 16}{2}\right) = 272.$$