1. (Answer: 275)

   Under the current system, there are $60 \cdot 24 = 1440$ minutes in a day. 6:36 AM is 396 minutes after midnight. Because there are 1000 metric minutes in a day, the equivalent metric time is $\frac{396}{1440} \cdot 1000 = 275$ metric minutes after midnight. The new digital alarm clock should be set at 2:75, and the requested answer is 275.

2. (Answer: 881)

   The given equation implies that $\log_{2^a}(\log_{2^b}(2^{1000})) = 1$, then $\log_{2^b}(2^{1000}) = 2^a$, and $2^{1000} = 2^{b \cdot 2^a}$. Therefore $b \cdot 2^a = 1000 = 125 \cdot 2^3$. The possible solutions for (a, b) are $(1, 500)$, $(2, 250)$, and $(3, 125)$. The requested sum is $501 + 252 + 128 = 881$.

3. (Answer: 350)

   Let x equal the number of centimeters burned since the candle was lit. The time required to burn down x centimeters when x is an integer is $10 + 20 + 30 + ... + 10x = 10 \cdot \frac{x(x+1)}{2} = 5x(x+1)$ seconds. Therefore $T = 5(119)(120)$ and $\frac{T}{2} = 5 \cdot 60 \cdot 119$.

   To find the number of centimeters, x, the candle has burned down in $\frac{T}{2}$ seconds, set $5x(x+1)$ equal to $5 \cdot 60 \cdot 119$ and note that this implies $x(x+1) = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 17 = (2^2 \cdot 3 \cdot 7) \cdot (5 \cdot 17) = 84 \cdot 85$. Thus $x = 84$, and $h = 119 - x = 119 - 84 = 35$. The requested product is $10h = 350$.

4. (Answer: 040)

   The midpoint of $\overline{AB}$ is $M = \left(\frac{3}{2}, \sqrt{3}\right)$. Then $\overrightarrow{AM} = \left\langle \frac{1}{2}, \sqrt{3} \right\rangle$ and $\overrightarrow{MP} = \frac{1}{\sqrt{3}} \left\langle \sqrt{3}, -\frac{1}{2} \right\rangle$. It follows that $P = M + \overrightarrow{MP} = \left(\frac{5}{2}, \frac{5\sqrt{3}}{6}\right)$. Thus $xy = \frac{25\sqrt{3}}{12}$. The requested sum is $p + q + r = 25 + 3 + 12 = 40$.

   OR

   In the complex plane $A = 1$, $B = 2 + 2\sqrt{3}i$, and $P = x + yi$. Note that a $\frac{2\pi}{3}$ counterclockwise rotation centered at P maps B to A. Hence

   $$e^{\frac{2\pi i}{3}}(B - P) = A - P \Rightarrow \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)(2 + 2\sqrt{3}i - x - yi) = 1 - x - yi$$

   $$\frac{x}{2} + \frac{\sqrt{3}}{2}y - 4 + \left(\frac{y}{2} - \frac{\sqrt{3}}{2} \cdot x\right)i = (1 - x) + i(-y).$$

This is equivalent to the conditions $3x + y\sqrt{3} = 10$ and $\sqrt{3}y = x$. It follows by simple substitution that $(x, y) = (\frac{5}{2}, \frac{5\sqrt{3}}{6})$.

OR

Let $O = (0, 0)$. Note that $\angle AOB + \angle APB = 60 + 120 = 180$. Hence $AOBP$ is a cyclic quadrilateral. Let $AB = \sqrt{3}t$. Now by properties of a $30-30-120°$ triangle, $t = AP = BP$. By Ptolemy's Theorem, $t \cdot 1 + t \cdot 4 = t\sqrt{3} \cdot OP$ and $OP = \frac{5}{\sqrt{3}}$. Because $AOBP$ is cyclic, $\angle ABP = \angle POA = 30°$. So $xy = OP^2 \cos 30 \cdot \sin 30 = \frac{25}{3} \cdot \frac{\sqrt{3}}{4} = \frac{25\sqrt{3}}{12}$.

5. (Answer: 020)

Let $\theta = \angle DAE$. Suppose, without loss of generality, that the triangle has sides of length 6 and that E is between D and C. Applying the Law of Cosines to $\triangle AEC$ gives $AE = \sqrt{6^2 + 2^2 - 2 \cdot 6 \cdot 2 \cdot \cos 60°} = 2\sqrt{7}$. The area $[DAE] = \frac{1}{3}[ABC] = \frac{6^2\sqrt{3}}{3 \cdot 4}$ is also $\frac{AD \cdot AE \sin\theta}{2} = \frac{(2\sqrt{7})^2 \sin\theta}{2}$. Therefore $\sin\theta = \frac{6^2\sqrt{3}}{3 \cdot 4} \cdot \frac{2}{4 \cdot 7} = \frac{3\sqrt{3}}{14}$. The requested sum is $3+3+14 = 20$.

OR

Let $\theta = \angle DAE$. Suppose, without loss of generality, that the triangle has sides of length 6 and that E is between D and C. Let M be the midpoint of $\overline{DE}$. Then $DM = 1$, $AM = 3\sqrt{3}$, and $\tan\frac{\theta}{2} = \frac{1}{3\sqrt{3}}$. Because $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$,

$$1 + \cos\theta = 2\cos^2\frac{\theta}{2} = 2\frac{1}{\sec^2\frac{\theta}{2}} = \frac{2}{1 + \tan^2\frac{\theta}{2}} = \frac{27}{14}.$$

Finally, $\sin\theta = (1 + \cos\theta)\tan\frac{\theta}{2} = \frac{27}{14}\frac{1}{3\sqrt{3}} = \frac{3\sqrt{3}}{14}$.

6. (Answer: 282)

Assume that integer x satisfies $x^2 \leq 1000 \cdot N < 1000 \cdot N + 1000 \leq (x + 1)^2$. Observe that x^2 and $(x + 1)^2$ differ by $2x + 1$. Because $(x + 1)^2 - x^2$ must exceed 1000, $2x + 1 > 1000$, and thus $x \geq 500$.

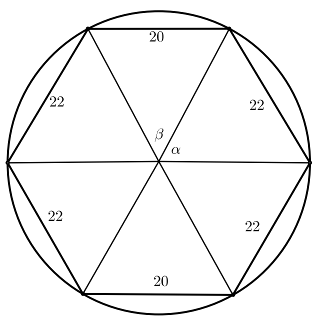
Let $x = 500 + a$, for some nonnegative integer a. Then $x^2 = 250000 + 1000a + a^2$ and $(x + 1)^2 = x^2 + 1001 + 2a$. Because x^2 is greater than a multiple of 1000 by the amount of a^2 and $(x + 1)^2$ is $1001 + 2a$ greater than x^2, the requirement will be satisfied by the least value of a such that $a^2 < 1000$ and $a^2 + 1001 + 2a \geq 2000$ or equivalently $a^2 + 2a - 999 \geq 0$. The least such positive integer value of a is 31.

Thus $x^2 = 250000 + 31000 + 961 = 281961 < 282000$ and $(x+1)^2 = 281961 + 1001 + 62 = 283024 \geq 283000$. The answer is therefore 282.

7. (Answer: 945)

   Let n represent the original number of clerks and k the number of clerks reassigned at the end of each hour. Because a clerk sorts one file every 2 minutes, in 10 minutes each clerk sorts 5 files, so $30n + 30(n-k) + 30(n-2k) + 5(n-3k) = 1775$. Then $95n - 105k = 1775$ and $19n - 21k = 355$. Thus $n = \frac{21k+355}{19}$, and because n and k must both be positive integers, $21k + 355$ must be a multiple of 19. The only such value of k for which $n, n-k, n-2k$, and $n-3k$ are also positive integers is $k = 3$. Then $n = 22$, and the number of files sorted in an hour and a half is $30 \times 22 + 15 \times 19 = 945$.

8. (Answer: 272)



   Let r be the radius of the circle, let α be the central angle of each arc cut off by chords of length 22, and let β be the central angle of each arc cut off by chords of length 20. Then $4\alpha + 2\beta = 2\pi$. Thus α and $\beta/2$ are complementary angles, so that $\cos\alpha = \sin(\beta/2)$.

   By the Law of Cosines, $22^2 = r^2 + r^2 - 2r^2\cos\alpha = 2r^2(1-\cos\alpha)$, so $\cos\alpha = 1 - \frac{22^2}{2r^2}$. Note that $\sin(\beta/2) = \frac{10}{r}$.

Together these observations give the equation

$$1 - \frac{22^2}{2r^2} = \frac{10}{r},$$

which implies

$$r^2 - 10r - 242 = 0.$$

The solutions are $r = 5 \pm \sqrt{267}$. Only the positive sign gives a positive value of r. The requested sum is $5 + 267 = 272$.

OR

Note that any diagonal of the hexagon that connects opposite vertices is a diameter of the circle. Let $ABCD$ be the isosceles trapezoid where $AD = BC = 22$, $AB = 20$, and $\overline{CD}$ is a diameter of the circle. Let the center of the circle be O, and let H be a point on $\overline{CD}$ such that $\overline{AH} \perp \overline{CD}$. Apply the Pythagorean Theorem to $\triangle AOH$ to get $r^2 = AH^2 + OH^2$, and to $\triangle ADH$ to get $(r - OH)^2 + AH^2 = AD^2$. Given that $OH = \frac{1}{2}AB = 10$ and $AD = 22$, conclude that $r^2 = 22^2 - (r-10)^2 + 10^2$ and $r^2 - 10r - 242 = 0$, which is the same equation as in the previous solution.

9. (Answer: 106)

With k colors of tile available there are k ways to decide how to color the first square. For each successive square, there are $k + 1$ options: one can either extend the current tile, or start a new tile of any of k colors. Thus there are a total of $k(k + 1)^6$ colored tilings.

The total number of colored tilings in red, blue, and/or green is thus $3 \cdot 4^6 = 12288$, which includes the $2 \cdot 3^6 = 1458$ tilings with no green tiles, and similarly for blue and red. By the Inclusion-Exclusion Principle, the number N of specified tilings is therefore $12288 - 3 \cdot 1458 + 3 \cdot 2^6 = 8106$. The requested remainder is 106.

10. (Answer: 146)

Assume, without loss of generality, that point K lies between points L and A. Observe that

$$\frac{[OAK]}{[BAK]} = \frac{[OAL]}{[BAL]} = \frac{4 + \sqrt{13}}{4},$$

so

$$\frac{[OKL]}{[BKL]} = \frac{[OAL] - [OAK]}{[BAL] - [BAK]} = \frac{4 + \sqrt{13}}{4}$$

and

$$[BKL] = [OKL]\frac{4}{4+\sqrt{13}}.$$

Then $[OKL] = \frac{13}{2}\sin\angle KOL$, and therefore the maximum possible value for $[OKL]$ occurs when $\angle KOL = 90^\circ$ and $[OKL] = \frac{13}{2}$. Thus the maximum value for $[BKL]$ is $\frac{13}{2}\cdot\frac{4}{4+\sqrt{13}} = \frac{104-26\sqrt{13}}{3}$. The requested sum is $104 + 26 + 13 + 3 = 146$.

11. (Answer: 399)

Suppose that f is a function from A to A such that $f(f(x))$ is constant. Let $a \in A$ be the constant value of $f(f(x))$. Suppose that $S = \{x \in A \mid f(x) = a\}$ has k elements. Note that $a \in S$, and for $x \notin S$, $f(x) \in S - \{a\}$. It follows that one can choose such a function f by selecting a in one of 7 ways, selecting the other elements of S in $\binom{6}{k-1}$ ways, and selecting values for $f(x)$ for each $x \notin S$ in $(k-1)^{7-k}$ ways. Thus the number of functions is

$$\begin{aligned}&\sum_{k=1}^{7} 7\cdot\binom{6}{k-1}\cdot(k-1)^{7-k}\\ &\quad = 0 + 7\cdot 6 + 7\cdot 240 + 7\cdot 540 + 7\cdot 240 + 7\cdot 30 + 7\cdot 1\\ &\quad = 7\cdot 1057.\end{aligned}$$

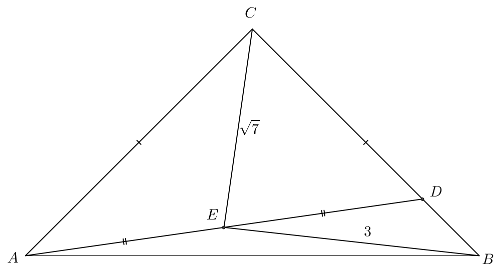
The requested remainder is then $7\cdot 57 = 399$.

12. (Answer: 540)

Let $f \in S$ only have roots with modulus (absolute value) 20 or 13. If f has all real roots, then they must come from the set $\{-20, -13, 13, 20\}$. There are 20 possible ways to choose three elements from this set, with replacement, where order is not important.

Suppose f has a nonreal root $z_0 = r + si$, where r and s are real. Then $r - si$ is another root, and there are four possibilities for the third (integer) root, k. Because $(z-(r+si))(z-(r-si))(z-k) = z^3 - (2r+k)z^2 + (2kr + r^2 + s^2)z - (r^2+s^2)k$, it follows that $2r$ must be an integer. If $|z_0| = 20$, then there are 79 possible values for $r : 0, \pm\frac{1}{2}, \pm\frac{2}{2}, \ldots, \pm\frac{39}{2}$. If $|z_0| = 13$, then there are 51 possible values for $r : 0, \pm\frac{1}{2}, \pm\frac{2}{2}, \ldots, \pm\frac{25}{2}$. Therefore there are $79\cdot 4 + 51\cdot 4 + 20 = 540$ polynomials in S that only have roots with modulus 20 or 13.

13. (Answer: 010)



Let $AB = 2x$, and $AC = BC = y$. Then $\cos\angle BAC = \cos\angle ABC = \frac{x}{y}$. Applying the Law of Cosines to $\triangle ABD$ yields

$$\begin{aligned} AD^2 &= BD^2 + AB^2 - 2BD \cdot AB \cdot \cos\angle ABC \\ &= \frac{y^2}{16} + 4x^2 - 2 \cdot \frac{y}{4} \cdot 2x \cdot \frac{x}{y} \\ &= 3x^2 + \frac{y^2}{16}. \end{aligned}$$

Because $\overline{CE}$ is a median in $\triangle ADC$, Stewart's Theorem shows that $4CE^2 = 2CD^2 + 2AC^2 - AD^2$, or $28 = \frac{18y^2}{16} + 2y^2 - 3x^2 - \frac{y^2}{16}$. Hence $\frac{49}{16}y^2 - 3x^2 = 28$. Similarly, in $\triangle ABD$, $4BE^2 = 2BD^2 + 2AB^2 - AD^2 = \frac{2y^2}{16} + 8x^2 - 3x^2 - \frac{y^2}{16}$. Hence $\frac{y^2}{16} + 5x^2 = 36$. Solving the system

$$\frac{49}{16}y^2 - 3x^2 = 28$$

$$\frac{y^2}{16} + 5x^2 = 36$$

yields $y^2 = 16$, $x^2 = 7$. Thus the area of $\triangle ABC$ is $x\sqrt{y^2 - x^2} = 3\sqrt{7}$, and the requested sum is $3 + 7 = 10$.

14. (Answer: 512)

If $m > 1$, then

$$\begin{aligned} f(3m, m+1) &= m - 2, \\ f(3m+1, m+1) &= m - 1, \text{ and} \\ f(3m+2, m+1) &= m. \end{aligned}$$

Furthermore,

$$\begin{aligned} f(3m,k) &< m-2 \text{ for } k \le m-2,\\ f(3m,m-1) &= 3,\\ f(3m,m) &= 0, \text{ and}\\ f(3m,k) &= 3m-2k < m-2 \text{ for } m+1 < k \le \frac{3m}{2}. \end{aligned}$$

Therefore $F(3m) = m-2$. Because $F(n+1) \le F(n)+1$ for all n, it follows that $F(3m+1) = m-1$ and $F(3m+2) = m$. Thus

$$\begin{aligned} \sum_{n=20}^{100} F(n) &= \sum_{m=7}^{33} (F(3m-1) + F(3m) + F(3m+1))\\ &= \sum_{m=7}^{33} ((m-1) + (m-2) + (m-1))\\ &= 3\sum_{m=7}^{33} (m-1) - 27\\ &= 3 \cdot \frac{27(6+32)}{2} - 27 = 1512. \end{aligned}$$

The requested remainder is 512.

15. (Answer: 222)

Notice that

$$\cos^2 A + \cos^2 B + 2\sin A \sin B \cos C = \frac{15}{8}$$

implies

$$\sin^2 A + \sin^2 B - 2\sin A \sin B \cos C = \frac{1}{8}.$$

Let R be the circumradius and a, b, and c be the sides opposite A, B, and C, respectively. By the Extended Law of Sines

$$\frac{1}{4R^2}\left(a^2 + b^2 - 2ab\cos C\right) = \frac{1}{8},$$

and by the Law of Cosines

$$\frac{1}{4R^2} \cdot c^2 = \frac{1}{8}.$$

Thus

$$\sin^2 C = \frac{1}{8}.$$

Similarly, the second given equation yields $\sin^2 A = \frac{4}{9}$, and a similar argument applied to $\cos^2 C + \cos^2 A + 2\sin C \sin A \cos B$ shows that it equals $2 - \sin^2 B$. Because

$$\sin B = \sin A \cos C + \sin C \cos A = \frac{2\sqrt{14}}{12} + \frac{\sqrt{10}}{12},$$

it follows that

$$2 - \sin^2 B = \frac{111 - 4\sqrt{35}}{72}.$$

The requested sum is $p + q + r + s = 111 + 4 + 35 + 72 = 222$.