MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

30th Annual

AIME II

American Invitational Mathematics Examination II
Wednesday, March 28, 2012

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted.**
4. A combination of the AIME and the American Mathematics Contest 12 are used to determine eligibility for participation in the USA Mathematical Olympiad (USAMO). A combination of the AIME and the American Mathematics Contest 10 are used to determine eligibility for participation in the USA Junior Mathematical Olympiad (USAJMO). The USAMO & the USAJMO will be given in your school on TUESDAY and WEDNESDAY, April 24 & 25, 2012.
5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

1. Find the number of ordered pairs of positive integer solutions (m, n) to the equation

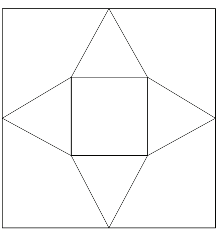
$$20m + 12n = 2012\,.$$

2. Two geometric sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ have the same common ratio, with $a_1 = 27$, $b_1 = 99$, and $a_{15} = b_{11}$. Find a_9.

3. At a certain university, the division of mathematical sciences consists of the departments of mathematics, statistics, and computer science. There are two male and two female professors in each department. A committee of six professors is to contain three men and three women and must also contain two professors from each of the three departments. Find the number of possible committees that can be formed subject to these requirements.

4. Ana, Bob, and Cao bike at constant rates of 8.6 meters per second, 6.2 meters per second, and 5 meters per second, respectively. They all begin biking at the same time from the northeast corner of a rectangular field whose longer side runs due west. Ana starts biking along the edge of the field, initially heading west, Bob starts biking along the edge of the field, initially heading south, and Cao bikes in a straight line across the field to a point D on the south edge of the field. Cao arrives at point D at the same time that Ana and Bob arrive at D for the first time. The ratio of the field's length to the field's width to the distance from point D to the southeast corner of the field can be represented as $p : q : r$, where p, q, and r are positive integers with p and q relatively prime. Find $p + q + r$.

5. In the accompanying figure, the outer square S has side length 40. A second square S' of side length 15 is constructed inside S with the same center as S and with sides parallel to those of S. From each midpoint of a side of S, segments are drawn to the two closest vertices of S'. The result is a four-pointed starlike figure inscribed in S. The star figure is cut out and then folded to form a pyramid with base S'. Find the volume of this pyramid.



6. Let $z = a + bi$ be the complex number with $|z| = 5$ and $b > 0$ such that the distance between $(1 + 2i)z^3$ and z^5 is maximized, and let $z^4 = c + di$. Find $c + d$.

7. Let S be the increasing sequence of positive integers whose binary representation has exactly 8 ones. Let N be the 1000th number in S. Find the remainder when N is divided by 1000.

8. The complex numbers z and w satisfy the system

$$\begin{aligned} z + \frac{20i}{w} &= 5 + i \\ w + \frac{12i}{z} &= -4 + 10i. \end{aligned}$$

Find the smallest possible value of $|zw|^2$.

9. Let x and y be real numbers such that $\dfrac{\sin x}{\sin y} = 3$ and $\dfrac{\cos x}{\cos y} = \dfrac{1}{2}$. The value of $\dfrac{\sin 2x}{\sin 2y} + \dfrac{\cos 2x}{\cos 2y}$ can be expressed in the form $\dfrac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

10. Find the number of positive integers n less than 1000 for which there exists a positive real number x such that $n = x\lfloor x \rfloor$.

    Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

11. Let $f_1(x) = \frac{2}{3} - \frac{3}{3x+1}$, and for $n \geq 2$, define $f_n(x) = f_1(f_{n-1}(x))$. The value of x that satisfies $f_{1001}(x) = x - 3$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

12. For a positive integer p, define the positive integer n to be *p-safe* if n differs in absolute value by more than 2 from all multiples of p. For example, the set of 10-safe numbers is $\{3, 4, 5, 6, 7, 13, 14, 15, 16, 17, 23, \ldots\}$. Find the number of positive integers less than or equal to 10,000 which are simultaneously 7-safe, 11-safe, and 13-safe.

13. Equilateral $\triangle ABC$ has side length $\sqrt{111}$. There are four distinct triangles AD_1E_1, AD_1E_2, AD_2E_3, and AD_2E_4, each congruent to $\triangle ABC$, with $BD_1 = BD_2 = \sqrt{11}$. Find $\displaystyle\sum_{k=1}^{4}(CE_k)^2$.

14. In a group of nine people each person shakes hands with exactly two of the other people from the group. Let N be the number of ways this handshaking can occur. Consider two handshaking arrangements different if and only if at least two people who shake hands under one arrangement do not shake hands under the other arrangement. Find the remainder when N is divided by 1000.

15. Triangle ABC is inscribed in circle ω with $AB = 5$, $BC = 7$, and $AC = 3$. The bisector of angle A meets side $\overline{BC}$ at D and circle ω at a second point E. Let γ be the circle with diameter $\overline{DE}$. Circles ω and γ meet at E and a second point F. Then $AF^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Your Exam Manager will receive a copy of the 2012 AIME Solution Pamphlet with the scores.

CONTACT US -- *Correspondence about the problems and solutions for this AIME and orders for any of our publications should be addressed to:*

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2011 USA(J)MO -- THE USA MATHEMATICAL OLYMPIAD (USAMO) and the USA MATHEMATICAL JUNIOR OLYMPIAD (USAJMO) are each a 6-question, 9-hour, essay-type examination. The USA(J)MO will be held in your school on Tuesday and Wednesday, April 24 & 25, 2012. Your teacher has more details on who qualifies for the USA(J)MO in the AMC 10/12 and AIME Teachers' Manuals. The best way to prepare for the USA(J)MO is to study previous years of these exams. Copies may be ordered from the web sites indicated below.

PUBLICATIONS -- For a complete listing of available publications please visit the following web sites:

AMC -- amc.maa.org/d-publication/publication.shtml
MAA -- www.maa.org/subpage_2.html

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