MAA

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AIME II

Solutions Pamphlet

American Invitational Mathematics Examination II Solutions Pamphlet
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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:

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1. (Answer: 034)

   Note that $20m+12n=2012$ if and only if $5m+3n=503$, so it is sufficient to find the number of positive integer solutions to the second equation. One positive integer solution to this equation is $(m,n)=(100,1)$. If (x,y) is another such solution, then $5(100-x)+3(1-y)=0$, so for some integer k, $100-x=3k$ and $1-y=-5k$. Thus the solutions to $5m+3n=503$ have the form $(m,n)=(100-3k,5k+1)$, and substituting these pairs into the original equation shows that they are all solutions. Both of these values are positive if and only if $0\le k\le 33$. Thus there are 34 solutions with positive integer values of the original equation.

2. (Answer: 363)

   Let r be the shared common ratio of the two sequences. Because the sequences are geometric, $27r^{15-1}=a_{15}=b_{11}=99r^{11-1}$, which implies that $r^4=\frac{11}{3}$. It follows that $a_9=27r^8=27\left(\frac{11}{3}\right)^2=3\cdot 121=363$.

3. (Answer: 088)

   If exactly one male professor is chosen from each department, then one female professor from each department must also be chosen. There are $2^6=64$ ways to form the committee in this manner. If two male professors are chosen from one of the departments, then two female professors must be chosen from one of the other two departments. The committee must then be completed by choosing one of the two male professors and one of the two female professors from the remaining department. There are $3\cdot 2\cdot 2\cdot 2=24$ ways to form the committee in this manner. Thus the number of committees that meet the requirements is $64+24=88$.

4. (Answer: 061)

   Let L be the length and W the width of the field in meters with $L>W$, and let x be the distance in meters from point D to the southeast corner of the field. According to the problem specifications, Ana bikes $2L+W-x$ meters, Bob bikes $W+x$ meters, and Cao bikes $\sqrt{W^2+x^2}$ meters, all in the same time. Thus it must be true that

   $$\frac{2L+W-x}{8.6}=\frac{W+x}{6.2}=\frac{\sqrt{W^2+x^2}}{5}.$$

   Simplifying the first equality yields $L=\frac{6}{31}W+\frac{37}{31}x$. Squaring both sides of the second equality and simplifying yields $168W^2-625Wx+168x^2=0$. The left side factors into $(24W-7x)(7W-24x)$, and so $x=\frac{24}{7}W$ or $x=\frac{7}{24}W$. Substituting into the expression for L yields

   $$L=\frac{6}{31}W+\frac{37}{31}\cdot\frac{24}{7}W=\frac{30}{7}W,$$

or

$$L = \frac{6}{31}W + \frac{37}{31} \cdot \frac{7}{24}W = \frac{13}{24}W.$$

The second value for L violates the requirement that $L > W$, so $L = \frac{30}{7}W$ and $x = \frac{24}{7}W$. The smallest value of W for which all of L, W, and x are integers is 7. Thus the required ratio is $30 : 7 : 24$, and $p + q + r = 61$.

5. (Answer: 750)

Let V be the vertex of the pyramid, let M be the center of S', and let P be the midpoint of one of the sides of S'. Then the height of the pyramid is VM, and the volume of the pyramid is $\frac{1}{3}(15)^2 \cdot VM$. Note that $\triangle PMV$ is a right triangle with hypotenuse $PV = 20 - \frac{15}{2}$ and leg $PM = \frac{15}{2}$. Therefore

$$\begin{aligned} VM &= \sqrt{PV^2 - PM^2} \\ &= \sqrt{(PV + PM)(PV - PM)} \\ &= \sqrt{20(20 - 15)} = 10\,. \end{aligned}$$

Hence the volume of the pyramid is $\frac{1}{3}(15)^2 \cdot 10 = 750$.

6. (Answer: 125)

The number z maximizes $|(1 + 2i)z^3 - z^5| = |z|^3 \cdot |1 + 2i - z^2|$. Because $|z|^3 = 125$ is fixed, it follows that z^2 points in the direction of $-1 - 2i$ with length 25, and thus $z^2 = -\delta(1 + 2i)$ for some positive real number δ. Thus $25 = \delta|1 + 2i| = \delta\sqrt{5}$ and $\delta^2 = 125$. Finally, $z^4 = 125(1 + 2i)^2 = 125(-3 + 4i)$, and the requested sum is $-375 + 500 = 125$.

7. (Answer: 032)

Because $\binom{12}{8} = 495$ and $\binom{13}{8} = 1287$, the 1000th integer in S must have 13 digits. There are $\binom{13}{8} = 1287$ integers in S less than 2^{13}, and exactly $\binom{11}{5} = 462$ of these have 13 digits and begin with 11. The 1000th integer in S must be the number in position $462 - (1287 - 1000) = 175$ among the numbers beginning with 11. Of these, $\binom{9}{3} = 84$ of them begin with 1100, another $\binom{8}{3} = 56$ of them begin with 11010, and $\binom{7}{3} = 35$ of them begin with 110110. It follows that the largest number in S beginning with 110110 must be in position $84 + 56 + 35 = 175$ among the elements of S that begin with 11. Thus the 1000th integer in S is 1101101111000. This number has the value

$$N = 2^{12} + 2^{11} + 2^9 + 2^8 + 2^6 + 2^5 + 2^4 + 2^3 = 7032\,.$$

The requested remainder when N is divided by 1000 is 32.

8. (Answer: 040)

Multiply corresponding sides of the given two equations to yield

$$\begin{aligned}\left(z+\frac{20i}{w}\right)\left(w+\frac{12i}{z}\right) &= (5+i)(-4+10i), \text{and}\\ zw+32i-\frac{240}{zw} &= -30+46i\,.\end{aligned}$$

Letting $v = zw$ gives $v^2-(-30+14i)v-240=0$. Thus

$$\begin{aligned}v &= \frac{-30+14i\pm\sqrt{(30-14i)^2+960}}{2}\\ &= -15+7i\pm\sqrt{(15-7i)^2+240}\\ &= -15+7i\pm\sqrt{416-210i}\,.\end{aligned}$$

Letting $416-210i=(a+bi)^2$ and equating the real and imaginary parts results in $ab=-105$ and $a^2-b^2=416$. The real solutions to this system are $(a,b)=(21,-5)$ or $(-21,5)$. Thus $v=-15+7i\pm(21-5i)$, so $v=6+2i$ or $-36+12i$. Thus the smallest possible value of $|zw|^2$ is $6^2+2^2=40$, and this value is attained when $w=2+4i$ and $z=1-i$.

9. (Answer: 107)

Let $\dfrac{\sin x}{\sin y}=A$, and let $\dfrac{\cos x}{\cos y}=B$. Note that

$$\frac{\sin 2x}{\sin 2y}=\frac{2\sin x\cos x}{2\sin y\cos y}=\frac{\sin x}{\sin y}\cdot\frac{\cos x}{\cos y}=AB.$$

Furthermore, $\sin x = A\sin y$ and $\cos x = B\cos y$. Square each of the last two equations and add the resulting equations to obtain $1=\sin^2 x+\cos^2 x = A^2\sin^2 y+B^2\cos^2 y = A^2(\sin^2 y+\cos^2 y)+(B^2-A^2)\cos^2 y$. Therefore $\cos^2 y=\dfrac{1-A^2}{B^2-A^2}$. Thus

$$\begin{aligned}\frac{\cos 2x}{\cos 2y} &= \frac{2\cos^2 x-1}{2\cos^2 y-1}=\frac{2B^2\cos^2 y-1}{2\cos^2 y-1}\\ &= \frac{2B^2\left(\frac{1-A^2}{B^2-A^2}\right)-1}{2\left(\frac{1-A^2}{B^2-A^2}\right)-1}=\frac{B^2-2A^2B^2+A^2}{2-A^2-B^2}\,.\end{aligned}$$

Substituting $A=3$ and $B=\dfrac{1}{2}$ produces $\dfrac{\sin 2x}{\sin 2y}=AB=\dfrac{3}{2}$, and $\dfrac{\cos 2x}{\cos 2y}=-\dfrac{19}{29}$. Thus $\dfrac{\sin 2x}{\sin 2y}+\dfrac{\cos 2x}{\cos 2y}=\dfrac{49}{58}$, and $p+q=107$.

10. (Answer: 496)

Let a be a positive integer. Because the distance from the integer a^2 to the integer $a(a+1)$ is $a(a+1) - a^2 = a$, there are a positive integers n that satisfy $a^2 \leq n < a(a+1)$. For each such n there is a rational number $x = \frac{n}{a}$ such that $n = xa = x\lfloor x \rfloor$. The condition $a(a+1) \leq 1000$ implies that $a \leq 31$. All values of n corresponding to $a = 1, 2, 3, \ldots, 31$ satisfy the conditions of the problem, and thus there are $1 + 2 + 3 + \cdots + 31 = \frac{31(31+1)}{2} = 496$ such values of n.

11. (Answer: 008)

Note that for $x \neq \frac{2}{3}$ or $-\frac{1}{3}$,

$$f_1(x) = \frac{2}{3} - \frac{3}{3x+1} = \frac{6x-7}{9x+3},$$

$$f_2(x) = f_1(f_1(x)) = \frac{6 \cdot \left(\frac{6x-7}{9x+3}\right) - 7}{9 \cdot \left(\frac{6x-7}{9x+3}\right) + 3} = \frac{-3x-7}{9x-6}, \text{ and}$$

$$f_3(x) = f_1(f_2(x)) = \frac{6 \cdot \left(\frac{-3x-7}{9x-6}\right) - 7}{9 \cdot \left(\frac{-3x-7}{9x-6}\right) + 3} = x.$$

Thus it follows that $f_{3k}(x) = x$ for all positive integers k. Thus $f_{1001}(x) = f_2(x) = \frac{-3x-7}{9x-6} = x - 3$, which is equivalent to $9x^2 - 30x + 25 = 0$, and the only solution to this quadratic is $x = \frac{5}{3}$. Thus $m + n = 5 + 3 = 8$.

Challenge: Show that if $p + q = 1$ and $f_1(x) = p - \frac{1}{x+q}$, then $f_3(x) = x$. The given problem highlights a special case of this fact in which $p = \frac{2}{3}$ and $q = \frac{1}{3}$.

12. (Answer: 958)

Because $\text{lcm}(7, 11, 13) = 1001$, the Chinese Remainder Theorem implies that there is a one-to-one correspondence between the integers from 0 to 1000 and the ordered triples $\{(a, b, c) : 0 \leq a \leq 6, 0 \leq b \leq 10, 0 \leq c \leq 12\}$, where the number n corresponds to the triple (a, b, c) if the remainders when n is divided by 7, 11, and 13 are a, b, and c, respectively. The number n is 7-safe if $3 \leq a \leq 4$, 11-safe if $3 \leq b \leq 8$, and 13-safe if $3 \leq c \leq 10$. Thus there are $(4-2) \cdot (8-2) \cdot (10-2) = 96$ positive integers less than 1001 that are 7-safe, 11-safe, and 13-safe. The same can be said for the integers in the range from 1001 to 2001, those from 2002 to 3002, those from 3003 to 4003, and so forth. It follows that there are $10 \cdot 96 = 960$ numbers between 0 and 10,009 that are simultaneously

7-safe, 11-safe, and 13-safe. The numbers 10,006 and 10,007 are the only integers between 10,000 and 10,009 that are 7-safe, 11-safe, and 13-safe, so the required number is $960 - 2 = 958$.

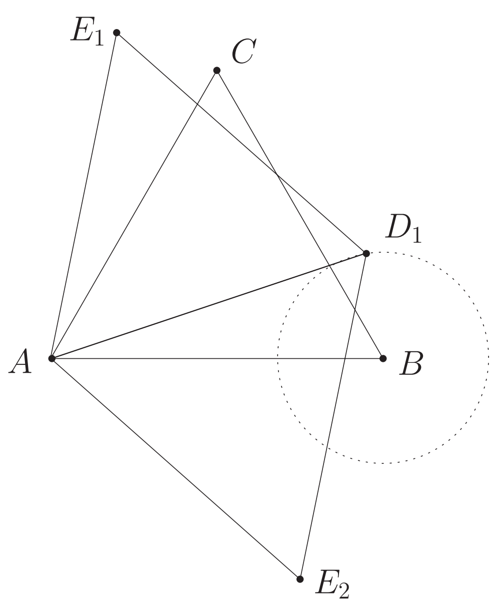
13. (Answer: 677)

Let $s = \sqrt{111}$ and $r = \sqrt{11}$, let θ be the common value of $\angle BAD_1$ and $\angle BAD_2$, so that $\angle CAE_k = \theta$, $120^\circ - \theta$, θ, and $120^\circ + \theta$ for $k = 1, 2, 3$, and 4, respectively. Applying the Law of Cosines to each of the triangles ACE_k gives

$$\begin{aligned} CE_1^2 &= CE_3^2 = 2s^2(1-\cos\theta), \\ CE_2^2 &= 2s^2\left(1-\cos\left(120^\circ-\theta\right)\right) = 2s^2\left(1-\cos\left(240^\circ+\theta\right)\right), \text{ and} \\ CE_4^2 &= 2s^2\left(1-\cos\left(120^\circ+\theta\right)\right). \end{aligned}$$

Because $\cos\theta + \cos(120^\circ+\theta) + \cos(240^\circ+\theta) = 0$, the sum $\sum_{k=1}^{4} CE_k^2$ can be simplified to $2s^2(4-\cos\theta)$. Furthermore, because $\angle BAD_1 = \angle CAE_1$, it follows that $r^2 = BD_1^2 = CE_1^2 = 2s^2(1-\cos\theta)$. Thus

$$\sum_{k=1}^{4} CE_k^2 = 2s^2(4-\cos\theta) = r^2 + 6s^2 = 11 + 6\cdot 111 = 677.$$



14. (Answer: 016)

For any particular method of conducting the handshakes, define a *minimal cycle* to be any subset of the 9 people in which

(a) no handshake occurs between a person in the subset and a person outside the subset, and

(b) no proper subset has property (a).

For any particular method of conducting the handshakes select any one of the nine people, A. Intersect all subsets of people both containing A and satisfying condition (a). This intersection will satisfy condition (b) and therefore will be minimal. It follows that for any particular method of conducting the handshakes, the set of nine people can be uniquely partitioned into minimal cycles.

The minimum size of a minimal cycle is 3, so the 9 people can be partitioned into 3 minimal cycles of 3 people each, a minimal cycle of 3 people and a minimal cycle of 6 people, a minimal cycle of 4 people and a minimal cycle of 5 people, or a minimal cycle of 9 people. Consider a minimal cycle of size $n \geq 3$. Select one person A from this set. The two people with whom A shakes hands (call them B and C) can be selected in $\binom{n-1}{2}$ ways. The set being minimal implies that B shakes hands with a fourth person D (for $n > 3$), D shakes hands with a fifth person E (for $n > 4$), and so on, with the last person in the set shaking hands with C. Thus there are $\binom{n-1}{2} \cdot (n-3)!$ methods of assigning the handshakes within this minimal cycle. When $n = 3$, 4, 5, 6, and 9, the number of methods are 1, 3, 12, 60, and 20160, respectively. It remains only to determine the number of ways the partitions can be formed.

The 3-3-3 partition can be done in $\frac{\binom{9}{3}\binom{6}{3}}{3!} = 280$ ways, the 3-6 partition in $\binom{9}{3} = 84$ ways, the 4-5 partition in $\binom{9}{4} = 126$ ways, and the 9 partition in 1 way. Thus the total number of methods is $280 \cdot 1 \cdot 1 \cdot 1 + 84 \cdot 1 \cdot 60 + 126 \cdot 3 \cdot 12 + 1 \cdot 20160 = 30016$, and the requested remainder is 16.

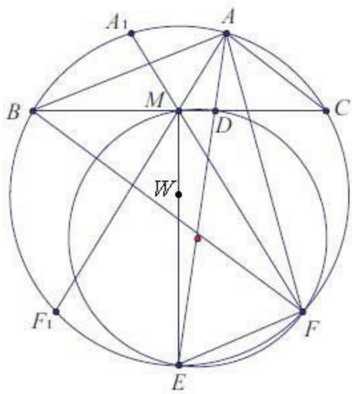
15. (Answer: 919)

Let M be the midpoint of $\overline{BC}$. Because $\overline{AE}$ bisects $\angle BAC$, point E is the midpoint of $\widehat{BC}$ and thus line ME is the perpendicular bisector of $\overline{BC}$. Then line ME passes through the center W of circle ω. Therefore $\angle EMD = 90^\circ$ and M lies on γ. Let rays AM and FM intersect ω at F_1 and A_1, respectively. Because $EFDM$ is cyclic,

$$\begin{aligned}\frac{\angle A_1WE}{2} &= \angle A_1FE = \angle MFE = \angle MDE \\ &= \angle BDE = \frac{\angle BWE + \angle AWC}{2},\end{aligned}$$

implying that $\widehat{BA_1} = \widehat{AC}$. Therefore A_1 and A are symmetric across line ME, from which it follows that F_1 and F are also symmetric across line ME and $\widehat{BF_1} = \widehat{CF}$. Thus $\angle BAF_1 = \angle FAC$ and

$$\begin{aligned}\angle ABF = \angle ABC + \angle CBF &= \frac{\angle AWC + \angle CWF}{2} \\ &= \frac{\angle AWC + \angle BWF_1}{2} = \angle AMC.\end{aligned}$$



Let R denote the radius of ω. By the Extended Law of Sines, $AF = 2R\sin\angle ABF = 2R\sin\angle AMC$. Applying the Law of Sines to $\triangle AMC$ gives

$$\frac{AC}{\sin\angle AMC} = \frac{AM}{\sin\angle ACB} \quad \text{or} \quad \sin\angle AMC = \frac{AC\sin\angle ACB}{AM}.$$

Consequently,

$$AF = \frac{2R \cdot AC\sin\angle ACB}{AM} = \frac{AB \cdot AC}{AM},$$

by the Extended Law of Sines. By the formula for the length of a median,

$$AM^2 = \frac{2AB^2 + 2AC^2 - BC^2}{4} = \frac{19}{4},$$

so $AF^2 = 900/19$. The requested sum is $900 + 19 = 919$.

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