MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

30th Annual

AIME I

Solutions Pamphlet

American Invitational Mathematics Examination I Solutions Pamphlet
Thursday, March 15, 2012

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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1. (Answer: 040)

   Because any multiple of 4 is even, it must be the case that both a and c are even. Furthermore, an integer xyz is a multiple of 4 if and only if the integer yz is a multiple of 4. Thus both $10b+c$ and $10b+a$ are multiples of 4. It follows that $(10b+a)-(10b+c)=a-c$ is a multiple of 4, so a and c leave the same remainder upon division by 4. Thus $\{a,c\}$ must be a subset of either $\{2,6\}$ or $\{4,8\}$.

   If $a=c=2$ or $a=c=6$ or $a=2$, $c=6$ or $a=6$, $c=2$, then b can be any of the digits 1, 3, 5, 7, or 9, for a total of 20 possibilities.

   If $a=c=4$ or $a=c=8$ or $a=4$, $c=8$ or $a=8$, $c=4$, then b can be any of the digits 0, 2, 4, 6, or 8, also for a total of 20 possibilities.

   Thus there are $20+20=40$ positive integers satisfying the conditions.

2. (Answer: 195)

   If the original sequence has n terms, then the sum of the added values is $1+3+5+\cdots+(2n-1)=n^2=836-715=121$, and $n=11$. It follows that 715 is 11 times the average value of the terms of the original sequence. This average, which is $\frac{715}{11}=65$, is the value of the middle term. The first and last terms of the sequence must add to twice 65, so the sum of the first, last, and middle terms is $3\cdot 65=195$.

3. (Answer: 216)

   Choose one person to receive the correct meal; this can be done in 9 ways. Then select 2 people to receive the other two meals of that type. If these meals are both given to people who originally ordered the same type of meal, there are 6 ways of choosing these people, and the remaining meals are then completely determined. If these two other meals go to people who ordered different types of meals, all remaining meals are determined except for the other diners who should have gotten the type of meal that was served correctly. There are $3\cdot 3$ ways to choose the recipients of the first two of these meals, and then 2 ways to distribute the last 2 meals. Thus there are $9(6+3\cdot 3\cdot 2)=216$ ways to distribute the meals.

4. (Answer: 279)

   Sparky, Butch, and Sundance take 10, 15, and 24 minutes, respectively, to walk one mile. Let x be the number of miles Butch walks. Then he rides $n-x$ miles and needs a total of $15x+10(n-x)$ minutes to cover n miles. Similarly, Sundance needs $24(n-x)+10x$ minutes. Equating these expressions gives $19x=14n$. Because x and n are positive integers, the smallest solutions are $x=14$ and $n=19$. Thus Butch walks 14 miles and rides $19-14=5$ miles, taking $15\cdot 14+10\cdot 5=260$ minutes to cover 19 miles, so $n+t=19+260=279$.

5. (Answer: 330)

   Let m and n be elements of B with $n = m + 1$. Let x_m denote the last two binary digits of m and x_n the last two binary digits of n. There are four cases to consider:

$$\begin{array}{lcl} x_m = 00 & \text{and} & x_n = 01 \\ x_m = 01 & \text{and} & x_n = 10 \\ x_m = 10 & \text{and} & x_n = 11 \\ x_m = 11 & \text{and} & x_n = 00 \end{array}$$

   The first and third cases are not possible, because then m and n would have different numbers of ones, so m and n could not both be in B. In the fourth case, if an element m of B ends in 11, then adding 1 to m results in a number with fewer digits equal to one than m, that is, the result of the addition cannot be in B. However, if $x_m = 01$ and $x_n = 10$, then m and $m + 1$ have the same number of digits equal to one, so if m is in B, then so is $m + 1 = n$. Given these last two digits, the eleven preceding digits consist of seven ones and four zeros. These can be arranged in $\binom{11}{4} = 330$ ways. Thus there are 330 pairs in B for which the difference is 1.

6. (Answer: 071)

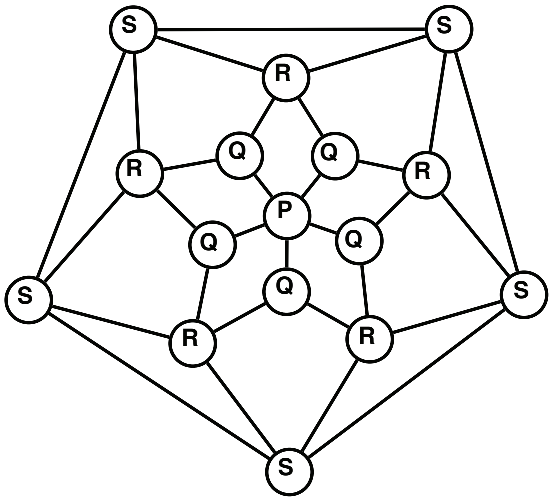
   It follows from the given conditions that $z = w^{11} = (z^{13})^{11} = z^{143}$ which implies that $z^{142} = 1$. All solutions of this equation satisfying the given conditions can be represented as $\cos\theta + i\sin\theta$, where $\theta = \frac{2k\pi}{142} = \frac{k\pi}{71}$ for some positive integer $k < 71$. It is straightforward to verify that for all θ of this form, $z = e^{i\theta}$ and $w = e^{11i\theta}$ satisfy the original equations. The number 71 is prime, so k and 71 are relatively prime, and it follows that $n = 71$.

7. (Answer: 280)

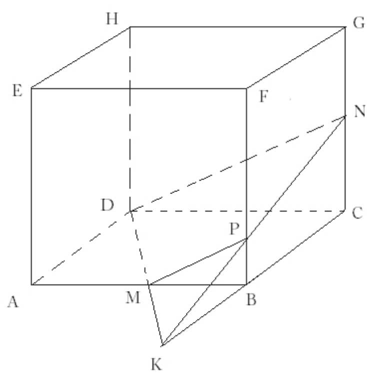
   Let p, q, r, and s be the sums of the numbers of coins held by all the students standing at all the circles in the diagram below labeled P, Q, R, and S, respectively. Using the numbers of connections in the diagram, it follows that $p = \frac{q}{3}$, $q = p + \frac{r}{2}$, $r = \frac{s}{2} + \frac{2q}{3}$, and $s = \frac{r}{2} + \frac{s}{2}$. From the first equation it follows that $q = 3p$. The second equation then gives $r = 4p$. Finally, both the third and the fourth equation give $s = 4p$. The total number of coins is $3360 = p + q + r + s = p + 3p + 4p + 4p = 12p$. Thus, $p = \frac{3360}{12} = 280$.

   Note that if each student with k neighbors starts with $56k$ coins, the conditions of the problem are satisfied.

   Query: Are there other initial distributions of the coins which satisfy the conditions of the problem?



8. (Answer: 089)



Let the plane DMN intersect $\overline{BF}$ at point P and the extension of $\overline{BC}$ at point K. Because M is the midpoint of $\overline{AB}$ and $\overline{MB}$ is parallel to $\overline{CD}$, it follows that B is the midpoint of $\overline{CK}$ and therefore $CK = 2$. Hence the volume of the pyramid $DCNK$ is equal to $\frac{1}{3}[CDN] \cdot CK = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 2 = \frac{1}{6}$. Because $DCNK$ is similar to $MBPK$ with a constant of proportionality of 2, the volume of $MBPK$ is $\frac{1}{2^3} \cdot \frac{1}{6} = \frac{1}{48}$, and the volume of $DCNMBP$ is $\frac{1}{6} - \frac{1}{48} = \frac{7}{48}$. Thus the volume of the larger solid is $1 - \frac{7}{48} = \frac{41}{48}$, and $p + q = 89$.

9. (Answer: 049)

Let $x = 2^a, y = 2^b$, and $z = 2^c$. Then the given equality becomes

$$2 \cdot \left(\frac{b+1}{a}\right) = 2 \cdot \left(\frac{c+2}{a+1}\right) = \frac{b+c+3}{4a+1}.$$

Because $\frac{b+1}{a} = \frac{c+2}{a+1}$, it follows that $\frac{b+1}{a} = \frac{c+2}{a+1} = \frac{b+1+c+2}{a+a+1}$. Thus $2 \cdot \left(\frac{b+c+3}{2a+1}\right) = \frac{b+c+3}{4a+1}$, and because $b+c+3 \neq 0$, this implies that $8a+2 = 2a+1$, so $a = -\frac{1}{6}$. Thus $\frac{b+1}{-\frac{1}{6}} = \frac{c+2}{-\frac{1}{6}+1}$, which implies that $c = -5b-7$. Every triple of the form $(x, y, z) = (2^{-1/6}, 2^b, 2^{-5b-7})$ is a solution to the given system, and hence $xy^5z = \frac{1}{2^{43/6}}$. Thus $p+q = 43+6 = 49$.

10. (Answer: 170)

Let n^2 be an element of $\mathcal{S}$. Then for some integer k, $n^2 = 1000k+256$ and $1000k = 2^3 \cdot 5^3 k = (n-16)(n+16)$. Because 5 cannot divide both $n-16$ and $n+16$, $5^3 = 125$ must divide one of them. Also, because $n^2 - 256$ is a multiple of 2^3, n must be a multiple of 4. It follows that the elements of $\mathcal{S}$ are precisely the numbers of the form $(500m \pm 16)^2$. The smallest such number is 256 when $m = 0$. The tenth smallest element of $\mathcal{S}$ is therefore $(500 \cdot 5 - 16)^2$, and the corresponding element of $\mathcal{T}$ is $6250 - 80 = 6170$. Hence the desired remainder is 170.

11. (Answer: 373)

Note that $x_{n+1} + y_{n+1} = x_n + y_n \pmod 3$ and $x_{n+1} - y_{n+1} = x_n - y_n \pmod 5$, so each reachable point is at the intersection of a line $x+y = 3j$ and a line $x-y = 5k$ with $-33 \le j \le 33$ and $-20 \le k \le 20$. Furthermore, j and k must have the same parity. The number of points that meet these conditions is $33 \cdot 21 + 34 \cdot 20 = 1373$. Call these points *good*.

To see that each good point is reachable, note first that it is possible to move from the line $x - y = 5k$ to either of the lines $x - y = 5(k+1)$ or $x - y = 5(k-1)$ by moving from $(x, x-5k)$ to $(x-5, x-5k-10)$ or $(x+2, x-5k+7)$, respectively. Therefore there is at least one reachable point on each line $x - y = 5k$ for $-20 \le k \le 20$. It is also possible to go from (x, y) to either $(x+9, y+9)$ or $(x-15, y-15)$ in two moves, so it is possible to make a sequence of moves from $(x, x+5k)$ to either $(x+3, x+5k+3)$ or $(x-3, x+5k-3)$. Thus if any good point on the line $x - y = 5k$ is reachable, so is every good point on that line. This implies that $M = 1373$, and the remainder when M is divided by 1000 is 373.

12. (Answer: 018)

The Angle Bisector Theorem gives $\frac{CD}{BC} = \frac{DE}{BE} = \frac{8}{15}$, so without loss of generality assume $BC = 15$ and $CD = 8$. Applying the Law of Cosines to $\triangle BCD$ gives $BD^2 = 8^2 + 15^2 - 2 \cdot 8 \cdot 15 \cdot \cos 60^\circ = 169$, so $BD = 13$. Again applying the Law of Cosines to $\triangle BCD$ gives $8^2 = 13^2 + 15^2 - 2 \cdot 13 \cdot 15 \cdot \cos\angle B$, showing that $\cos\angle B = \frac{11}{13}$. It then follows that $\sin\angle B = \sqrt{1 - \left(\frac{11}{13}\right)^2} = \frac{4\sqrt{3}}{13}$. Then $\tan\angle B = \frac{4\sqrt{3}}{11}$. The requested sum is $4+3+11 = 18$.

13. (Answer: 041)

Let O be the common center of the circles and let the triangle be ABC, where $OA = 3$, $OB = 4$, and $OC = 5$. If O lies outside $\triangle ABC$, then the triangle is contained in a semicircle of radius 5. Because $\angle ACB = 60°$ has its vertex C on the arc of the semicircle, the altitude of the equilateral triangle is limited by 5, so the side of the triangle is less than or equal to $\frac{10}{\sqrt{3}}$. If O lies inside $\triangle ABC$, consider the rotation $\mathbf{R}$ of $60°$ centered at A that sends B to C. Let $\mathbf{R}(O) = P$. Then $\triangle AOP$ is equilateral, $\mathbf{R}(\triangle ABO) = \triangle ACP$, and these two triangles are congruent. In $\triangle OPC$, $OP = 3$, $PC = 4$ and $CO = 5$, implying that $\angle OPC = 90°$ and $\angle APC = \angle APO + \angle OPC = 150°$. Applying the Law of Cosines to $\triangle APC$ gives $s^2 = AC^2 = AP^2 + PC^2 - 2AC \cdot PC \cos \angle APC = 3^2 + 4^2 + 3 \cdot 4\sqrt{3} = 25 + 12\sqrt{3}$. The area is $(25 + 12\sqrt{3}) \cdot \frac{\sqrt{3}}{4} = 9 + \frac{25}{4}\sqrt{3}$, and the requested sum is $9 + 25 + 4 + 3 = 41$.

OR

Place $\triangle ABC$ in a coordinate plane with $A = \left(-\frac{s}{2}, 0\right)$, $B = \left(\frac{s}{2}, 0\right)$, and $C = \left(0, \frac{\sqrt{3}s}{2}\right)$. The point $O = (x, y)$ satisfies $\left(x + \frac{s}{2}\right)^2 + y^2 = 9$, $\left(x - \frac{s}{2}\right)^2 + y^2 = 16$, and $x^2 + \left(y - \frac{\sqrt{3}s}{2}\right)^2 = 25$. Subtracting the second equation from the first gives $x = -\frac{7}{2s}$, and subtracting the third from the first gives $sx + \sqrt{3}sy = -16 + \frac{s^2}{2}$, from which $y = \frac{s^2 - 25}{2\sqrt{3}s}$. Then

$$9 = OA^2 = \left(-\frac{7}{2s} + \frac{s}{2}\right)^2 + \left(\frac{s^2 - 25}{2\sqrt{3}s}\right)^2,$$

from which $s^4 - 50s^2 + 193 = 0$, so $s^2 = 25 + 12\sqrt{3}$, and the result follows.

14. (Answer: 375)

Let a, b, and c correspond to points A, B, and C in the complex plane, respectively, and assume that $\angle ABC$ is right. Let the midpoint D of $\overline{AC}$ correspond to the number $d = \frac{a+c}{2}$. Because the coefficient of z^2 in $P(z)$ is 0, $a + b + c = 0$ and $b = -(a + c)$. Because D is the circumcenter of $\triangle ABC$, the distances DA, DB, and DC are equal, so $|b - d| = \frac{|a-c|}{2}$. Thus

$$\left| -(a + c) - \frac{a + c}{2} \right| = \frac{|a - c|}{2},$$

implying that $|a - c| = 3|a + c|$. Note that $|a|^2 + |c|^2 = \frac{|a-c|^2}{2} + \frac{|a+c|^2}{2}$ for any two complex numbers a and c. It follows that

$$250 = \frac{|a - c|^2}{2} + \frac{|a + c|^2}{2} + |a + c|^2 = 6|a + c|^2,$$

and $h^2 = |a - c|^2 = 9|a + c|^2 = \frac{9 \cdot 250}{6} = 375$.

15. (Answer: 332)

If $\gcd(a, n) = d > 1$, then the mathematician seated in seat n/d before the break will be seated in seat $an/d = n(a/d)$, which is seat n, after the break. This is impossible because a mathematician seated in seat n before the break will also be seated in seat n after the break, violating condition (1). On the other hand, if $\gcd(a, n) = 1$, then $\{a, 2a, \ldots, na\}$ is a complete set of residue classes modulo n, and hence condition (1) is satisfied.

Condition (2) holds if and only if for all i and j with $1 \le i < j \le n$, $ai - aj \not\equiv i - j \pmod{n}$ and $ai - aj \not\equiv j - i \pmod{n}$, that is, $(a-1)(i-j) \not\equiv 0 \pmod{n}$ and $(a+1)(i-j) \not\equiv 0 \pmod{n}$. Thus both $a-1$ and $a+1$ are relatively prime to n.

Combining the above, a pair (n, a) of positive integers satisfies the conditions of the problem if and only if

$$\gcd(a(a-1)(a+1), n) = 1.$$

Call such a pair a *good pair*.

Note that 6 divides $a(a-1)(a+1)$ for every positive integer a. Hence if $\gcd(n, 6) \neq 1$, then (n, a) is not a good pair. On the other hand, if $\gcd(n, 6) = 1$, then $(n, 3)$ is a good pair. Therefore all integers n with $1 < n < 1000$ and $\gcd(n, 6) = 1$ satisfy the conditions of the problem. These are exactly the integers of form either $6k + 1$ or $6k - 1$ with $1 \le k \le 166$, and there are $166 + 166 = 332$ such integers.

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